

**ECE-320: Linear Control Systems**  
Homework 8

Due: Friday May 8 at the beginning of class

1) Consider the discrete-time state variable model  $\underline{x}(k+1) = G(T)\underline{x}(k) + H(T)u(k)$

where the explicit dependence of  $G$  and  $H$  on the sampling time  $T$  has been emphasized. Here

$$G(T) = e^{AT}$$

$$H(T) = \int_0^T e^{A\lambda} d\lambda B$$

a) Show that if  $A$  is invertible, we can write  $H(T) = [e^{AT} - I]A^{-1}B$

b) Show that if  $A$  is invertible and  $T$  is small we can write the state model as

$$\underline{x}(k+1) = [I + AT]\underline{x}(k) + BTu(k)$$

c) Show that if we use the approximation

$$\dot{\underline{x}}(t) \approx \frac{\underline{x}((k+1)T) - \underline{x}(kT)}{T} = A\underline{x}(kT) + Bu(kT)$$

we get the same answer as in part **b**, but using this approximation we do not need to assume  $A$  is invertible.

d) Show that if we use two terms in the approximation for  $e^{AT}$  (and no assumptions about  $A$  being invertible), we can write the state equations as

$$\underline{x}(k+1) = [I + AT]\underline{x}(k) + [IT + \frac{1}{2}AT^2]Bu(k)$$

2) For the state variable system

$$\dot{\underline{x}}(t) = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

a) Show that

$$e^{AT} = \begin{bmatrix} 2e^{2T} - e^{3T} & e^{2T} - e^{3T} \\ 2e^{3T} - 2e^{2T} & 2e^{3T} - e^{2T} \end{bmatrix}$$

b) Derive the equivalent ZOH discrete-time system

$$\underline{x}(k+1) = G\underline{x}(k) + Hu(k)$$

for  $T = 0.1$  (integrate each entry in the matrix  $e^{A\lambda}$  separately.) Compare your answer with that given by Matlab's **c2d** command,  $[G,H] = c2d(A,B,T)$ .

3) For the matrix  $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$  show that  $e^{At} = \begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix}$

4) Consider the discrete-time state variable model  $\underline{x}(k+1) = G\underline{x}(k) + Hu(k)$  with the initial state  $x(0) = 0$ . Let

$$G = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, H = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, C = [1 \quad 0], D = 0$$

a) Determine the corresponding transfer function for the system.

b) Using state variable feedback with  $u(k) = G_{pf}r(k) - Kx(k)$  show that the transfer function is given by

$$F(z) = \frac{Y(z)}{R(z)} = C(zI - \tilde{G})^{-1} \tilde{H} = \frac{G_{pf}(z+1)}{(z+k_1)(z+k_2) - (k_1-1)(k_2-1)}$$

c) Show that if  $G_{pf} = 1$  and  $k_1 = k_2 = 0$ , the transfer function reduces to that found in part a.

d) Is the system controllable? That is, is it possible to find  $k_1$  and  $k_2$  to place the poles of the closed loop system where ever we want? For example, can we make both poles be zero?

5) Consider the discrete-time state variable model

$$\underline{x}(k+1) = G\underline{x}(k) + Hu(k)$$

with the initial state  $x(0) = 0$ . Let

$$G = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, H = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = [0 \quad 1], D = 0$$

a) Determine the corresponding transfer function for the system.

b) Using state variable feedback with  $u(k) = G_{pf}r(k) - Kx(k)$  show that the transfer function is given by

$$F(z) = \frac{Y(z)}{R(z)} = \frac{G_{pf}(z-1)}{(z-1)(z+k_2-1)}$$

c) Show that if  $G_{pf} = 1$  and  $k_1 = k_2 = 0$ , the transfer function reduces to that found in part a.

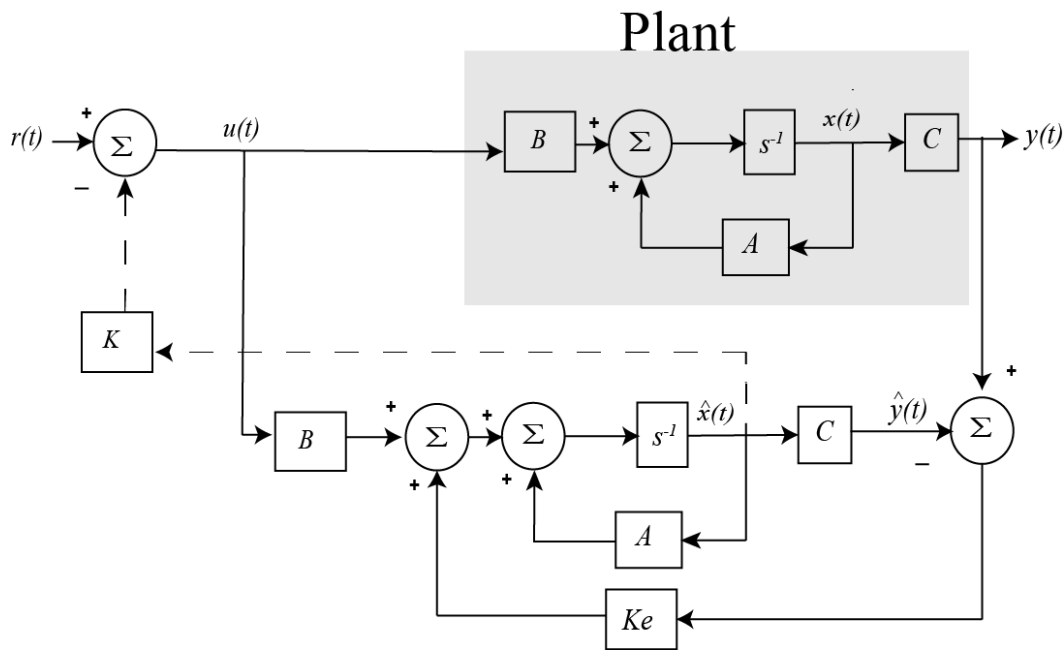
d) Is the system controllable?

For the following two problems we need the following information. If we have a state variable model with state feedback,  $\dot{x}(t) = Ax(t) + Bu(t)$   $u(t) = G_{pf}r(t) - Kx(t)$ , and we combine them, then we have  $\dot{x}(t) = [A - BK]x(t) + [BG_{pf}]r(t)$ . The poles of the system are determined by the eigenvalues of  $A - BK$ . The Matlab command **place** assumes we have a system in this form and we indicate the  $A$  and  $B$  matrices, indicate where we want the eigenvalues to be, and it returns  $K$ . So in the problems below, one of our goals is to put the system into this form.

Most of the answers to the following questions are very short.

6) In this problem we show how to use an observer to estimate the states of a feedback system so we do not have to measure them all. In this formulation, we only need a model of the plant, the input and the output (we need the input and the output for transfer function control too!)

Consider the following continuous-time state variable system with an observer:



For this system we have the plant representation

$$\dot{x}(t) = Ax(t) + Bu(t) \quad y(t) = Cx(t)$$

and the observer

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + K_e[y(t) - \hat{y}(t)] \quad \hat{y}(t) = C\hat{x}(t)$$

Note that the observer follows the same system dynamics as the plant (assuming we know the  $A$  and  $B$  matrices) plus a term that “correct” for a difference between the true output of the system  $y(t)$  and the estimated output from the observer  $\hat{y}(t)$  by an amount  $K_e$ . We use the estimated state  $\hat{x}(t)$  in the state variable feedback.

a) Assuming the output is a scalar and the state variable is an  $n \times 1$  vector, what is the size of  $K_e$ ?

b) Show that we can rewrite the observer equation as

$$\dot{\hat{x}}(t) = (A - K_e C)\hat{x}(t) + Bu(t) + K_e C x(t)$$

c) Show that if we subtract the equation in part (b) from the equation representing the plant we get

$$\dot{x}(t) - \dot{\hat{x}}(t) = [A - K_e C](x(t) - \hat{x}(t))$$

d) Now let's define the error vector as the difference between the true state and the estimated state,  $e(t) = x(t) - \hat{x}(t)$ , and with this definition show that the equation in part c is

$$\dot{e}(t) = [A - K_e C]e(t)$$

We want our estimate to equal the true state (eventually) so clearly we want  $e(t) \rightarrow 0$  as  $t \rightarrow \infty$  so we want to choose  $K_e$  so this happens.

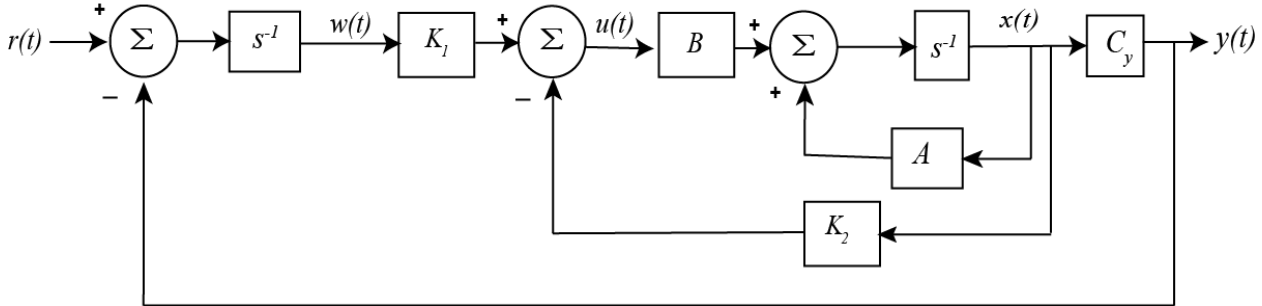
e) At this point we are basically done with everything we need to do except use the following little trick from linear algebra:  $(ABC)^T = C^T B^T A^T$ . With this trick show that we can write the above equations in the form

$$\dot{\hat{e}}(t)^T = [\tilde{A} - \tilde{B}\tilde{K}]e(t)^T$$

and determine  $\tilde{A}$ ,  $\tilde{B}$ , and  $\tilde{K}$ .

7) In this problem we show how to make a state variable feedback system into a type one system.

Consider the following continuous-time state variable system:



For this system we have

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = C_y x(t)$$

$$u(t) = -K_2 x(t) + K_1 w(t)$$

$$\dot{w}(t) = r(t) - y(t)$$

- a) Show that we can represent the open loop system in the following form, and determine the  $\alpha_i$

$$\begin{bmatrix} \dot{x}(t) \\ \dot{w}(t) \end{bmatrix} = \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{bmatrix} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} + \begin{bmatrix} \alpha_5 \\ \alpha_6 \end{bmatrix} u(t) + \begin{bmatrix} \alpha_7 \\ \alpha_8 \end{bmatrix} r(t)$$

- b) Define

$$x_e(t) = x(t) - x(\infty)$$

$$w_e(t) = w(t) - w(\infty)$$

$$u_e(t) = u(t) - u(\infty)$$

and assume  $r(t) = r(\infty) = 1$ . Show that we can write this new system in the following form and determine the  $\beta_i$

$$\begin{bmatrix} \dot{x}_e(t) \\ \dot{w}_e(t) \end{bmatrix} = \begin{bmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{bmatrix} \begin{bmatrix} x_e(t) \\ w_e(t) \end{bmatrix} + \begin{bmatrix} \beta_5 \\ \beta_6 \end{bmatrix} u_e(t)$$

- c) Show that we can write  $u_e(t) = -K_2 x_e(t) + K_1 w_e(t)$ . Don't just use the results from part a, you also need to use part b.

d) Now define

$$e(t) = \begin{bmatrix} x_e(t) \\ w_e(t) \end{bmatrix}$$

Show that we can now write the equations

$$\dot{e}(t) = \hat{A}e(t) + \hat{B}u_e(t)$$

$$u_e(t) = -\hat{K}e(t)$$

What are the  $\hat{A}$ ,  $\hat{B}$ , and  $\hat{K}$ ?

e) Show that we can then write  $\dot{e}(t) = (\hat{A} - \hat{B}\hat{K})e(t)$  which is the standard form for pole placement of a state variable feedback system. Once we have found  $\hat{K}$  we can extract  $K_1$  and  $K_2$