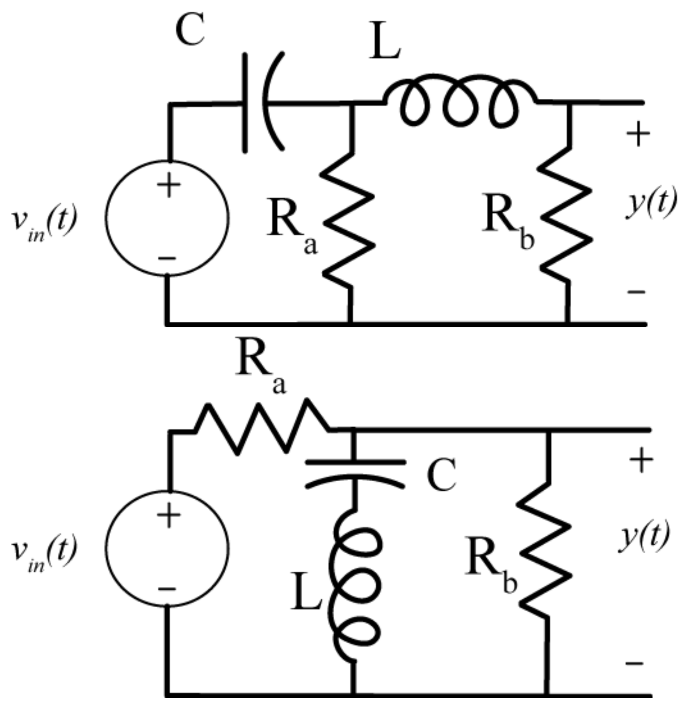
ECE-320: Linear Control Systems
Homework 7

Due: Friday May 1 at the beginning of class
EXAM 2, Friday May 1

1) For the following two circuits,

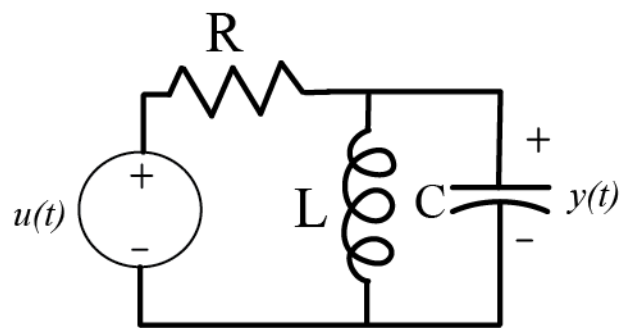


show that the state variable descriptions are given by

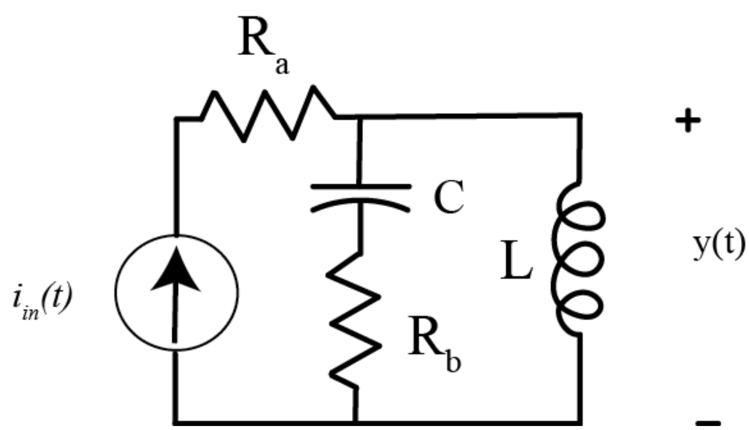
$$\frac{d}{dt}\begin{bmatrix} i_L(t) \\ v_c(t) \end{bmatrix} = \begin{bmatrix} -\frac{R_b}{L} & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{R_aC} \end{bmatrix}\begin{bmatrix} i_L(t) \\ v_c(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ \frac{1}{R_aC} \end{bmatrix} v_{in}(t) \quad y(t) = \begin{bmatrix} R_B & 0 \end{bmatrix}\begin{bmatrix} i_L(t) \\ v_c(t) \end{bmatrix} + [0]v_{in}(t)$$

$$\frac{d}{dt}\begin{bmatrix} i_L(t) \\ v_c(t) \end{bmatrix} = \begin{bmatrix} -\frac{R_aR_b}{L(R_a+R_b)} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}\begin{bmatrix} i_L(t) \\ v_c(t) \end{bmatrix} + \begin{bmatrix} \frac{R_b}{L(R_a+R_b)} \\ 0 \end{bmatrix} v_{in}(t) \quad y(t) = \begin{bmatrix} -\frac{R_aR_b}{R_a+R_b} & 0 \end{bmatrix}\begin{bmatrix} i_L(t) \\ v_c(t) \end{bmatrix} + \begin{bmatrix} \frac{R_b}{R_a+R_b} \end{bmatrix} v_{in}(t)$$

2) For the following circuit, the state variables are the current through the inductor and the voltage across the capacitor. Determine a state variable model for this system. Specifically, you need to identify the A, B, C, and D matrices/vectors/scalars.



3) For the following circuit, the state variables are the current through the inductor and the voltage across the capacitor. Determine a state variable model for this system. Specifically, you need to identify the A, B, C, and D matrices/vectors/scalars.



4) For the plant $G_p(s) = \dfrac{K}{\dfrac{1}{\omega_n^2}s^2 + \dfrac{2\zeta}{\omega_n}s + 1}$

a) If the plant input is $u(t)$ and the output is $x(t)$, show that we can represent this system with the differential equation

$$\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^2 x(t) = K\omega_n^2 u(t)$$

b) Assuming we use states $q_1(t) = x(t)$ and $q_2(t) = \dot{x}(t)$, and the output is $x(t)$, show that we can write the state variable description of the system as

$$\frac{d}{dt}\begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix}\begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ K\omega_n^2 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} u(t)$$

or

$$\dot{q}(t) = Aq(t) + Bu(t) \qquad y(t) = Cq(t) + Du(t)$$

Determine the A, B, C and D matrices.

c) Assume we use state variable feedback of the form $u(t) = G_{pf}r(t) - kq(t)$, where $r(t)$ is the new input to the system, G_{pf} is a prefilter (for controlling the steady state error), and k is the state variable feedback gain vector. Show that the state variable model for the closed loop system is

$$\dot{q}(t) = (A - Bk)q(t) + (BG_{pf})r(t)$$
$$y(t) = (C - Dk)q(t) + (DG_{pf})r(t)$$

or

$$\dot{q}(t) = \tilde{A}q(t) + \tilde{B}r(t)$$
$$y(t) = \tilde{C}q(t) + \tilde{D}r(t)$$

d) Show that the transfer function (matrix) for the closed loop system between input and output is given by

$$G(s) = \frac{Y(s)}{R(s)} = (C - Dk)(sI - (A - Bk))^{-1}BG_{pf} + DG_{pf}$$

and if D is zero this simplifies to

$$G(s) = \frac{Y(s)}{R(s)} = C(sI - (A - Bk))^{-1}BG_{pf}$$

e) Assume $r(t) = u(t)$ and $D = 0$. Show that, in order for $\lim_{t\to\infty} y(t) = 1$, we must have

$$G_{pf} = \frac{-1}{C(A - Bk)^{-1}B}$$

Note that the prefilter gain is a function of the state variable feedback gain!

If matrix P is given as

$$P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then

$$P^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

and the determinant of P is given by $ad - bc$. This determinant will also give us the characteristic polynomial of the system.

5) For each of the systems below:

- determine the transfer function when there is state variable feedback
- determine if k_1 and k_2 exist ($k = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$) to allow us to place the closed loop poles anywhere. That is, can we make the denominator look like $s^2 + a_1 s + a_0$ for any a_1 and any a_0. If this is true, the system is said to be ***controllable***.

a) Show that for

$$\dot{q} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} q + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} q + [0]u$$

the closed loop transfer function with state variable feedback is $G(s) = \dfrac{(s-1)G_{pf}}{(s-1)(s-1+k_2)}$

b) Show that for

$$\dot{q} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} q + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} q + [0]u$$

the closed loop transfer function with state variable feedback is $G(s) = \dfrac{sG_{pf}}{s^2 + (k_2 - 1)s + k_1}$

c) Show that for

$$\dot{q} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} q + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} q + [0]u$$

the closed loop transfer function with state variable feedback is $G(s) = \dfrac{G_{pf}}{s^2 + (k_2 - 1)s + (k_1 - 1)}$