ECE-320: Linear Control Systems
Homework 6

Due: Friday April 24 at the beginning of class

1) Consider the continuous-time plant with transfer function

$$G_p(s) = \frac{1}{(s+1)(s+2)}$$

We want to determine the discrete-time equivalent to this plant, $G_p(z)$, by assuming a zero order hold is placed before the continuous-time plant to convert the discrete-time control signal to a continuous time control signal.

Show that if we assume a sampling interval of T, the equivalent discrete-time plant is

$$G_p(z) = \frac{z(0.5 - e^{-T} + 0.5\mathrm{e}^{-2T}) + (0.5e^{-T} - e^{-2T} + 0.5e^{-3T})}{(z - e^{-T})(z - e^{-2T})}$$

Note that we have poles were we expect them to be, but we have introduced a zero in going from the continuous time system to the discrete-time system.

2) Consider a system with closed loop transfer function $G_o(s) = \dfrac{\alpha k_p}{s + \alpha + k_p}$. The *nominal* values for the parameters are $k_p = 1$ and $\alpha = 2$.

a) Determine an expression for the sensitivity of the closed loop system to variations in k_p. Your final answer should be written as numbers and the complex variable *s*.
b) Determine an expression for the sensitivity of the closed loop system to variations in α. Your final answer should be written as numbers and the complex variable *s*.
c) Determine expressions for the *magnitude* of the sensitivity functions in terms of frequency, ω
d) As $\omega \to \infty$ the system is more sensitive to which of the two parameters?

3) The file **zoh_files.slx**and **zoh_driver.m** illustrate how to model discrete-time transfer function systems from continuous-time transfer function systems in both Matlab and Simulink.

- The first system in **zoh_files.slx** illustrates how to utilize a sample and hold and a zero order hold to allow modelling a continuous-time system as a discrete-time system.
- The second system is the equivalent discrete-time constructed in Matlab. If you run the file **zoh_driver.m** you should see the results of the two simulations lie on top of each other.

For this problem turn in your plot of the outputs (the Matlab and Simulink should be the same) and print your **zoh_files.slx** file (so I can see it). This problem should be very simple, but I want you to have to see how to do some of these things in Matlab and Simulink.

4) Consider the plant

$$G_p(s) = \frac{\alpha_0}{s+\alpha_1} = \frac{3}{s+0.5}$$

where 3 is the nominal value of α_0 and 0.5 is the nominal value of α_1. In this problem we will investigate the sensitivity of closed loop systems with various types of controllers to these two parameters. We will assume we want the settling time of our system to be 0.5 seconds and the steady state error for a unit step input to be less than 0.1.

a) (*ITAE Model Matching*) Since this is a first order system, we will use the first order ITAE model,

$$G_o(s) = \frac{\omega_o}{s+\omega_o}$$

i) For what value of ω_o will we meet the settling time requirements and the steady state error requirements?

ii) Determine the corresponding controller $G_c(s)$.

iii) Show that the closed loop transfer function (using the parameterized form of $G_p(s)$ and the controller from part ii) is

$$G_o(s) = \frac{\frac{8}{3}\alpha_0(s+0.5)}{s(s+\alpha_1)+\frac{8}{3}\alpha_0(s+0.5)}$$

iv) Show that the sensitivity of $G_o(s)$ to variations in α_0 is given by $S_{\alpha_0}^{G_0} = \frac{s}{s+8}$

v) Show that the sensitivity of $G_o(s)$ to variations in α_1 is given by $S_{\alpha_1}^{G_o} = \frac{-0.5s}{s^2+8.5s+4}$

b) (*Proportional Control*) Consider a proportional controller, with $k_p = 2.5$.

i) Show that the closed loop transfer function is $G_o(s) = \frac{2.5\alpha_0}{s+\alpha_1+2.5\alpha_0}$

ii) Show that the sensitivity of $G_o(s)$ to variations in α_0 is given by $S_{\alpha_0}^{G_0} = \frac{s+0.5}{s+8}$

iii) Show that the sensitivity of $G_o(s)$ to variations in α_1 is given by $S_{\alpha_1}^{G_o} = \frac{-0.5}{s+8}$

c) (*Proportional+Integral Control*) Consider a PI controller with $k_p = 4$ and $k_i = 40$.

i) Show that the closed loop transfer function is $G_o(s) = \dfrac{4\alpha_0(s+10)}{s(s+\alpha_1)+4\alpha_0(s+10)}$

ii) Show that the sensitivity of $G_o(s)$ to variations in α_0 is given by $S_{\alpha_0}^{G_0} = \dfrac{s(s+0.5)}{s^2+12.5s+120}$

iii) Show that the sensitivity of $G_o(s)$ to variations in α_1 is given by $S_{\alpha_1}^{G_o} = \dfrac{-0.5s}{s^2+12.5s+120}$

d) Using Matlab, simulate the unit step response of each type of controller. Plot all responses on one graph. Use different line types and a legend. Turn in your plot and code. ***Do not*** *make separate graphs for each system!*

e) Using Matlab and subplot, plot the sensitivity to α_0 for each type of controller on ***one graph*** at the top of the page, and the sensitivity to α_1 on one graph on the bottom of the page. Be sure to use different line types and a legend. Turn in your plot and code. Only plot up to about 8 Hz (50 rad/sec) using a semilog scale with the sensitivity in dB (see next page). ***Do not*** *make separate graphs for each system!*

In particular, these results should show you that the model matching method, which essentially tries and cancel the plant, are generally more sensitive to getting the plant parameters correct than the PI controller for low frequencies. However, for higher frequencies the methods are all about the same.

Hint: If $T(s) = \dfrac{2s}{s^2+2s+10}$, *plot the magnitude of the frequency response using:*

```
T = tf([2 0],[1 2 10]);
w = logspace(-1,1.7,1000);
[M,P]= bode(T,w);
Mdb = 20*log10(M(:));
semilogx(w,Mdb); grid;
xlabel('Frequency (rad/sec)');
ylabel('dB');
```