ECE-320: Linear Control Systems
Homework 9

Due: Thursday May 12 at the beginning of class

1) Consider the discrete-time state variable model

$$\underline{x}(k+1) = G(T)\underline{x}(k) + H(T)u(k)$$

where the explicit dependence of G and H on the sampling time T has been emphasized. Here

$$G(T) = e^{AT}$$
$$H(T) = \int_0^T e^{A\lambda} d\lambda B$$

a) Show that if A is invertible, we can write $H(T) = [e^{AT} - I]A^{-1}B$

b) Show that if A is invertible and T is small we can write the state model as

$$\underline{x}(k+1) = [I + AT]\underline{x}(k) + BTu(k)$$

c) Show that if we use the approximation

$$\dot{\underline{x}}(t) \approx \frac{\underline{x}((k+1)T) - \underline{x}(kT)}{T} = Ax(kT) + Bu(kT)$$

we get the same answer as in part **b,** but using this approximation we do not need to assume A is invertible.

d) Show that if we use two terms in the approximation for e^{AT} (and no assumptions about A being invertible), we can write the state equations as

$$\underline{x}(k+1) = \left[I + AT\right]\underline{x}(k) + [T + \frac{1}{2}AT^2]Bu(k)$$

2) For the matrix $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ show that $e^{At} = \begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix}$

3) For the state variable system

$$\underline{\dot{x}}(t) = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

a) Show that

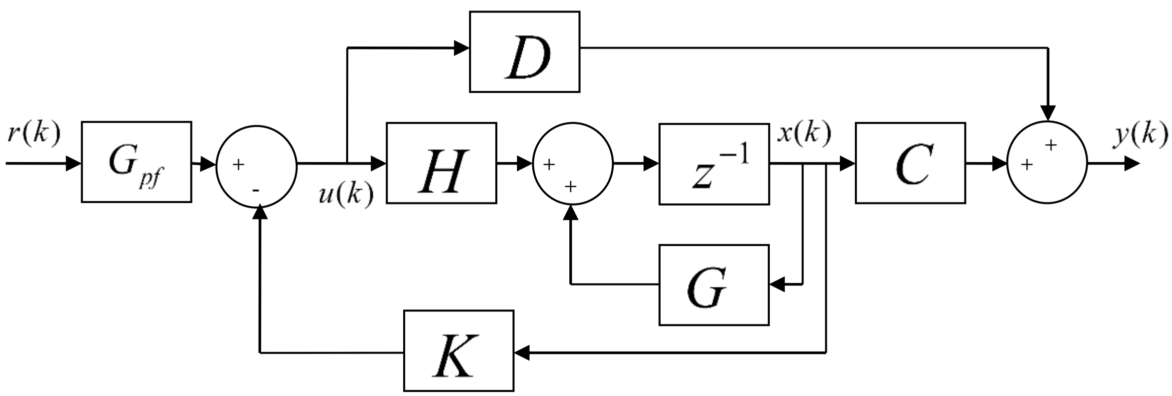
$$e^{AT} = \begin{bmatrix} 2e^{2T} - e^{3T} & e^{2T} - e^{3T} \\ 2e^{3T} - 2e^{2T} & 2e^{3T} - e^{2T} \end{bmatrix}$$

b) Derive the equivalent ZOH discrete-time system

$$\underline{x}(k+1) = G\underline{x}(k) + Hu(k)$$

for $T = 0.1$ (integrate each entry in the matrix $e^{A\lambda}$ separately.) Compare your answer with that given by Matlab's **c2d** command.

4) Consider the following state variable system



Here G_{pf} is the *prefilter gain*, $r(k)$ is the *reference input*, and $K = [k_1 \quad k_2]$ is the *feedback gain matrix*. The state variable model for the plant is assumed to be

$$x(k+1) = Gx(k) + Hu(k)$$
$$y(k) = Cx(k) + Du(k)$$

From the diagram we have $u(k) = G_{pf} r(k) - Kx(k)$.

a) Determine an expression for the transfer function between the input $R(z)$ and the output $Y(z)$.

b) Assuming $D = 0$ and using the Final Value Theorem, show that for a single input single output system to have a zero steady state error for a unit step input we need to choose the prefilter to be

$$G_{pf} = \frac{1}{C(I-G+HK)^{-1}H}$$

Hint: In order to have a zero steady state error for a unit step input, the final value of the output, $y(\infty) = y_{ss}$ *, should be 1.*

5) Consider the discrete-time state variable model

$$\underline{x}(k+1) = G\underline{x}(k) + Hu(k)$$

with the initial state $x(0) = 0$. Let

$$G = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, H = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}, D = 0$$

a) Determine the corresponding transfer function for the (open loop) system.

b) Now assume we are using state variable feedback with a <u>*prefilter gain*</u> G_{pf} , so $u(k) = G_{pf}r(k) - Kx(k)$. Here $r(k)$ is the <u>*reference input*</u> and $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$ is the <u>*feedback gain matrix*</u>. With this form of state variable feedback, we have the system

$$x(k+1) = Gx(k) + H\left[G_{pf}r(k) - Kx(k)\right] = [G - HK]x(k) + [HG_{pf}]r(k)$$

or

$$x(k+1) = \tilde{G}x(k) + \tilde{H}r(k)$$

Note that now the system input is the reference input $r(k)$. Show that for $D = 0$ the transfer matrix is given by

$$F(z) = \frac{Y(z)}{R(z)} = C(zI - \tilde{G})^{-1}\tilde{H} = \frac{G_{pf}(z+1)}{(z+k_1)(z+k_2) - (k_1-1)(k_2-1)}$$

c) Show that if $G_{pf} = 1$ and $k_1 = k_2 = 0$, the transfer function reduces to that found in part **a.**

d) Is it possible to find k_1 and k_2 to place the poles of the closed loop system where ever we want? For example, can we make both poles be zero?

6) Consider the discrete-time state variable model

$$\underline{x}(k+1) = G\underline{x}(k) + Hu(k)$$

with the initial state $x(0) = 0$. Let

$$G = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, H = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}, D = 0$$

a) Determine the corresponding transfer function for the (open loop) system.

b) Now assume we are using state variable feedback with $u(k) = G_{pf} r(k) - Kx(k)$. Show that for $D = 0$ the transfer matrix is given by

$$F(z) = \frac{Y(z)}{R(z)} = \frac{G_{pf}}{z^2 + (k_2 - 1)z + (k_1 - 1)}$$

c) Show that if $G_{pf} = 1$ and $k_1 = k_2 = 0$, the transfer function reduces to that found in part **a.**

d) Is it possible to find k_1 and k_2 to place the poles of the closed loop system where ever we want? For example, can we make both poles be zero? If we want the poles to be at p_1 and p_2 show that $k_1 = 1 - (p_1 + p_2)$ and $k_2 = 1 + p_1 p_2$.