

**ECE-320: Linear Control Systems**  
Homework 7

Due: Tuesday April 27, 2010 at the beginning of class

1) For the plant

$$G_p(s) = \frac{K}{\frac{1}{\omega_n^2} s^2 + \frac{2\zeta}{\omega_n} s + 1}$$

a) If the plant input is  $u(t)$  and the output is  $x(t)$ , show that we can represent this system with the differential equation

$$\frac{1}{\omega_n^2} \ddot{x}(t) + \frac{2\zeta}{\omega_n} \dot{x}(t) + x(t) = Ku(t)$$

b) Assuming we use states  $q_1(t) = x(t)$  and  $q_2(t) = \dot{x}(t)$ , and the output is  $x(t)$ , show that we can write the state variable description of the system as

$$\frac{d}{dt} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ K\omega_n^2 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} u(t)$$

or

$$\dot{q}(t) = Aq(t) + Bu(t)$$

$$y(t) = Cq(t) + Du(t)$$

Determine the A, B, C and D matrices.

c) Assume we use state variable feedback of the form  $u(t) = G_{pf}r(t) - kq(t)$ , where  $r(t)$  is the new input to the system,  $G_{pf}$  is a prefilter (for controlling the steady state error), and  $k$  is the state variable feedback gain vector. Show that the state variable model for the closed loop system is

$$\dot{q}(t) = (A - Bk)q(t) + (BG_{pf})r(t)$$

$$y(t) = (C - Dk)q(t) + (DG_{pf})r(t)$$

or

$$\dot{q}(t) = \tilde{A}q(t) + \tilde{B}r(t)$$

$$y(t) = \tilde{C}q(t) + \tilde{D}r(t)$$

d) Show that the transfer function (matrix) for the closed loop system between input and output is given by

$$G(s) = \frac{Y(s)}{R(s)} = (C - Dk)(sI - (A - Bk))^{-1} BG_{pf} + DG_{pf}$$

and if  $D$  is zero this simplifies to

$$G(s) = \frac{Y(s)}{R(s)} = C(sI - (A - Bk))^{-1} BG_{pf}$$

e) Assume  $r(t) = 1$  and  $D = 0$ . Show that, in order for  $\lim_{t \rightarrow \infty} y(t) = 1$ , we must have

$$G_{pf} = \frac{-1}{C(A - Bk)^{-1} B}$$

Note that the prefilter gain is a function of the state variable feedback gain!

If matrix  $P$  is given as

$$P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then

$$P^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

and the determinant of  $P$  is given by  $ad - bc$ . This determinant will also give us the characteristic polynomial of the system.

2) For each of the systems below:

- determine the transfer function when there is state variable feedback
- determine if  $k_1$  and  $k_2$  exist ( $k = [k_1 \quad k_2]$ ) to allow us to place the closed loop poles anywhere. That is, can we make the denominator look like  $s^2 + a_1s + a_0$  for any  $a_1$  and any  $a_0$ . If this is true, the system is said to be **controllable**.

a) Show that for

$$\begin{aligned} \dot{q} &= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} q + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= [0 \quad 1] q + [0] u \end{aligned}$$

the closed loop transfer function with state variable feedback is

$$G(s) = \frac{(s-1)G_{pf}}{(s-1)(s-1+k_2)}$$

b) Show that for

$$\dot{q} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} q + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = [0 \quad 1]q + [0]u$$

the closed loop transfer function with state variable feedback is

$$G(s) = \frac{sG_{pf}}{s^2 + (k_2 - 1)s + k_1}$$

c) Show that for

$$\dot{q} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} q + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = [1 \quad 0]q + [0]u$$

the closed loop transfer function with state variable feedback is

$$G(s) = \frac{G_{pf}}{s^2 + (k_2 - 1)s + (k_1 - 1)}$$

## Preparation for Lab 7

3) *You will be using this code and the following designs in Lab 7, so come prepared! This prelab is really pretty mindless, so just follow along*

a) The one degree of freedom Simulink model (**Basic\_1dof\_State\_Variable\_Model.mdl**) implements a state variable model for a one degree of freedom system. This model uses the Matlab code **Basic\_1dof\_State\_Variable\_Model\_Driver.m** to drive it. Both of these programs are available on the course website.

a) Get the state variable model files for the systems you modeled in lab. *Since you will be implementing these controllers during lab 9, if you have any clue at all you and your lab partner will do different systems!*

*You will need to have **Basic\_1dof\_State\_Variable\_Model\_Driver.m** load the correct state model into the system!*

b) You need to set the **saturation\_level** to the correct level for the rectilinear (model 210) or torsional (model 205) system. Assume we have an input step of 1 cm or 15 degrees (be sure to convert to radians!)

c) Design a state variable feedback system using **pole placement** for either your torsional or your rectilinear system. For this method, we basically guess the pole locations and simulate the system. To set the location of the closed loop poles, find the part of the code that assigns poles to the variable p, and change the elements of p. You will need to

choose the closed loop pole locations (This is a guess and check sort of thing. The biggest problem is making sure the control effort is not too large.) Your resulting design must have a settling time of 0.5 seconds or less and must have a percent overshoot of 10% or less. Your design should not saturate the system (control effort) and you should use a **constant prefilter**.

d) Run your simulation for 2.0 seconds. Plot both the system output (from 0 to 2 seconds) and the control effort (from 0 to 0.2 seconds). Plot the control effort only out to 0.2 seconds since the control effort is usually largest near the initial time. If your control effort reaches its limits, you need to go back and modify your design. Turn in your plot with your closed loop poles and your gains (you can just write these on your plot).

e) An alternative method for determining the feedback gains is based on what is called a **linear quadratic regulator**. The linear quadratic regulator finds the gain  $K$  to minimize

$$J = \int_0^{\infty} [x^T(t)Qx(t) + u(t)^T Ru(t)] dt$$

where

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$u(t) = -Kx(t)$$

For our one degree of freedom systems,  $Q$  is a 2x2 positive definite matrix, and  $R$  is a scalar. Since we will use a diagonal matrix for  $Q$  and for our system  $u(t)$  is a scalar, we can rewrite  $J$  as

$$J = \int_0^{\infty} [q_1 x_1^2(t) + q_2 \dot{x}_1^2(t) + Ru^2(t)] dt$$

A large value of  $R$  penalizes a large control signal, a large value of  $q_1$  will penalize the position of the first cart, while a large value of  $q_2$  will penalize a large value of the velocity of the first cart. All of the  $q_i$  should be zero or positive.

It's easiest to find  $K$  using the following command in Matlab:  $K = lqr(A, B, diag([q_1 \quad q_2]), R)$

Try different values of the  $q_i$  to find an acceptable controller. Turn in your plot with your closed loop poles and your gains (you can just write these on your plot).

*Turn in your plots and your code. Be sure your plots are accurately labeled!*