ECE-320: Linear Control Systems
Homework 2

Due: Thursday March 18 *at the beginning of class*

1) For the following transfer functions

$$H(s)=\frac{2}{s^2+2s+2} \quad H(s)=\frac{3}{s^2+4s+6}$$
$$H(s)=\frac{5}{s^2+6s+10} \quad H(s)=\frac{4}{s^2-4s+7}$$

By computing the inverse Laplace transform show that the step responses are given by

$$y(t)=\left[1-e^{-t}\cos(t)-e^{-t}\sin(t)\right]u(t)$$
$$y(t)=\left[\frac{1}{2}-\frac{1}{\sqrt{2}}e^{-2t}\sin(\sqrt{2}t)-\frac{1}{2}e^{-2t}\cos(\sqrt{2}t)\right]u(t)$$
$$y(t)=\left[\frac{1}{2}-\frac{3}{2}e^{-3t}\sin(t)-\frac{1}{2}e^{-3t}\cos(t)\right]u(t)$$
$$y(t)=\left[\frac{4}{7}+\frac{8\sqrt{3}}{21}e^{2t}\sin(\sqrt{3}t)-\frac{4}{7}e^{2t}\cos(\sqrt{3}t)\right]u(t)$$

2) For the following transfer functions, determine the static gain and the steady state output for a step input of amplitude 2.

$$G_1(s) = \frac{s+2}{s^2+s+1}$$
$$G_2(s) = \frac{s+1}{s+2}$$
$$G_3(s) = \frac{s-1}{(s+1)(s+3)}$$

3) For a system with transfer function

$$H(s)=\frac{a}{s^2+2s+3}$$

what is the range of values of a so that the absolute value of the steady state error for a unit step input is less than 0.2? What value of a will produce a finite error for a unit ramp input? (*Ans.* $2.4 < a < 3.6$,3)

4) For a system with transfer function

$$H(s) = \frac{bs+3}{s^2+2s+3}$$

what is the range of values of b so that the absolute value of the steady state error for a unit ramp input is less than 0.1? What is the steady state error for a unit step input? ($Ans. 1.7 < b < 2.3, 0$)

5) For systems with the following transfer functions:

$$H_a(s) = \frac{1}{s+2}$$

$$H_b(s) = \frac{s+6}{(s+2)(s+3)}$$

a) Determine the unit step and unit ramp response for each system using Laplace transforms. Your answer should be time domain functions $y_a(t)$ and $y_b(t)$.

b) From these time domain functions, determine the steady state errors for a unit step and unit ramp input.

c) Using the equations derived in class (and in the notes), determine the steady state errors for a unit step and a unit ramp input directly from the transfer functions.

The following Matlab code can be used to estimate the step and ramp response for 5 seconds for transfer function $H_b(s)$.

```
H = tf([1 6],[1 5 6]);                    % enter the transfer function
t = [0:0.01:5];                           % t goes from 0 to 5 by increments of 0.01
ustep = ones(1,length(t));                % the step input is all ones, u(t) = 1;
uramp = t;                                % the ramp input is has the input u(t) = t;
ystep = lsim(H,ustep,t);                  % find the step response
yramp = lsim(H,uramp,t);                  % find the ramp response
figure;                                   % make a new figure
orient tall                               % or orient landscape, use more of the page
subplot(2,1,1);                           % put two graphs on one piece of paper
plot(t,ustep,'.-',t,ystep,'-');           % plot input/output with different line types
grid;                                     % put a grid on the graph
legend('Step Input','Step Response',4);   % put a legend on the graph
subplot(2,1,2);                           % second of two graphs on one piece of paper
plot(t,uramp,'.-',t,yramp,'-');           % plot input/output with different line types
grid;                                     % put a grid on the graph
legend('Ramp Input','Ramp Response',4);   % put a legend on the graph
```

d) Plot the step and ramp response for both systems (a and b) and indicate the steady state errors on the graph. Draw on the graph to show you know what the steady state errors are.
Ans. Steady state errors for a unit step input: 0.5,0; for a unit ramp input : infinity and 0.666

6) For systems A and B, with step responses shown in Figure 1, estimate

- the percent overshoot
- the settling time
- the steady state error for the step input shown
- the steady state error for a unit ramp input

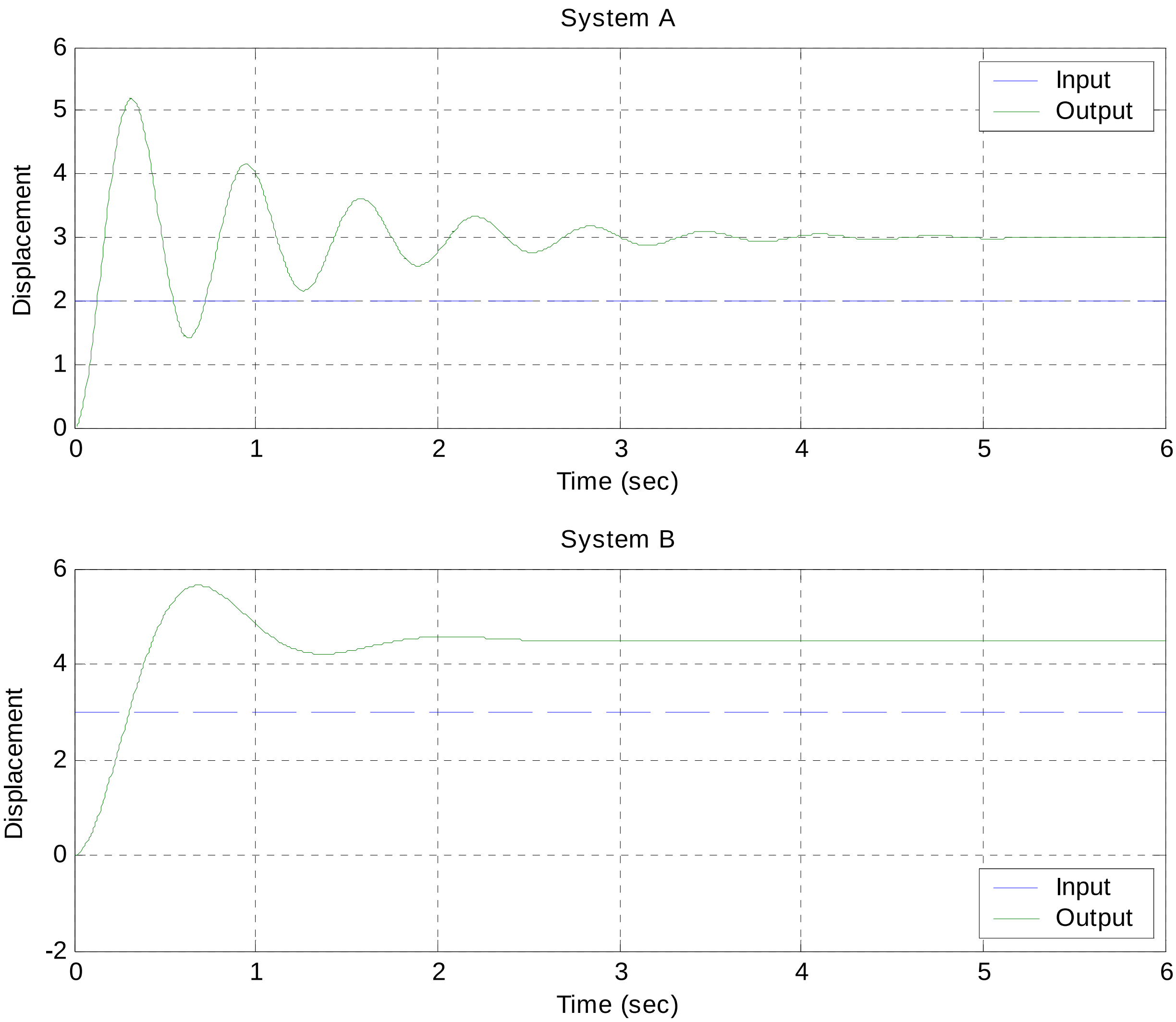


Figure 1. Step responses of system A and system B.

7) For systems C and D, with ramp responses shown in Figure 2, determine

- the steady state error for the ramp input shown
- the steady state error for a unit step input

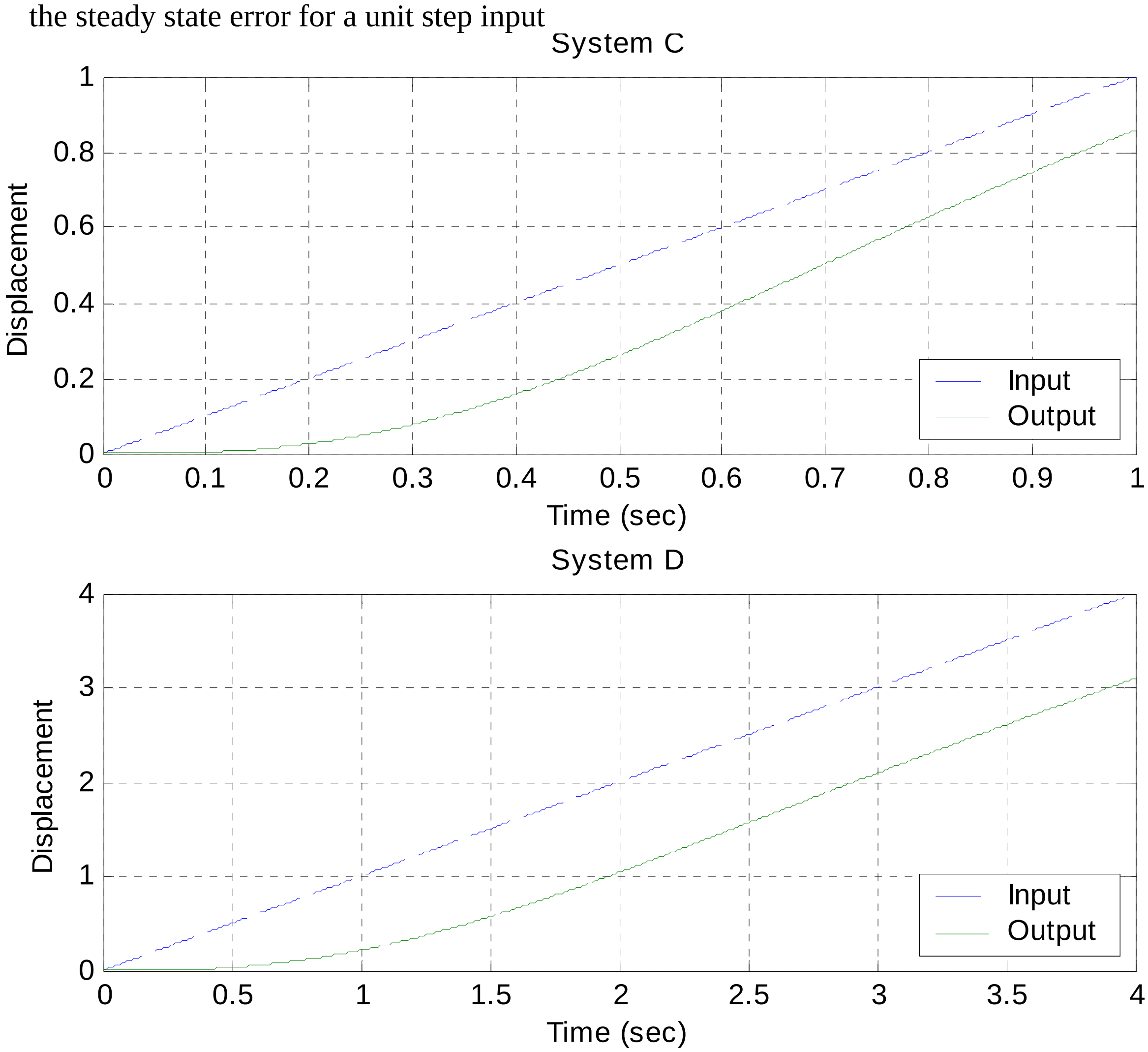


Figure 2. Ramp responses for systems C and D.

Answers for 6 and 7 (in no particular order, your approximations should be close, but they probably won't match.) 0,0,73%, 25%, 2, 3.5, -1, -1.2,1,0.16, ∞ *,* ∞

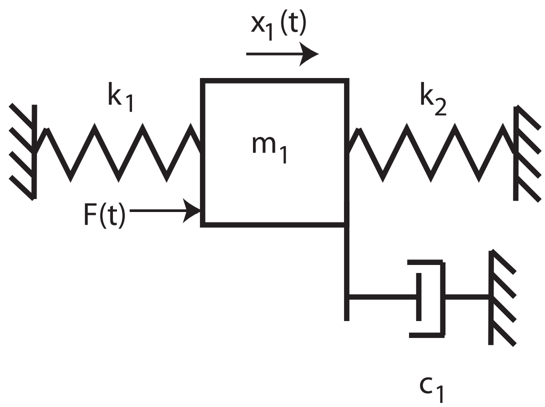
8) An ideal second order system has the transfer function $G_o(s)$. The system specifications for a step input are as follows:

a) Percent Overshoot < 5%
b) Settling Time < 4 seconds (2% criteria)
c) Peak Time < 1 second

Show the permissible area for the poles of $G_o(s)$ in order to achieve the desired response.

Preparation for Lab 2

9) Consider the following one degree of freedom system we will be utilizing this term:



a) Draw a freebody diagram of the forces on the mass.

b) Show that the equations of motion can be written:

$$m_1\ddot{x}_1(t) + c_1\dot{x}(t) + (k_1 + k_2)x_1(t) = F(t)$$

or

$$\frac{1}{\omega_n^2}\ddot{x}_1(t) + \frac{2\zeta}{\omega_n}\dot{x}(t) + x_1(t) = KF(t)$$

c) What are the damping ratio ζ, the natural frequency ω_n, and the static gain K in terms of m_1, k_1, k_2, and c_1?

d) Show that the transfer function for the *plant* is given by

$$G_p(s) = \frac{X_1(s)}{F(s)} = \frac{K}{\frac{1}{\omega_n^2}s^2 + \frac{2\zeta}{\omega_n}s + 1}$$

10) One of the methods we will be using to identify ζ and ω_n is the *log-decrement* method, which we will review/derive in this problem. If our system is at rest and we provide the mass with an initial displacement away from equilibrium, the response due to this displacement can be written

$$x_1(t) = Ae^{-\zeta\omega_n t}\cos(\omega_d t + \theta)$$

where

$x_1(t)$ = displacement of the mass as a function of time
ζ = damping ratio

ω_n = natural frequency

ω_d = damped frequency = $\omega_n \sqrt{1-\zeta^2}$

After the mass is released, the mass will oscillate back and forth with period given by $T_d = \dfrac{2\pi}{\omega_d}$, so if we measure the period of the oscillation (T_d) we can estimate ω_d.

Let's assume t_0 is the time of one peak of the cosine. Since the cosine is periodic, subsequent peaks will occur at times given by $t_n = t_0 + nT_d$, where n is an integer.

a) Show that

$$\frac{x_1(t_0)}{x_1(t_n)} = e^{\zeta \omega_n T_d n}$$

b) If we define the log decrement as

$$\delta = \ln\left[\frac{x_1(t_0)}{x_1(t_n)}\right]$$

show that we can compute the damping ratio as

$$\zeta = \frac{\delta}{\sqrt{4n^2\pi^2 + \delta^2}}$$

c) Given the initial condition response shown in the Figures 3 and 4 on the next page, estimate the damping ratio and natural frequency using the log-decrement method. (*You should get answers that include the numbers 15, 0.2, 0.1 and 15, approximately*.)

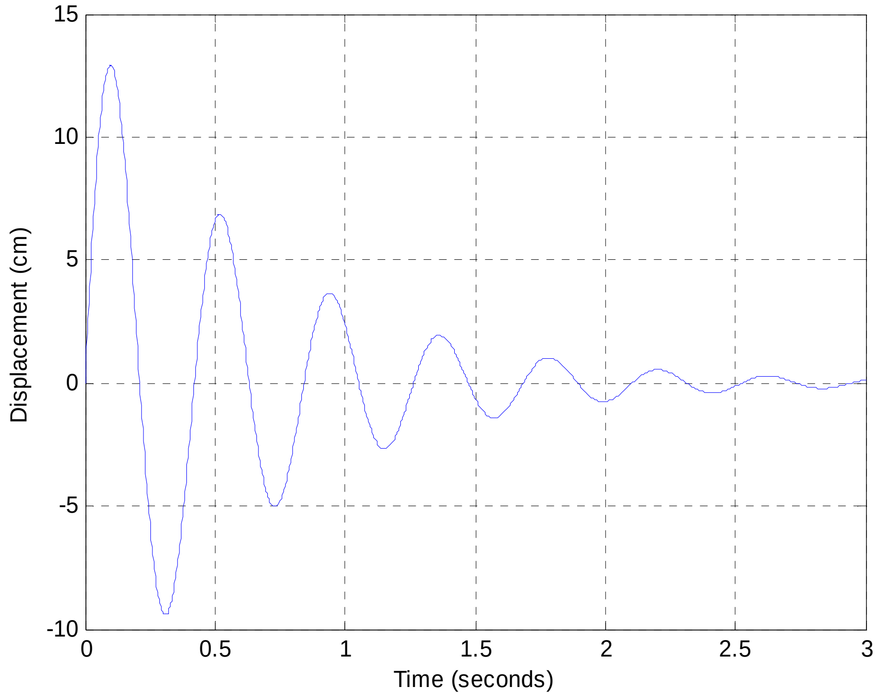


Figure 3. Initial condition response for second order system A.

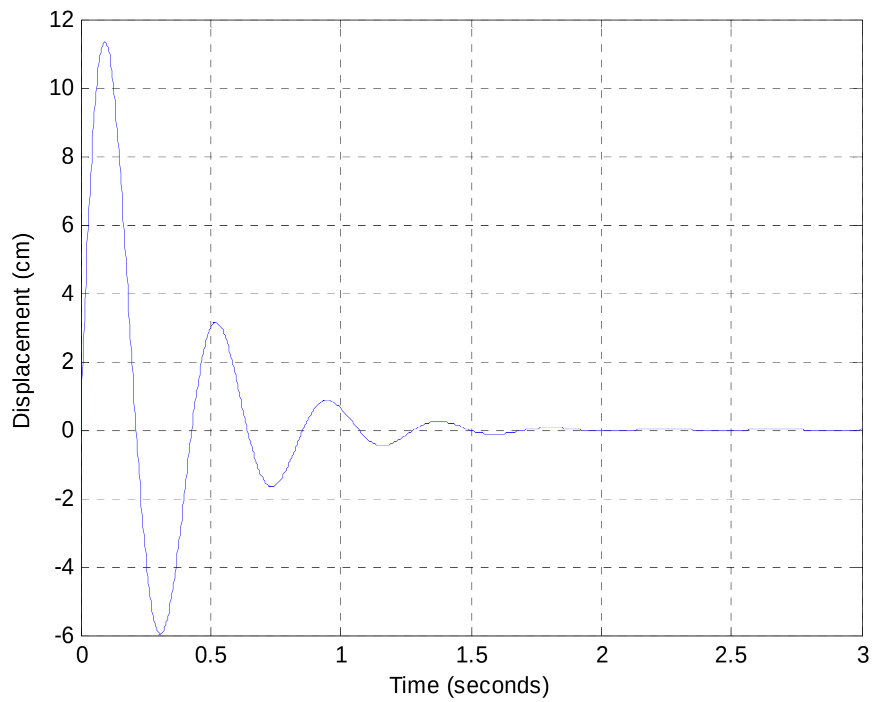


Figure 4. Initial condition response for second order system B.