Percent Overshoot: $P.O. = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100\%$

If $\beta = \frac{PO^{max}}{100}$, then

$$\zeta = \frac{\frac{-\ln(\beta)}{\pi}}{\sqrt{1 + \left(\frac{-\ln(\beta)}{\pi}\right)^2}}$$

Time to Peak: $T_p = \frac{\pi}{\omega_d}$

Settling Time: $T_s = \frac{4}{\zeta\omega_n}$

Consider a plant with proper transfer function $G_p(s) = N(s)/D(s)$ where

- $N(s)$ and $D(s)$ have no common factors
- The leading coefficient of $D(s)$ (the coefficient of the highest power of s in $D(s)$) is 1.

An implementable transfer function $G_0(s)$ that minimizes the performance index

$$J = \int_0^\infty \left\{ q \left[y(t) - r(t) \right]^2 + u^2(t) \right\} dt$$

where $r(t) = 1$ (a unit step) and $q > 0$ is given by

$$G_0(s) = \frac{qN(0)N(s)}{D_0(0)D_0(s)}$$

where

$$Q(s) = D(s)D(-s) + qN(s)N(-s) = D_0(s)D_0(-s)$$

Root Locus Construction

Once each pole has been paired with a zero, we are done!

1. Loci Branches

$$\textbf{poles}\ (k=0) \rightarrow \textbf{zeros}\ (k=\infty)$$

Continuous curves, which comprise the locus, start at each of the n poles of $G(s)$ for which $k=0$. As k approaches ∞, the branches of the locus approach the m zeros of $G(s)$. Locus branches for excess poles extend to infinity.

The root locus is symmetric about the real axis.

2. Real Axis Segments

The root locus includes all points along the real axis to the left of an odd number of poles plus zeros of $G(s)$.

3. Asymptotic Angles

As $k \rightarrow \infty$, the branches of the locus become asymptotic to straight lines with angles

$$\theta = \frac{180^o + i360^o}{n-m}, \quad i = 0, \pm 1, \pm 2, ...$$

until all $(n-m)$ angles not differing by multiples of 360^o are obtained. n is the number of poles of $G(s)$ and m is the number of zeros of $G(s)$.

4. Centroid of the Asymptotes

The starting point on the real axis from which the asymptotic lines radiate is given by

$$\sigma_c = \frac{\sum_i p_i - \sum_j z_j}{n-m}$$

where p_i is the i^{th} pole of $G(s)$, z_j is the j^{th} zero of $G(s)$, n is the number of poles of $G(s)$ and m is the number of zeros of $G(s)$. This point is terms the centroid of the asymptotes.

5. Leaving/Entering the Real Axis

When two branches of the root locus leave or enter the real axis, they usually do so at angles of ± 90 degrees.

Controller Types

Proportional (P), $G_c(s) = k$

Integral (I), $G_c(s) = k/s$

Proportional + Integral (PI), $G_c(s) = k(s + z)/s$

Proportional + Derivative (PD), $G_c(s) = k(s + z)$

Proportional + Integral + Derivative (PID), $G_c(s) = k(s + z_1)(s + z_2)/s$

Lead, $G_c(s) = k(s + z)/(s + p)$, $p > z$

Lag, $G_c(s) = (s + z)/(s + p)$, $z > p$

Diophantine Equations

For plant $G_p(s) = N(s)/D(s)$, controller $G_c(s) = B(s)/A(s)$, and desired characteristic equation $D_0(s)$ we will have to solve the equation

$$A(s)D(s) + B(s)N(s) \quad = \quad D_0(s)$$

This is called the *Diophantine* equation. We solve this equation by equating powers of s, setting up a system of equations, and then solving. The closed-loop transfer function will be

$$G_0(s) \quad = \quad \frac{B(s)N(s)}{D_0(s)}$$

where $B(s)$ contains the zeros we have added to the system.

Theorem *Strictly Proper Plant* Assume we have a strictly proper n^{th} order plant transfer function, $G_p(s) = N(s)/D(s)$. Since $G_p(s)$ is strictly proper we have the degree of $N(s) <$ the degree of $D(s)$. Since $G_p(s)$ is n^{th} order the degree of $D(s) = n$. Assume also that $N(s)$ and $D(s)$ have no common factors. Then for any polynomial $D_0(s)$ of degree $n + m$ a proper controller $G_c(s) = B(s)/A(s)$ of degree m exists so that the characteristic equation of the resulting closed-loop system is equal to $D_0(s)$. If $m = n - 1$, the controller is unique. If $m \geq n$, the controller is not unique and some of the coefficients can be used to achieve other design objectives.

Theorem *Special case: degree $N(s)$ = degree $D(s)$.* Assume we have a proper n^{th} order plant transfer function, $G_p(s) = N(s)/D(s)$, where the degree of $D(s)$ = degree $N(s) = n$. Assume also that $N(s)$ and $D(s)$ have no common factors. Then for any polynomial $D_0(s)$ of degree $n + m$ a proper controller $G_c(s) = B(s)/A(s)$ of degree m exists so that the characteristic equation of the resulting closed-loop system is equal to $D_0(s)$. If $m = n$, and the controller is chosen to be strictly proper, the controller is unique. If $m \geq n + 1$, the controller is not unique and some of the coefficients can be used to achieve other design objectives.

$f(z)$	Linear Approximation
$(1+z)^a$	$1+az$
e^{az}	$1+az$
$\cos(az)$	1
$\sin(az)$	az
$\ln(1+z)$	z
$\cos(\alpha+z)$	$\cos(\alpha) - z\sin(\alpha)$
$\sin(\alpha+z)$	$\sin(\alpha) + z\cos(\alpha)$

Table 1: Functions and their linear approximation near $z = 0$.

Taylor Series

Assume we have a function $f(z)$ and we want to approximate the function near $z = 0$. The Taylor series approximation near $z = 0$ is

$$f(z) \quad \approx \quad f(0) + f'(0)z + \text{higher order terms}$$

You should be able to derive all of the entries in Table . This approximation is only valid for z near 0. The further away from zero we go, the worse the approximation is likely to be.

Linearization Procedure

Our goal here is to find a linear model that we can use to determine the transfer function of a system. The procedure we will go through is listed below, and will be followed with a few examples.

Step 1 Determine the nominal *operating point* of the system and the equation that these operating points solve. We will assume the operating points are the static equilibrium points. At the static equilibrium points, all derivatives are zero. For the linearization to be valid, the system must not stray very far from this operating point. Label these points x_0, y_0, u_0, etc. *These points are assumed to be constants.*

Step 2 Look at variations from these operation points. For example, we assume

$$\begin{aligned}
x(t) &= x_0 + \Delta x(t) \\
y(t) &= y_0 + \Delta y(t) \\
u(t) &= u_0 + \Delta u(t)
\end{aligned}$$

Note that only $\Delta x(t)$, $\Delta y(t)$, etc. vary with time. x_0, y_0, etc. are constants. Now we have two cases to consider:

Step 2a If our functions are *arguments* to other standard functions, we leave this approximation as it is. For example, $\cos(x(t))$ would be rewritten $\cos(x_0 + \Delta x(t))$. Similarly for all other trigonometric functions and exponentials.

Step 2b If our functions are not arguments to standard functions, we rewrite the functions as

$$
\begin{aligned}
x(t) &= x_0 + \Delta x(t) = x_0 \left(1 + \frac{\Delta x(t)}{x_0}\right) \\
y(t) &= y_0 + \Delta y(t) = y_0 \left(1 + \frac{\Delta y(t)}{y_0}\right) \\
u(t) &= u_0 + \Delta u(t) = u_0 \left(1 + \frac{\Delta u(t)}{u_0}\right)
\end{aligned}
$$

We rewrite the functions in this way because this is the form we will use the Taylor series on. Here our *small* z will be $\frac{\Delta x(t)}{x_0}$, $\frac{\Delta y(t)}{y_0}$, etc.

Step 3 Substitute our expressions for $x(t)$, $y(t)$, etc., into the dynamics, and simplify where possible.

Step 4 Using Taylor series, expand out all nonlinear terms.

Step 5 Put the Taylor series expansion into the defining differential equation and multiply out all terms.

Step 6 Drop all second order (or higher) terms. Thus terms of the form $\left(\frac{\Delta x(t)}{x_0}\right)^2$, $\left(\frac{\Delta x(t)}{x_0}\right)\left(\frac{\Delta y(t)}{y_0}\right)$, etc. will be dropped.

Step 7 Using the relationships found in step 1, try and remove all constant terms in the model. If there are any constant terms left over, you have made an error. All of the remaining terms should be Δ terms.

Step 8 Find the resulting transfer function.

Table of Laplace Transforms

$f(t)$	$F(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
$tu(t)$	$\frac{1}{s^2}$
$\frac{t^{n-1}}{(n-1)!}u(t)\ (n = 1,\ 2,\ 3...)$	$\frac{1}{s^n}$
$t^n u(t)\ (n = 1,\ 2,\ 3,\ ...)$	$\frac{n!}{s^{n+1}}$
$e^{-at}u(t)$	$\frac{1}{s+a}$
$te^{-at}u(t)$	$\frac{1}{(s+a)^2}$
$\frac{1}{(n-1)!}t^{n-1}e^{-at}u(t)\ (n = 1,\ 2,\ 3,\ ...)$	$\frac{1}{(s+a)^n}$
$t^n e^{-at}u(t)\ (n = 1,\ 2,\ 3,\ ...)$	$\frac{n!}{(s+a)^{n+1}}$
$\sin(bt)u(t)$	$\frac{b}{s^2+b^2}$
$\cos(bt)u(t)$	$\frac{s}{s^2+b^2}$
$e^{-at}\sin(bt)u(t)$	$\frac{b}{(s+a)^2+b^2}$
$e^{-at}\cos(bt)u(t)$	$\frac{(s+a)}{(s+a)^2+b^2}$