ECE-320: Linear Control Systems
Homework 1

Due: Tuesday March 15 at 10 AM

1) The (one sided) Laplace transform is defined as

$$X(s) = \int_0^\infty x(t)e^{-st}\,dt$$

or the transform pair

$$x(t) \Leftrightarrow X(s)$$

a) By differentiating both sides of the above equation with respect to s, show that

$$-\frac{dX(s)}{ds} = \int_0^\infty tx(t)e^{-st}\,dt$$

$$\frac{d^2X(s)}{ds^2} = \int_0^\infty t^2 x(t)e^{-st}\,dt$$

or

$$tx(t) \quad \Leftrightarrow \quad -\frac{dX(s)}{ds}$$

$$t^2 x(t) \quad \Leftrightarrow \quad \frac{d^2X(s)}{ds^2}$$

b) By evaluating the integral, show that, if the real part of a is negative,

$$x(t) = e^{-at}u(t) \Leftrightarrow \frac{1}{s+a}$$

c) Combining parts (a) and (b), and a little bit of Maple free calculus, show that

$$x(t) = e^{-at}u(t) \quad \Leftrightarrow \quad \frac{1}{s+a}$$

$$tx(t) = te^{-at}u(t) \quad \Leftrightarrow \quad \frac{1}{(s+a)^2}$$

$$\frac{1}{2}t^2 x(t) = \frac{1}{2}t^2 e^{-at}u(t) \quad \Leftrightarrow \quad \frac{1}{(s+a)^3}$$

The moral of this problem is if a pole is repeated in the Laplace (s) domain, we just multiply by t in the time domain to get the shape of the time response.

2) Starting from the definition of the Laplace transform, show

$$x(t - t_0) \Leftrightarrow e^{-st_0} X(s)$$

Note that in this problem, we are assuming $x(t) = x(t)u(t)$ and that $x(t - t_0) = x(t - t_0)u(t - t_0)$. That is, we assume $x(t)$ is zero for $t < 0$ and $x(t - t_0)$ is zero for $t < t_0$.

The moral of this problem is that any time you see an e^{-st_0} in the Laplace domain, there is a delay, or transport lag, in the time domain.

3) Starting from the definition of the Laplace transform, show
$$x(t)e^{-at} \Leftrightarrow X(s + a)$$
Using this result, and completing the square in the denominator, show that

$$\frac{As + B}{s^2 + 2as + c} \Leftrightarrow e^{-at}\left[A\cos(bt) + \frac{B - Aa}{b}\sin(bt)\right]u(t)$$

where $b^2 = c - a^2$ and b^2 is assumed to be positive (complex conjugate roots)

This is an amoral problem

4) Determine the impulse response of the following using partial fractions as necessary. You are expected to be able to do all of these with the Laplace transform Table in the notes, and the properties above. You may check your answers with Maple.

a) $H(s) = \dfrac{e^{-s}}{s^2 + 3}$

b) $H(s) = \dfrac{s+1}{s^2 + 2}$

c) $H(s) = \dfrac{s+1}{(s+2)^2 + 4}$

d) $H(s) = \dfrac{s+3}{(s+1)(s+2)}$

e) $H(s) = \dfrac{s}{(s+1)^2(s+3)}$

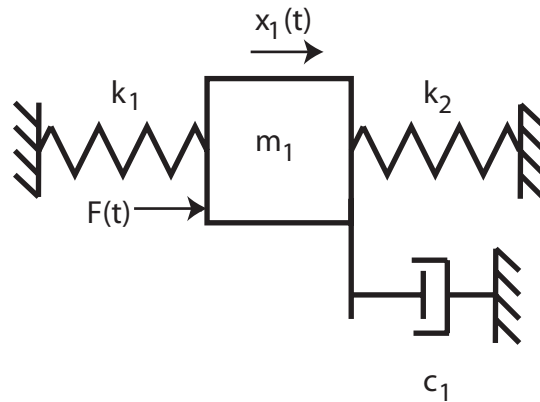f) $H(s) = \dfrac{1}{(s+1)(s+2)(s+3)(s+4)}$

g) $H(s) = \dfrac{s}{(s^2 + 2)(s+1)}$

h) $H(s) = \dfrac{1}{s^2 + s + 1}$

i) $H(s) = \dfrac{s}{2s^2 + s + 3}$

j) $H(s) = \dfrac{s}{s^2 + 3s + 2}$

Preparation for Lab 2.

5) Consider the following one degree of freedom system we will be utilizing this term:

a) Draw a freebody diagram of the forces on the mass.

b) Show that the equations of motion can be written:

$$m_1 \ddot{x}_1(t) + c_1 \dot{x}(t) + (k_1 + k_2) x_1(t) = F(t)$$

or

$$\frac{1}{\omega_n^2} \ddot{x}_1(t) + \frac{2\zeta}{\omega_n} \dot{x}(t) + x_1(t) = KF(t)$$

c) What are the damping ratio ζ, the natural frequency ω_n, and the static gain K in terms of m_1, k_1, k_2, and c_1?

d) Show that the transfer function for the *plant* is given by

$$G_p(s) = \frac{X_1(s)}{F(s)} = \frac{K}{\dfrac{1}{\omega_n^2} s^2 + \dfrac{2\zeta}{\omega_n} s + 1}$$

6) One of the methods we will be using to identify ζ and ω_n is the *log-decrement* method, which we will review/derive in this problem. If our system is at rest and we provide the mass with an initial displacement away from equilibrium, the response due to this displacement can be written

$$x_1(t) = Ae^{-\zeta\omega_n t}\cos(\omega_d t + \theta)$$

where

$x_1(t) =$ displacement of the mass as a function of time
ζ = damping ratio
ω_n = natural frequency
ω_d = damped frequency = $\omega_n\sqrt{1-\zeta^2}$

After the mass is released, the mass will oscillate back and forth with period given by $T_d = \dfrac{2\pi}{\omega_d}$, so if we measure the period of the oscillation (T_d) we can estimate ω_d.

Let's assume t_0 is the time of one peak of the cosine. Since the cosine is periodic, subsequent peaks will occur at times given by $t_n = t_0 + nT_d$, where n is an integer.

a) Show that

$$\frac{x_1(t_0)}{x_1(t_n)} = e^{\zeta\omega_n T_d n}$$

b) If we define the log decrement as

$$\delta = \ln\left[\frac{x_1(t_0)}{x_1(t_n)}\right]$$

show that we can compute the damping ratio as

$$\zeta = \frac{\delta}{\sqrt{4n^2\pi^2 + \delta^2}}$$

c) Given the initial condition response shown in the Figures 1 and 2 on the next page, estimate the damping ratio and natural frequency using the log-decrement method. (*You should get answers that include the numbers 15, 0.2, 0.1 and 15, approximately.*)
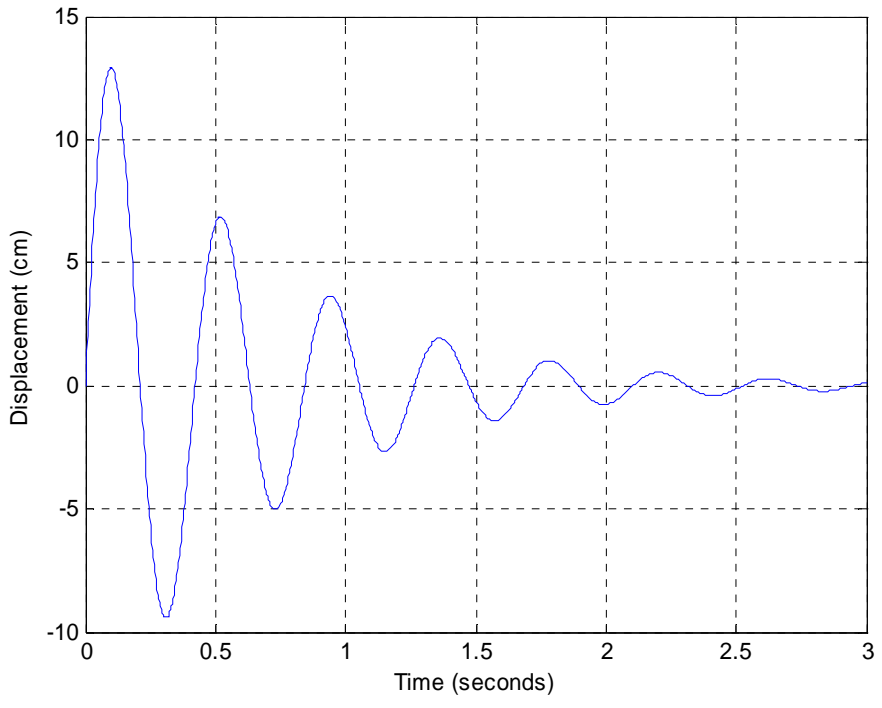
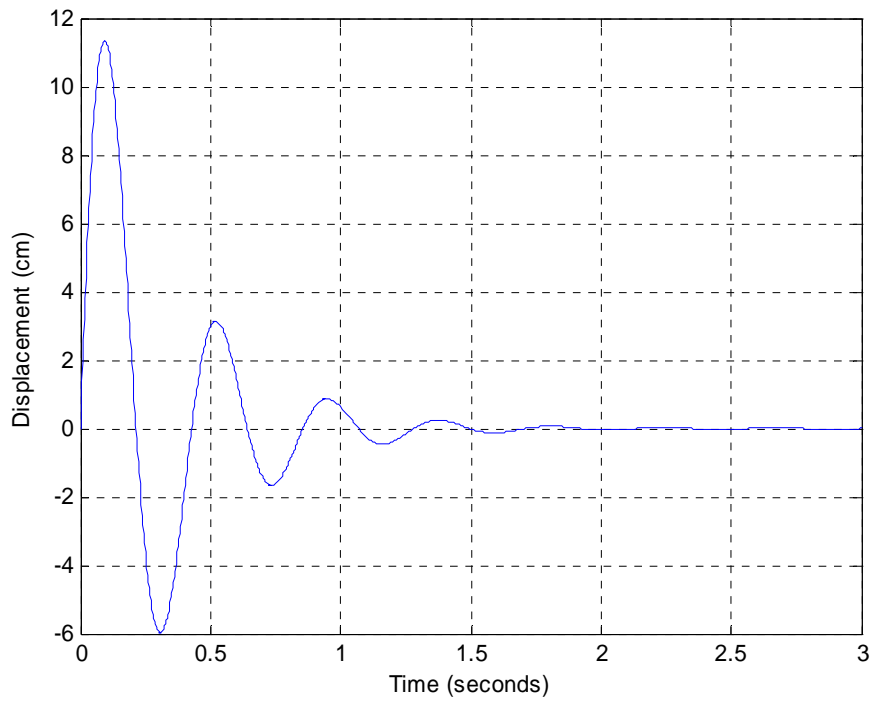Figure 1. Initial condition response for second order system A.

Figure 2. Initial condition response for second order system B.

7) (*Note: To do this problem you need to either be using Matlab 6.5.1 or will have to make a new version of openloop.mdl that works with the version of Matlab you are using. If you are using Matlab 7.0 you most likely cannot read/utilize the Simulink files on the class website.*)

Download the files **waytoolong.mat** and **compare1.m** from the class website. The file **waytoolong.mat** represents data that was collected from an unknown system that was subjected to a step input of amplitude 0.1 cm. In this problem you need to try and identify the system using the files you developed and utilized in lab 1.

a) In Matlab, type **load waytoolong.mat**. This should load two new arrays into your workspace, **time** and **x1**.

b) Type **plot(time,x1); grid;** in Matlab to be sure you have something that looks like the response of a second order system.

c) Edit the beginning of the file **openloop_driver.m** as follows:

comment out these lines by using a % sign

```
%  load state_model_1dof
% C = [1 0];
%[num_Gp,den_Gp] = ss2tf(A,B,C,D);
```

add these lines to try and model the unknown system

```
zeta = 0.1;                              % estimate of damping ratio
omega_n = 10.0;                          % estimate of natural frequency
K = 20;                                   % estimate of static gain
num_Gp = [K];                            % the new numerator
den_Gp = [1/omega_n^2 2*zeta/omega_n 1];  % the new denominator
```

set the input amplitude Amp to 0.1 cm, and the final time to something convenient that allows you to see alot of what is happening.

d) Run **openloop_driver.m**, then run **compare1.m.** The file **compare1.m** is a routine we will be using alot in lab to compare the response of a model (**m_time** and **m_y** from the Simulink model) with that of a real system (**time** and **x1**), so you might want to figure out how it works. You will need to modify it in Lab 2. If everything has gone correctly, you should get a plot like that shown in Figure 3 for Tf =2.0 seconds.
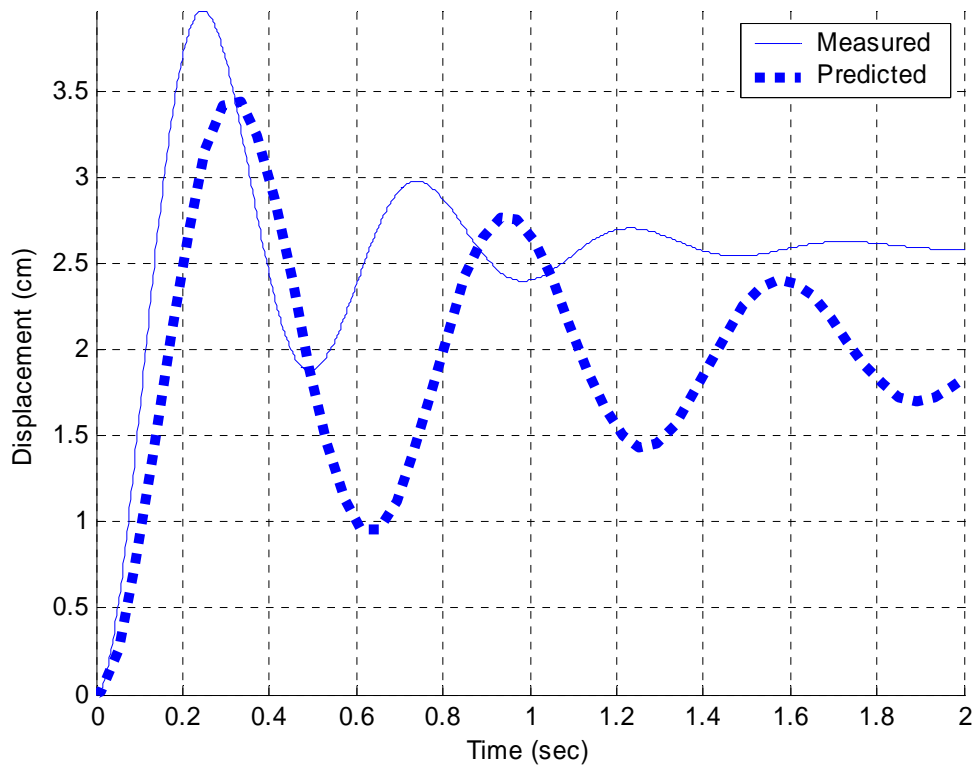
Figure 3: Step response of mystery second order system

e) You now need to adjust the static gain (K), the damping ratio (zeta), and the natural frequency (omega_n) in the model file to try and make the model match the real system. You should pay attention to how changing K, zeta, and omega_n changes the response of the model, since this is the kind of stuff that will be showing up on quizes and exams. When you are done, turn in your final plot comparing the real system to your model, and also your modified **openloop_driver.m** file. (*Note: This system was easier to model than the systems we have in lab, as you will see....*)