**ECE-320**: Linear Control Systems
Homework 1

Due: Tuesday September 8 *at the beginning of class*

**1)** The (one sided) Laplace transform is defined as

$$X(s) = \int_0^\infty x(t)e^{-st}dt$$

or the transform pair

$$x(t) \Leftrightarrow X(s)$$

**a)** By differentiating both sides of the above equation with respect to $s$, show that

$$-\frac{dX(s)}{ds} = \int_0^\infty tx(t)e^{-st}dt$$

$$\frac{d^2X(s)}{ds^2} = \int_0^\infty t^2x(t)e^{-st}dt$$

or

$$tx(t) \quad \Leftrightarrow \quad -\frac{dX(s)}{ds}$$

$$t^2x(t) \quad \Leftrightarrow \quad \frac{d^2X(s)}{ds^2}$$

**b)** By evaluating the integral, show that, if the real part of $a$ is positive,

$$x(t) = e^{-at}u(t) \Leftrightarrow \frac{1}{s+a}$$

**c)** Combining parts (a) and (b), and a little bit of Maple free calculus,
show that

$$x(t) = e^{-at}u(t) \quad \Leftrightarrow \quad \frac{1}{s+a}$$

$$tx(t) = te^{-at}u(t) \quad \Leftrightarrow \quad \frac{1}{(s+a)^2}$$

$$\frac{1}{2}t^2x(t) = \frac{1}{2}t^2e^{-at}u(t) \quad \Leftrightarrow \quad \frac{1}{(s+a)^3}$$

*The moral of this problem is if a pole is repeated in the Laplace ( $s$ ) domain, we just multiply by $t$ in the time domain to get the shape of the time response. (There is still some scaling involved, but we are mostly concerned with the shape of the signal.)*

**2)** Starting from the definition of the Laplace transform, show

$$x(t-t_0) \Leftrightarrow e^{-st_0} X(s)$$

Note that in this problem, we are assuming $x(t) = x(t)u(t)$ and that $x(t-t_0) = x(t-t_0)u(t-t_0)$. That is, we assume $x(t)$ is zero for $t<0$ and $x(t-t_0)$ is zero for $t<t_0$.

*The moral of this problem is that any time you see an* $e^{-st_0}$ *in the Laplace domain, there is a delay, or transport lag, in the time domain.*

**3)** Starting from the definition of the Laplace transform, show $x(t)e^{-at} \Leftrightarrow X(s+a)$
Using this result, and completing the square in the denominator, show that

$$\frac{As+B}{s^2+2as+c} \Leftrightarrow e^{-at}\left[A\cos(bt)+\frac{B-Aa}{b}\sin(bt)\right]u(t)$$

where $b^2 = c-a^2$ and $b^2$ is assumed to be positive (complex conjugate roots)

**4)** Determine the impulse response of the following using partial fractions as necessary. You are expected to be able to do all of these with the Laplace transform Table in the notes, and the properties above. You may *check* your answers with Maple.

**a)** $H(s) = \frac{e^{-s}}{s^2+3}$ *(Hint: Ignore the exponential term (assume it is a 1) and find the inverse Laplace transform, then include the transport delay.)*

**b)** $H(s) = \frac{s+1}{s^2+2}$

**c)** $H(s) = \frac{s+1}{(s+2)^2+4}$

**d)** $H(s) = \frac{s+3}{(s+1)(s+2)}$

**e)** $H(s) = \frac{s}{(s+1)^2(s+3)}$

**f)** $H(s) = \frac{1}{(s+1)(s+2)(s+3)(s+4)}$

**g)** $H(s) = \frac{s}{(s^2+2)(s+1)}$

**h)** $H(s) = \frac{1}{s^2+s+1}$

**i)** $H(s) = \frac{s}{2s^2+s+3}$

**j)** $H(s) = \frac{s}{s^2+3s+2}$