ECE-320

Equation Sheet

Second Order System Properties

Percent Overshoot: $P.O. = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100\%$,

If $\beta = \dfrac{PO^{max}}{100}$ then $\zeta = \dfrac{\dfrac{-\ln(\beta)}{\pi}}{\sqrt{1 + \left(\dfrac{-\ln(\beta)}{\pi}\right)^2}}$, $\theta = cos^{-1}(\zeta)$

Time to Peak: $T_p = \dfrac{\pi}{\omega_d}$, $\omega_d = \omega_n\sqrt{1-\zeta^2}$

2% Settling Time: $T_s = \dfrac{4}{\zeta\omega_n} = 4\tau$

Model Matching

Assume we have a proper plant $G_p(s) = N(s)/D(s)$ and we want the closed loop system to have the transfer function $G_o(s) = N_0(s)/D_0(s)$. We can find a controller

$$G_c(s) = \frac{N_0(s)D(s)}{N(s)[D_0(s) - N_0(s)]}$$

under the following conditions:

- degree $D_0(s)$ - degree $N_0(s) \geq$ degree $D(s)$ - degree $N(s)$
- The right half plane zeros of $N(s)$ are retained in $N_0(s)$
- $G_0(s)$ is stable

Quadratic Optimal Control

Consider a plant with a *proper* transfer function

$$G_p(s) = \frac{N(s)}{D(s)}$$

where
- $N(s)$ and $D(s)$ have no common factors
- The leading coefficient of $D(s)$ is 1 (the coefficient of the highest power of s in $D(s)$ is 1)

An implementable closed loop transfer function $G_0(s)$ that minimizes the performance index

$$J = \int_0^\infty \left\{ q[y(t) - r(t)]^2 + u^2(t) \right\} dt$$

where $r(t) = 1$ (a unit step) is given by

$$G_0(s) = \frac{qN(0)N(s)}{D_0(0)D_0(s)}$$

where

$$Q(s) = D(s)D(-s) + q\, N(s)N(-s) = D_0(s)D_0(-s)$$

Controller Types

Proportional (P), $G_c(s) = k$

Integral (I), $G_c(s) = \dfrac{k}{s}$

Proportional + Integral (PI), $G_c(s) = \dfrac{k(s+z)}{s}$

Proportional + Derivative (PD), $G_c(s) = k(s+z)$

Proportional + Integral + Derivative (PID), $G_c(s) = \dfrac{k(s+z_1)(s+z_2)}{s}$

Lead, $G_c(s) = \dfrac{k(s+z)}{(s+p)}, \quad p > z$

Lag, $G_c(s) = \dfrac{k(s+z)}{(s+p)}, \quad z > p$

Diophantine Equations

For plant $G_p(s) = N(s)/D(s)$, controller $G_c(s) = B(s)/A(s)$, and desired characteristic Polynomial $D_0(s)$ we will have to solve the equation $A(s)D(s) + B(s)N(s) = D_0(s)$

This is called the *Diophantine* equation. We solve this equation by equating powers of s, setting up a system of equations, and then solving. The closed-loop transfer function will be

$$G_0(s) = \frac{B(s)N(s)}{D_0(s)}$$

where $B(s)$ contains the zeros we have added to the system.

Theorem (*Strictly Proper Plant*) Assume we have a strictly proper n^{th} order plant transfer function, $G_p(s) = N(s)/D(s)$. Since $G_p(s)$ is strictly proper we have the degree of $N(s) <$ the degree of $D(s)$. Since $G_p(s)$ is n^{th} order the degree of $D(s) = n$. Assume also that $N(s)$ and $D(s)$ have no common factors. Then for any polynomial $D_0(s)$ of degree $n+m$ a proper controller $G_c(s) = B(s)/A(s)$ of degree m exists so that the characteristic polynomial of the resulting closed-loop system is equal to $D_0(s)$.

If $m = n-1$, the controller is unique. If $m \geq n$, the controller is not unique and some of the coefficients can be used to achieve other design objectives.

Theorem (*Special case: degree $N(s)$ = degree $D(s)$*) Assume we have a proper n^{th} order plant transfer function, $G_p(s) = N(s)/D(s)$, where the degree of $D(s)$ = degree $N(s)$ = n. Assume also that $N(s)$ and $D(s)$ have no common factors. Then for any polynomial $D_0(s)$ of degree $n+m$ a proper controller $G_c(s) = B(s)/A(s)$ of degree m exists so that the characteristic equation of the resulting closed-loop system is equal to $D_0(s)$. If $n = m$, and the controller is chosen to be strictly proper, the controller is unique. If $m \geq n+1$, the controller is not unique and some of the coefficients can be used to achieve other design objectives.

Root Locus Construction

Once each pole has been paired with a zero, we are done

1. *Loci Branches*

$$poles \underset{k=0}{\longrightarrow} zeros \underset{k=\infty}{}$$

Continuous curves, which comprise the locus, start at each of the *n* poles of $G(s)$ for which $k = 0$. As k approaches ∞, the branches of the locus approach the *m* zeros of $G(s)$. Locus branches for excess poles extend to infinity.

The root locus is s**ymmetric** about the real axis.

2. *Real Axis Segments*

The root locus includes all points along the real axis to the left of an odd number of poles plus zeros of $G(s)$.

3. *Asymptotic Angles*

As $k \rightarrow \infty$ k , the branches of the locus become asymptotic to straight lines with angles

$$\theta = \frac{180^o + i360^o}{n - m}, i = 0, \pm 1, \pm 2,..$$

until all $(n-m)$ angles not differing by multiples of 360° are obtained. *n* is the number of poles of $G(s)$ and *m* is the number of zeros of $G(s)$.

4. *Centroid of the Asymptotes*

The starting point on the real axis from which the asymptotic lines radiate is given by

$$\sigma_c = \frac{\sum_i p_i - \sum_j z_j}{n - m}$$

where p_i is the ith pole of $G(s)$, z_j is the jth zero of $G(s)$, *n* is the number of poles of $G(s)$ and *m* is the number of zeros of $G(s)$. This point is termed the *centroid of the asymptotes*.

5. *Leaving/Entering the Real Axis*

When two branches of the root locus leave or enter the real axis, they usually do so at angles of $\pm 90^o$.

Frequency Domain Phase Lead

Consider the following feedback system
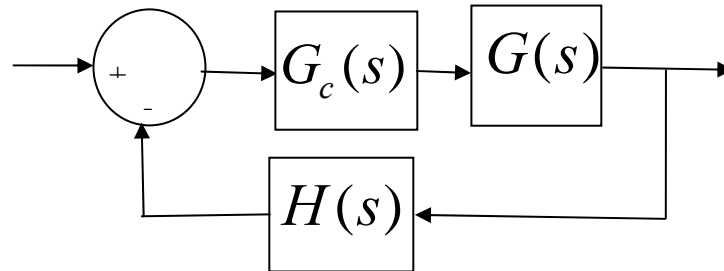
The primary function of the phase lead compensator is to reshape the frequency response curve to add phase to the system, thereby increasing the phase margin. This usually results in an increased bandwidth, and hence a faster response time.

Basic Procedure

1) Assume the phase lead compensator has the form

$$G_c(s) = K \frac{(Ts+1)}{(\alpha Ts+1)} = K_c \frac{s+\dfrac{1}{T}}{s+\dfrac{1}{\alpha T}}$$

where $K_c = \dfrac{K}{\alpha}$

Determine K to satisfy any steady state error requirements.

2) Using the value of K determined in step 1, draw the Bode plot of $KG(s)H(s)$.

3) Determine the necessary phase to be added to the system to achieve the desired phase margin. Add an additional 5° to 12° to the necessary phase to account for the fact that the phase lead compensator shifts the gain crossover frequency to the right and decreases the phase margin. The total phase we need to add is ϕ_m.

4) Determine α using the relationship $\alpha = \dfrac{1-\sin(\phi_m)}{1+\sin(\phi_m)}$. α should be larger than 0.05, or you need two or more compensators in series.

5) Determine the frequency ω_m at which

$$20\log_{10} | KG(j\omega_m)H(j\omega_m) | = -20\log_{10}(\frac{1}{\sqrt{\alpha}}) = 10\log_{10}(\alpha)$$

This is the new gain crossover frequency $\omega_m = \frac{1}{T\sqrt{\alpha}}$. Since we know ω_m and α, we can

compute $T = \frac{1}{\omega_m \sqrt{\alpha}}$

6) Determine the pole and zero of the compensator $z = \frac{1}{T}$ and $p = \frac{1}{T}\frac{1}{\alpha}$

7) Determine $K_c = \frac{K}{\alpha}$

8) Check the phase margin to see if it is acceptable. We will use *sisotool.*

When you use sisotool to tweak your compensators, you should

1) enter the transfer function for $G(s)H(s)$

2) instruct sisotool to use the Natural Frequency or Time Constant method of displaying the controller. To do this

edit → sisotool preferences →options
and select Natural Frequency or Time Constant

3) enter the pole and zero of the compensator

4) Set the gain to be K, the required gain for the steady state errors. This gain cannot change!

5) Play with the pole and/or zero to get the required phase margin

Laplace Transforms

$$\mathcal{L}\{\delta(t)\} = 1$$

$$\mathcal{L}\{u(t)\} = \frac{1}{s}$$

$$\mathcal{L}\{tu(t)\} = \frac{1}{s^2}$$

$$\mathcal{L}\left\{\frac{t^{m-1}}{(m-1)!}u(t)\right\} = \frac{1}{s^m}$$

$$\mathcal{L}\{e^{-at}u(t)\} = \frac{1}{s+a}$$

$$\mathcal{L}\{te^{-at}u(t)\} = \frac{1}{(s+a)^2}$$

$$\mathcal{L}\left\{\frac{t^{(m-1)}}{(m-1)!}e^{-at}u(t)\right\} = \frac{1}{(s+a)^m}$$

$$\mathcal{L}\{\cos(\omega_0 t)u(t)\} = \frac{s}{s^2 + \omega_0^2}$$

$$\mathcal{L}\{\sin(\omega_0 t)u(t)\} = \frac{\omega_0}{s^2 + \omega_0^2}$$

$$\mathcal{L}\{e^{-\alpha t}\cos(\omega_0 t)u(t)\} = \frac{s+\alpha}{(s+\alpha)^2 + \omega_0^2}$$

$$\mathcal{L}\{e^{-\alpha t}\sin(\omega_0 t)u(t)\} = \frac{\omega_0}{(s+\alpha)^2 + \omega_0^2}$$

$$\mathcal{L}\left\{\frac{dx(t)}{dt}\right\} = sX(s) - x(0^-)$$

$$\mathcal{L}\left\{\frac{d^2 x(t)}{dt^2}\right\} = s^2 X(s) - sx(0^-) - \dot{x}(0^-)$$

$$\mathcal{L}\{x(t-a)\} = e^{-as}X(s)$$

$$\mathcal{L}\{e^{-at}x(t)\} = X(s+a)$$

$$\mathcal{L}\left\{x\left(\frac{t}{a}\right), a > 0\right\} = aX(as)$$