Notes for ECE-320

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The following pages contain a first attempt at writing notes for ECE-320. The topics we cover in ECE-320 are not covered in any single book. These notes are not complete, especially the sections on root locus design and design using Bode plots.

The major sources for these notes are

- *Analog and Digital Control System Design*, by C. T. Chen. Sanders College Publishing. 1993.
- *Linear Control Systems*, by Rohrs, Melsa, and Schulz. McGraw-Hill, 1993.
- *Modern Control Engineering*, by Ogata. Prentice-Hall, 2002.
- *Modern Control Systems*, by Dorf and Bishop. Prentice-Hall, 2005.

Contents

1 Table of Laplace Transforms

$f(t)$	$F(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
$tu(t)$	$\frac{1}{s^2}$
$\frac{t^{n-1}}{(n-1)!}u(t)$ $(n = 1, 2, 3...)$	$\frac{1}{s^n}$
$t^n u(t)$ $(n = 1, 2, 3, ...)$	$\frac{n!}{s^{n+1}}$
$e^{-at}u(t)$	$\frac{1}{s+a}$
$te^{-at}u(t)$	$\frac{1}{(s+a)^2}$
$\frac{1}{(n-1)!}t^{n-1}e^{-at}u(t)$ $(n = 1, 2, 3, ...)$	$\frac{1}{(s+a)^n}$
$t^n e^{-at}u(t)$ $(n = 1, 2, 3, ...)$	$\frac{n!}{(s+a)^{n+1}}$
$\sin(bt)u(t)$	$\frac{b}{s^2+b^2}$
$\cos(bt)u(t)$	$\frac{s}{s^2+b^2}$
$e^{-at}\sin(bt)u(t)$	$\frac{b}{(s+a)^2+b^2}$
$e^{-at}\cos(bt)u(t)$	$\frac{(s+a)}{(s+a)^2+b^2}$

2 Laplace Transform Review

In this course we will be using Laplace transforms extensively. Although we do not often go from the s-plane to the time domain, it is important to be able to do this and to understand what is going on. In what follows is a brief review of some results with Laplace transforms.

2.1 Poles and Zeros

Assume we have the transfer function

$$H(s) = \frac{N(s)}{D(s)}$$

where $N(s)$ and $D(s)$ are polynomials in s with no common factors. The roots of $N(s)$ are the **zeros** of the system, while the roots of $D(s)$ are the **poles** of the system.

2.2 Proper and Strictly Proper Transfer Functions

The transfer function

$$H(s) = \frac{N(s)}{D(s)}$$

is **proper** if the degree of the polynomial $N(s)$ is less than or equal to the degree of the polynomial $D(s)$. The transfer function $H(s)$ is **strictly proper** if the degree of $N(s)$ is less than the degree of $D(s)$.

2.3 Impulse Response and Transfer Functions

If $H(s)$ is a transfer function, the inverse Laplace transform of $H(s)$ is call the **impulse response**, $h(t)$.

$$\begin{aligned} \mathcal{L}\{h(t)\} &= H(s) \\ h(t) &= \mathcal{L}^{-1}\{H(s)\} \end{aligned}$$

2.4 Partial Fractions with Distinct Poles

Let's assume we have a transfer function

$$H(s) \;=\; \frac{N(s)}{D(S)} \;=\; \frac{N(s)}{D(s)} \;=\; \frac{K(s+z_1)(s+z_2)...(s+z_m)}{(s+p_1)(s+p_2)...(s+p_n)}$$

where we assume $m < n$ (this makes $H(s)$ a **strictly proper** transfer function). The poles of the system are at $-p_1,\ -p_2,\ ...\ -p_n$ and the zeros of the system are at $-z_1,\ -z_2,\ ...\ -z_m$. Since we have distinct poles, $p_i \neq p_j$ for all i and j. Also, since we assumed $N(s)$ and $D(s)$ have no common factors, we know that $z_i \neq p_j$ for all i and j.
We would like to find the corresponding impulse response, $h(t)$. To do this, we assume

$$H(s) \;=\; \frac{N(s)}{D(s)} \;=\; a_1\frac{1}{s+p_1} + a_2\frac{1}{s+p_2} + ... + a_n\frac{1}{s+p_n}$$

If we can find the a_i, it will be easy to determine $h(t)$ since the only inverse Laplace transform we need is that of $\frac{1}{s+p}$, and we know (or can look up) $\frac{1}{s+p} \leftrightarrow e^{-pt}u(t)$. To find a_1, we first multiply by $(s+p_1)$,

$$(s+p_1)H(s) \;=\; a_1 + a_2\frac{s+p_1}{s+p_2} + ... + a_n\frac{s+p_1}{s+p_n}$$

and then let $s \rightarrow -p_1$. Since the poles are all distinct, we will get

$$\lim_{s\rightarrow -p_1}(s+p_1)H(s) \;=\; a_1$$

Similarly, we will get

$$\lim_{s\rightarrow -p_2}(s+p_2)H(s) \;=\; a_2$$

and in general

$$\lim_{s\rightarrow -p_i}(s+p_i)H(s) \;=\; a_i$$

<u>**Example 1**</u>. Let's assume we have

$$H(s) \;=\; \frac{s+1}{(s+2)(s+3)}$$

and we want to determine $h(t)$. Since the poles are distinct, we have

$$H(s) \;=\; \frac{(s+1)}{(s+2)(s+3)} \;=\; a_1\frac{1}{s+2} + a_2\frac{1}{s+3}$$

Then

$$a_1 \;=\; \lim_{s\rightarrow -2}(s+2)\frac{(s+1)}{(s+2)(s+3)} \;=\; \lim_{s\rightarrow -2}\frac{(s+1)}{(s+3)} = \frac{-1}{1} = -1$$

and

$$a_2 \quad = \quad \lim_{s \to -3} (s+3) \frac{(s+1)}{(s+2)(s+3)} \quad = \quad \lim_{s \to -3} \frac{(s+1)}{(s+2)} = \frac{-2}{-1} = 2$$

Then

$$H(s) \quad = \quad -1 \frac{1}{s+2} + 2 \frac{1}{s+3}$$

and hence

$$h(t) \quad = \quad -e^{-2t}u(t) + 2e^{-3t}u(t)$$

It is often unnecessary to write out all of the steps in the above example. In particular, when we want to find a_i we will always have a cancellation between $(s+p_i)$ in the numerator with the $(s+p_i)$ in the denominator. Using this fact, when we want to find a_i we can just ignore (or cover up) the factor $(s+p_i)$ in the denominator. For our example above, we then have

$$a_1 \quad = \quad \lim_{s \to -2} \frac{(s+1)}{\text{[redacted]}(s+3)} \quad = \quad \frac{-1}{1} = -1$$

$$a_2 \quad = \quad \lim_{s \to -3} \frac{(s+1)}{(s+2)\text{[redacted]}} \quad = \quad \frac{-2}{-1} = 2$$

where we have covered up the poles associated with a_1 and a_2, respectively.

Example 2. Let's assume we have

$$H(s) \quad = \quad \frac{s^2 - s + 2}{(s+2)(s+3)(s+4)}$$

and we want to determine $h(t)$. Since the poles are distinct, we have

$$H(s) \quad = \quad \frac{(s^2 - s + 2)}{(s+2)(s+3)(s+4)} \quad = \quad a_1 \frac{1}{s+2} + a_2 \frac{1}{s+3} + a_3 \frac{1}{s+4}$$

Using the coverup method, we then determine

$$a_1 \quad = \quad \lim_{s \to -2} \frac{(s^2 - s + 2)}{\text{[redacted]}(s+3)(s+4)} \quad = \quad \frac{8}{(1)(2)} = 4$$

$$a_2 \quad = \quad \lim_{s \to -3} \frac{(s^2 - s + 2)}{(s+2)\text{[redacted]}(s+4)} \quad = \quad \frac{14}{(-1)(1)} = -14$$

$$a_3 \quad = \quad \lim_{s \to -4} \frac{(s^2 - s + 2)}{(s+2)(s+3)\text{[redacted]}} \quad = \quad \frac{22}{(-2)(-1)} = 11$$

and hence

$$h(t) \quad = \quad 4e^{-2t}u(t) - 14e^{-3t}u(t) + 11e^{-4t}u(t)$$

Example 3. Let's assume we have

$$H(s) = \frac{1}{(s+1)(s+5)}$$

and we want to determine $h(t)$. Since the poles are distinct, we have

$$H(s) = \frac{1}{(s+1)(s+5)} = a_1 \frac{1}{s+1} + a_2 \frac{1}{s+5}$$

Using the coverup method, we then determine

$$\begin{aligned} a_1 &= \lim_{s \to -1} \frac{1}{\text{[redacted]}(s+5)} = \frac{1}{4} \\ a_2 &= \lim_{s \to -5} \frac{1}{(s+1)\text{[redacted]}} = \frac{1}{-4} \end{aligned}$$

and hence

$$h(t) = \frac{1}{4} e^{-t} u(t) - \frac{1}{4} e^{-5t} u(t)$$

Although we have only examined real poles, this method is also valid for complex poles, although there are usually easier ways to deal with complex poles, as we'll see.

2.5 Partial Fractions with Distinct and Repeated Poles

Whenever there are repeated poles, we need to use a different form for the partial fractions for those poles. This is probably most easily explained by means of examples.

Example 4. Assume we have the transfer function

$$H(s) \quad = \quad \frac{1}{(s+1)(s+2)^2}$$

and we want to find the corresponding impulse response, $h(t)$. To do this we look for a partial fraction expansion of the form

$$H(s) \quad = \quad \frac{1}{(s+1)(s+2)^2} \quad = \quad a_1 \frac{1}{s+1} + a_2 \frac{1}{s+2} + a_3 \frac{1}{(s+2)^2}$$

Example 5. Assume we have the transfer function

$$H(s) \quad = \quad \frac{s+1}{s^2(s+2)(s+3)}$$

and we want to find the corresponding impulse response, $h(t)$. To do this we look for a partial fraction expansion of the form

$$H(s) \quad = \quad \frac{s+1}{s^2(s+2)(s+3)} \quad = \quad a_1 \frac{1}{s} + a_2 \frac{1}{s^2} + a_3 \frac{1}{s+2} + a_4 \frac{1}{s+3}$$

Note that there are always as many unknowns (the a_i) as the degree of the denominator polynomial.

Now we need to be able to determine the expansion coefficients. We already know how to do this for distinct poles, so we do those first.
For **Example 4**,

$$a_1 \quad = \quad \lim_{s \to -1} \frac{1}{\text{[redacted]}(s+2)^2} \quad = \quad \frac{1}{1} \quad = 1$$

For **Example 5**,

$$\begin{aligned} a_3 \quad &= \quad \lim_{s \to -2} \frac{s+1}{s^2 \text{[redacted]} (s+3)} \quad = \quad \frac{-1}{(-2)^2(1)} \quad = -\frac{1}{4} \\ a_4 \quad &= \quad \lim_{s \to -3} \frac{s+1}{s^2(s+2) \text{[redacted]}} \quad = \quad \frac{-2}{(-3)^2(-1)} \quad = \frac{2}{9} \end{aligned}$$

The next set of expansion coefficients to determine are those with the highest power of the repeated poles.

For **Example 4**, multiply though by $(s+2)^2$ and let $s \to -2$,

$$a_3 \quad = \quad \lim_{s \to -2} (s+2)^2 \frac{1}{(s+1)(s+2)^2} \quad = \quad \lim_{s \to -2} \frac{1}{s+1} \quad = \quad -1$$

or with the coverup method

$$a_3 = \lim_{s \to -2} \frac{1}{(s+1)\text{[redacted]}} = \frac{1}{-1} = -1$$

For **<u>Example 5</u>**, multiply though by s^2 and let $s \to 0$

$$a_2 = \lim_{s \to 0} s^2 \frac{s+1}{s^2(s+2)(s+3)} = \lim_{s \to 0} \frac{s+1}{(s+2)(s+3)} = \frac{1}{6}$$

or with the coverup method

$$a_2 = \lim_{s \to 0} \frac{s+1}{\text{[redacted]}(s+2)(s+3)} = \frac{1}{6} = \frac{1}{6}$$

So far we have:

for **<u>Example 4</u>**

$$\frac{1}{(s+1)(s+2)^2} = \frac{1}{s+1} + a_2 \frac{1}{s+2} - \frac{1}{(s+2)^2}$$

and for **<u>Example 5</u>**

$$\frac{s+1}{s^2(s+2)(s+3)} = a_1 \frac{1}{s} + \frac{1}{6}\frac{1}{s^2} - \frac{1}{4}\frac{1}{s+2} + \frac{2}{9}\frac{1}{s+3}$$

We now need to determine any remaining coefficients. There are two common ways of doing this, both of which are based on the fact that both sides of the equation must be equal for any value of s. The two methods are

1. Multiply both sides of the equation by s and let $s \to \infty$. If this works it is usually very quick.

2. Select convenient values of s and evaluate both sides of the equation for these values of s

For **<u>Example 4</u>**, using Method 1,

$$\lim_{s \to \infty} \left[s \frac{1}{(s+1)(s+2)^2} \right] = \lim_{s \to \infty} \left[\frac{s}{s+1} + a_2 \frac{s}{s+2} - \frac{s}{(s+2)^2} \right]$$

or

$$0 = 1 + a_2 + 0$$

so a_2 = -1.

For **<u>Example 5</u>**, using Method 1,

$$\lim_{s \to \infty} \left[s \frac{s+1}{s^2(s+2)(s+3)} \right] = \lim_{s \to \infty} \left[a_1 \frac{s}{s} + \frac{1}{6}\frac{s}{s^2} - \frac{1}{4}\frac{s}{s+2} + \frac{2}{9}\frac{s}{s+3} \right]$$

or

$$0 = a_1 + 0 - \frac{1}{4} + \frac{2}{9}$$

so $a_1 = \frac{1}{4} - \frac{2}{9} = \frac{1}{36}$

For **Example 4**, using Method 2, let's choose $s = 0$ (note both sides of the equation must be finite!)

$$\lim_{s \to 0} \left[\frac{1}{(s+1)(s+2)^2} \right] = \lim_{s \to 0} \left[\frac{1}{s+1} + a_2 \frac{1}{s+2} - \frac{1}{(s+2)^2} \right]$$

or

$$\frac{1}{4} = 1 + \frac{a_2}{2} - \frac{1}{4}$$

so $a_2 = 2(\frac{1}{4} + \frac{1}{4} - 1) = -1$

For **Example 5**, using Method 2, let's choose $s = -1$ (note that $s = 0$, $s = -2$, or $s = -3$ will not work)

$$\lim_{s \to -1} \left[\frac{s+1}{s^2(s+2)(s+3)} \right] = \lim_{s \to -1} \left[a_1 \frac{1}{s} + \frac{1}{6}\frac{1}{s^2} - \frac{1}{4}\frac{1}{s+2} + \frac{2}{9}\frac{1}{s+3} \right]$$

or

$$0 = -a_1 + \frac{1}{6} - \frac{1}{4}\frac{1}{9}$$

so $a_1 = \frac{1}{6} - \frac{1}{4} + \frac{1}{9} = \frac{1}{36}$

Then for **Example 4**,

$$h(t) = e^{-t}u(t) - e^{-2t}u(t) - te^{-2t}u(t)$$

and for **Example 5**

$$h(t) = \frac{1}{36}u(t) + \frac{1}{6}tu(t) - \frac{1}{4}e^{-2t}u(t) + \frac{2}{9}e^{-3t}u(t)$$

In summary, for repeated and distinct poles, go through the following steps:

1. Determine the form of the partial fraction expansion. There must be as many unknowns as the highest power of s in the denominator.

2. Determine the coefficients associated with the distinct poles using the coverup method.

3. Determine the coefficient associated with the highest power of a repeated pole using the coverup method.

4. Determine the remaining coefficients by

   - Multiplying both sides by s and letting $s \rightarrow \infty$
   - Setting s to a convenient value in both sides of the equations. Both sides must remain finite

Example 6. Assuming

$$H(s) \quad = \quad \frac{s^2}{(s+1)^2(s+3)}$$

determine the corresponding impulse response $h(t)$.

First, we determine the correct form

$$H(s) \quad = \quad \frac{s^2}{(s+1)^2(s+3)} \quad = \quad a_1 \frac{1}{s+1} + a_2 \frac{1}{(s+1)^2} + a_3 \frac{1}{s+3}$$

Second, we determine the coefficient(s) of the distinct pole(s)

$$a_3 \quad = \quad \lim_{s \rightarrow -3} \frac{(s^2)}{(s+1)^2 \text{[redacted]}} \quad = \quad \frac{9}{4}$$

Third, we determine the coefficient(s) of the highest power of the repeated pole(s)

$$a_2 \quad = \quad \lim_{s \rightarrow -1} \frac{(s^2)}{\text{[redacted]}(s+3)} \quad = \quad \frac{1}{2}$$

Fourth, we determine any remaining coefficients

$$\lim_{s \rightarrow \infty} \left[s \frac{s^2}{(s+1)^2(s+3)} \right] \quad = \quad \lim_{s \rightarrow \infty} \left[a_1 \frac{s}{s+1} + \frac{1}{2} \frac{s}{(s+1)^2} + \frac{9}{4} \frac{s}{(s+3)} \right]$$

or

$$1 \quad = \quad a_1 + 0 + \frac{9}{4}$$

or $a_1 = 1 - \frac{9}{4} = -\frac{5}{4}$.

Putting it all together, we have

$$h(t) \quad = \quad -\frac{5}{4} e^{-t} u(t) + \frac{1}{2} t e^{-t} u(t) + \frac{9}{4} e^{-3t} u(t)$$

Example 7. Assume we have the transfer function

$$H(s) \quad = \quad \frac{s+3}{s(s+1)^2(s+2)^2}$$

find the corresponding impulse response, $h(t)$.

First we determine the correct form

$$H(s) \quad = \quad \frac{s+3}{s(s+1)^2(s+2)^2} \quad = \quad a_1\frac{1}{s} + a_2\frac{1}{s+1} + a_3\frac{1}{(s+1)^2} + a_4\frac{1}{s+2} + a_5\frac{1}{(s+2)^2}$$

Second, we determine the coefficient(s) of the distinct pole(s)

$$a_1 \quad = \quad \lim_{s\to 0} \frac{s+3}{\text{[redacted]}(s+1)^2(s+2)^2} \quad = \quad \frac{3}{(1)(4)} \quad = \frac{3}{4}$$

Third, we determine the coefficient(s) of the highest power of the repeated pole(s)

$$a_3 \quad = \quad \lim_{s\to -1} \frac{s+3}{s\,\text{[redacted]}(s+2)^2} \quad = \quad \frac{2}{(-1)(1)} \quad = \quad -2$$

$$a_5 \quad = \quad \lim_{s\to -2} \frac{s+3}{s(s+1)^2\text{[redacted]}} \quad = \quad \frac{1}{(-2)(1)} \quad = \quad -\frac{1}{2}$$

Fourth, we determine any remaining coefficients

$$\lim_{s\to\infty}\left[s\frac{s+3}{s(s+1)^2(s+2)^2}\right] \quad = \quad \lim_{s\to\infty}\left[\frac{3}{4}\frac{s}{s} + a_2\frac{s}{s+1} - 2\frac{s}{(s+1)^2} + a_4\frac{s}{s+2} - \frac{1}{2}\frac{s}{(s+2)^2}\right]$$

or

$$0 \quad = \quad \frac{3}{4} + a_2 + a_4$$

We need one more equation, so let's set $s = -3$

$$\lim_{s\to -3}\left[\frac{s+3}{s(s+1)^2(s+2)^2}\right] \quad = \quad \lim_{s\to -3}\left[\frac{3}{8}\frac{1}{s} + a_2\frac{1}{s+1} - 2\frac{1}{(s+1)^2} + a_4\frac{1}{s+2} - \frac{1}{2}\frac{1}{(s+2)^2}\right]$$

or

$$0 \quad = \quad -\frac{1}{4} - a_2\frac{1}{2} - \frac{1}{2} - a_4 - \frac{1}{2}$$

This gives us the set of equations

$$\begin{bmatrix} 1 & 1 \\ \frac{1}{2} & -1 \end{bmatrix}\begin{bmatrix} a_2 \\ a_4 \end{bmatrix} \quad = \quad \begin{bmatrix} \frac{-3}{4} \\ \frac{5}{4} \end{bmatrix}$$

with solution $a_2 = 1$ and $a_4 = -\frac{7}{4}$. Putting it all together we have

$$h(t) \quad = \quad \frac{3}{4}u(t) + e^{-t}u(t) - 2te^{-t}u(t) + \frac{-7}{4}e^{-2t}u(t) - \frac{1}{2}te^{-2t}u(t)$$

2.6 Complex Conjugate Poles: Completing the Square

Before using partial fractions on systems with complex conjugate poles, we need to review one property of Laplace transforms:

$$\textit{if } x(t) \Leftrightarrow X(s), \textit{ then } e^{-at}x(t) \Leftrightarrow X(s+a)$$

To show this, we start with what we are given:

$$\mathcal{L}\{x(t)\} \;=\; \int_0^\infty x(t)e^{-st}dt \;=\; X(s)$$

Then

$$\mathcal{L}\{e^{-at}x(t)\} \;=\; \int_0^\infty e^{-at}x(t)e^{-st}dt \;=\int_0^\infty x(t)e^{-(s+a)t}dt \;=\; X(s+a)$$

The other relationship we need are the Laplace transform pairs for sines and cosines

$$\cos(bt)u(t) \;\Leftrightarrow\; \frac{s}{s^2+b^2}$$
$$\sin(bt)u(t) \;\Leftrightarrow\; \frac{b}{s^2+b^2}$$

Finally, we need to put these together, to get the Laplace transform pair:

$$e^{-at}\cos(bt)u(t) \;\Leftrightarrow\; \frac{s+1}{(s+a)^2+b^2}$$
$$e^{-at}\sin(bt)u(t) \;\Leftrightarrow\; \frac{b}{(s+a)^2+b^2}$$

Complex poles always result in sines and cosines. We will be trying to make terms with complex poles look like these terms by competing the square in the denominator.

In order do get the denominators in the correct form when we have complex poles, we need to compete the square in the denominator. That is, we need to be able to write the denominator as

$$D(s) \;=\; (s+a)^2+b^2$$

To do this, we always first find a using the fact that the coefficient of s will be $2a$. The we use whatever is needed to construct b. A few example will hopefully make this clear.

Example 8. Let's assume

$$D(s) \;=\; s^2+s+2$$

and we want to write this in the correct form. First we recognize that the coefficient of s is 1, so we know $2a = 1$ or $a = \frac{1}{2}$. We then have

$$D(s) \;=\; s^2+s+2 \;=\; (s+\frac{1}{2})^2+b^2$$

To find b we expand the right hand side of the above equations, and then equate powers of s:

$$D(s) \;=\; s^2+s+2 \;=\; (s+\frac{1}{2})^2+b^2 \;=\; s^2+s+\frac{1}{4}+b^2$$

clearly $2 = b^2 + \frac{1}{4}$, or $b^2 = \frac{7}{4}$, or $b = \frac{\sqrt{7}}{2}$. Hence we have

$$D(s) \;=\; s^2+s+2 \;=\; (s+\frac{1}{2})^2+\left(\frac{\sqrt{7}}{2}\right)^2$$

and this is the form we need.

Example 9. Let's assume

$$D(s) \;=\; s^2+3s+5$$

and we want to write this in the correct form. First we recognize that the coefficient of s is 3, so we know $2a = 3$ or $a = \frac{3}{2}$. We then have

$$D(s) \;=\; s^2+3s+5 \;=\; (s+\frac{3}{2})^2+b^2$$

To find b we expand the right hand side of the above equations, and then equate powers of s:

$$D(s) \;=\; s^2+3s+5 \;=\; (s+\frac{3}{2})^2+b^2 \;=\; s^2+3s+\frac{9}{4}+b^2$$

clearly $5 = b^2 + \frac{9}{4}$, or $b^2 = \frac{11}{4}$, or $b = \frac{\sqrt{11}}{2}$. Hence we have

$$D(s) \;=\; s^2+3s+5 \;=\; (s+\frac{3}{2})^2+\left(\frac{\sqrt{11}}{2}\right)^2$$

and this is the form we need.

Now that we know how to complete the square in the denominator, we are ready to look at complex poles. We will start with two simple examples, and then explain how to deal with more complicated examples.

Example 10. Assuming

$$H(s) \;=\; \frac{1}{s^2+s+2}$$

and we want to find the corresponding impulse response $h(t)$. In this simple case, we first complete the square, as we have done above, to write

$$H(s) \;=\; \frac{1}{(s+\frac{1}{2})^2+\left(\frac{\sqrt{7}}{2}\right)^2}$$

This almost has the form we want, which is

$$e^{-at}\sin(bt)u(t) \quad \Leftrightarrow \quad \frac{b}{(s+a)^2+b^2}$$

However, to use this form we need b in the numerator. To achieve this we will multiply and divide by $b = \frac{\sqrt{7}}{2}$

$$\begin{aligned} H(s) &= \frac{1}{(s+\frac{1}{2})^2 + \left(\frac{\sqrt{7}}{2}\right)^2} \\ &= \frac{1}{\frac{\sqrt{7}}{2}} \frac{\frac{\sqrt{7}}{2}}{(s+\frac{1}{2})^2 + \left(\frac{\sqrt{7}}{2}\right)^2} \end{aligned}$$

or

$$h(t) = \frac{2}{\sqrt{7}} e^{-\frac{1}{2}t} \sin(\frac{\sqrt{7}}{2}t)u(t)$$

Example 11. Assuming

$$H(s) = \frac{s}{s^2+3s+5}$$

and we want to find the corresponding impulse response $h(t)$. In this simple case, we first complete the square, as we have done above, to write

$$H(s) = \frac{s}{(s+\frac{3}{2})^2 + \left(\frac{\sqrt{11}}{2}\right)^2}$$

This almost has the form we want, which is

$$e^{-at}\cos(bt)u(t) \quad \Leftrightarrow \quad \frac{(s+a)}{(s+a)^2+b^2}$$

However, to use this form we need $s+a$ in the numerator, not just s To achieve this we will add and subtract $a = \frac{3}{2}$ in the numerator

$$\begin{aligned} H(s) &= \frac{s+\frac{3}{2}-\frac{3}{2}}{(s+\frac{3}{2})^2 + \left(\frac{\sqrt{11}}{2}\right)^2} \\ &= \frac{s+\frac{3}{2}}{(s+\frac{1}{2})^2 + \left(\frac{\sqrt{11}}{2}\right)^2} - \frac{\frac{3}{2}}{(s+\frac{1}{2})^2 + \left(\frac{\sqrt{11}}{2}\right)^2} \end{aligned}$$

The first term is now what we want, and will produce a term of the form

$$e^{-\frac{3}{2}t}\cos(\frac{\sqrt{11}}{2}t)u(t)$$

The second term needs some work. It looks like a sine times a decaying exponential, but the scaling is wrong. Again, to put this term in the correct form we will multiply and divide by $\frac{\sqrt{11}}{2}$

$$H(s) \quad = \quad \frac{s+\frac{3}{2}}{(s+\frac{1}{2})^2 + \left(\frac{\sqrt{11}}{2}\right)^2} - \frac{3}{2}\frac{1}{\frac{\sqrt{11}}{2}}\frac{\frac{\sqrt{11}}{2}}{(s+\frac{1}{2})^2 + \left(\frac{\sqrt{11}}{2}\right)^2}$$

which gives

$$h(t) \quad = \quad e^{-\frac{3}{2}t}\cos(\frac{\sqrt{11}}{2}t)u(t) - \frac{3}{\sqrt{11}}e^{-\frac{3}{2}t}\sin(\frac{\sqrt{11}}{2}t)u(t)$$

Note that it is possible to combine the sine and cosine terms into a single cosine with a phase angle, but we will not pursue that here.

The examples we are done so far only contain complex roots. In general we need to be able to deal with systems that have both complex and real roots. Since we are dealing with real systems in this course, all complex poles will occur in complex conjugate pairs. Hence when we have complex poles we will look for quadratic factors of the general form

$$\frac{cs+d}{s^2 + +es + d}$$

Note that there are **<u>two</u>** unknown coefficients in this term. Since we need as many unknowns as the highest power of s in the denominator, and this term has 2 powers of s, we need two unknowns. We are now ready to do one more example.

<u>Example 12</u>. Assuming

$$H(s) \quad = \quad \frac{1}{(s+2)(s^2+s+1)}$$

and we want to determine the corresponding impulse response $h(t)$. First we need to find the correct form for the partial fractions

$$H(s) \quad = \quad \frac{1}{(s+2)(s^2+s+1)} \quad = a_1\frac{1}{s+2} + \frac{a_2 s + a_3}{s^2+s+1}$$

Note that we have three unknowns since the highest power of s in the denominator is 3. Since there is an isolated pole at -2, we find coefficient a_1 first using the coverup method

$$a_1 \quad = \quad \lim_{s\to -2}\frac{1}{\text{[redacted]}(s^2+s+1)} \quad = \quad \frac{1}{(-2)^2+(-2)+1} \quad = \quad \frac{1}{3}$$

To find a_2, let's use our trick of multiplying by s and letting $s \to \infty$

$$\lim_{s\to\infty}\left[s\frac{1}{(s+2)(s^2+s+1)}\right] \quad = \quad \lim_{s\to\infty}\left[\frac{1}{3}\frac{s}{s+2} + \frac{a_2 s^2 + a_3 s}{s^2+s+1}\right]$$

or

$$0 = \frac{1}{3} + a_2$$

so $a_2 = -\frac{1}{4}$. Now we have to find a_3 and the only trick we have left is choosing a value of s. For this particular transfer function, $s = 0$ is a good choice

$$\lim_{s\to 0}\left[\frac{1}{(s+2)(s^2+s+1)}\right] = \lim_{s\to 0}\left[\frac{1}{4}\frac{1}{s+2} + \frac{a_2 s + a_3}{s^2+s+1}\right]$$

or

$$\frac{1}{3} = \frac{1}{6} + a_3$$

or $a_3 = \frac{1}{3}$. So far we have

$$H(s) = \frac{1}{3}\frac{1}{s+2} + \frac{-\frac{1}{3}s + \frac{1}{3}}{s^2+s+1}$$

The first term is easy, now we need to work on the second term. First we complete the square in the denominator

$$s^2 + s + 1 = (s + \frac{1}{2})^2 + \left(\frac{\sqrt{3}}{2}\right)^2$$

so we have

$$H(s) = \frac{1}{3}\frac{1}{s+2} + \frac{-\frac{1}{3}s + \frac{1}{3}}{(s+\frac{1}{2})^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

The next thing to do is to add and subtract $\frac{1}{2}$, so the numerator has the correct form so we have

$$\begin{aligned}
H(s) &= \frac{1}{3}\frac{1}{s+2} + \frac{-\frac{1}{3}(s+\frac{1}{2}-\frac{1}{2}) + \frac{1}{3}}{(s+\frac{1}{2})^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \\
&= \frac{1}{3}\frac{1}{s+2} + \frac{-\frac{1}{3}(s+\frac{1}{2}) + (\frac{1}{6}+\frac{1}{3})}{(s+\frac{1}{2})^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \\
&= \frac{1}{3}\frac{1}{s+2} + \frac{-\frac{1}{3}(s+\frac{1}{2}) + \frac{1}{2}}{(s+\frac{1}{2})^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \\
&= \frac{1}{3}\frac{1}{s+2} - \frac{1}{3}\frac{(s+\frac{1}{2})}{(s+\frac{1}{2})^2 + \left(\frac{\sqrt{3}}{2}\right)^2} + \frac{1}{2}\frac{1}{(s+\frac{1}{2})^2 + \left(\frac{\sqrt{3}}{2}\right)^2}
\end{aligned}$$

Finally, we have to scale the final term to put it into the correct form

$$\begin{aligned}
H(s) &= \frac{1}{3}\frac{1}{s+2} - \frac{1}{3}\frac{(s+\frac{1}{2})}{(s+\frac{1}{2})^2 + \left(\frac{\sqrt{3}}{2}\right)^2} + \frac{1}{2}\frac{1}{\frac{\sqrt{3}}{2}}\frac{\frac{\sqrt{3}}{2}}{(s+\frac{1}{2})^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \\
&= \frac{1}{3}\frac{1}{s+2} - \frac{1}{3}\frac{(s+\frac{1}{2})}{(s+\frac{1}{2})^2 + \left(\frac{\sqrt{3}}{2}\right)^2} + \frac{1}{\sqrt{3}}\frac{\frac{\sqrt{3}}{2}}{(s+\frac{1}{2})^2 + \left(\frac{\sqrt{3}}{2}\right)^2}
\end{aligned}$$

So we finally have

$$h(t) = \frac{1}{3}e^{-2t}u(t) - \frac{1}{3}e^{-\frac{1}{2}t}\cos(\frac{\sqrt{3}}{2}t)u(t) + \frac{1}{\sqrt{3}}e^{-\frac{1}{2}t}\sin(\frac{\sqrt{3}}{2}t)u(t)$$

2.7 Complex Conjugate Poles-Again

It is very important to understand the basic structure of complex conjugate poles. For a system with complex poles at $-a \pm bj$, the characteristic equation (denominator of the transfer function) will be

$$\begin{aligned} D(s) &= [s-(-a+jb)][s-(-a-jb)] \\ &= [s+(a-jb)][s+(a+jb0] \\ &= s^2 + [(a-jb)+(a+jb)]s + (a-jb)(a+jb) \\ &= s^2 + 2as + a^2 + b^2 \\ &= (s+a)^2 + b^2 \end{aligned}$$

We know that this form leads to terms of the form $e^{-at}\cos(bt)u(t)$ and $e^{-at}\sin(bt)u(t)$. Hence we have the general relationship that complex poles at $-a \pm jb$ lead to time domain functions that

- decay like e^{-at} (the real part determines the decay rate)
- oscillate like $\cos(bt)$ or $\sin(bt)$ (the imaginary part determines the oscillation frequency)

These relationships relating the imaginary and real parts of the poles with corresponding time domain functions is very important to remember.

3 Final Value Theorem

The final value theorem for Laplace transforms can generally be stated as follows:
If $Y(s)$ has all of its poles in the open left half plane, with the possible exception of a single pole at the origin, then

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s)$$

provided the limits exist.

Example 1. For $y(t) = e^{-at}u(t)$ with $a > 0$ we have

$$\begin{aligned}
\lim_{t\to\infty} y(t) &= \lim_{t\to\infty} e^{-at} = 0 \\
\lim_{s\to 0} Y(s) &= \lim_{s\to 0} s\frac{1}{s+a} = \lim_{s\to 0} \frac{s}{s+a} = 0
\end{aligned}$$

Example 2. For $y(t) = \sin(bt)u(t)$ we have

$$\begin{aligned}
\lim_{t\to\infty} y(t) &= \lim_{t\to\infty} \sin(bt) \\
\lim_{s\to 0} Y(s) &= \lim_{s\to 0} s\frac{b}{s^2+b^2} = \lim_{s\to 0} \frac{sb}{s^2+b^2} = 0
\end{aligned}$$

Clearly $\lim_{t\to\infty} y(t) \neq \lim_s sY(s)$. Why? Because the final value theorem is not valid since $Y(s)$ has two poles on the $j\omega$ axis.

Example 3. For $y(t) = u(t)$ we have

$$\begin{aligned}
\lim_{t\to\infty} y(t) &= \lim_{t\to\infty} u(t) = 1 \\
\lim_{s\to 0} Y(s) &= \lim_{s\to 0} s\frac{1}{s} = \lim_{s\to 0} \frac{s}{s} = 1
\end{aligned}$$

Example 4. For $y(t) = e^{-at}\cos(bt)u(t)$ with $a > 0$ we have

$$\begin{aligned}
\lim_{t\to\infty} y(t) &= \lim_{t\to\infty} e^{-at}\cos(bt)u(t) = 0 \\
\lim_{s\to 0} Y(s) &= \lim_{s\to 0} s\frac{(s+a)}{(s+a)^2+b^2} = \lim_{s\to 0} \frac{s(s+a)}{(s+a)^2+b^2} = 0
\end{aligned}$$

We will use this final value theorem a great deal in this course.

4 Step Response and Position Error, Ramp Response and Velocity Error

In control systems, we are often most interested in the response of a system to the following types of inputs

- a step
- a ramp
- a sinusoid

Although in reality control systems have to respond to a large number of different inputs, these are usually good models for the range of input signals a control system is likely to encounter.

4.1 Step Response and Position Error

The *step response* of a system is the response of the system to a step input. In the time domain, we compute the step response as

$$y(t) = h(t) \star Au(t)$$

where A is the amplitude of the step and $u(t)$ is the unit step function and $\star$ is the convolution operator. In the s domain, we compute the step response as

$$\begin{aligned} Y(s) &= H(s)\frac{A}{s} \\ y(t) &= \mathcal{L}^{-1}\{Y(s)\} \end{aligned}$$

The *position error*, e_p, is the difference between the input step and the resulting response as $t \rightarrow \infty$,

$$\begin{aligned} e_p &= \lim_{t\to\infty} [Au(t) - y(t)] \\ &= A - \lim_{t\to\infty} y(t) \end{aligned}$$

Example 1. Consider the system with transfer function $H(s) = \frac{4}{s^2+2s+5}$. Determine step response and position error for this system.

First we find the step response,

$$\begin{aligned} Y(s) &= \frac{4}{s^2+2s+5}\frac{A}{s} = a_1\frac{1}{s} + \frac{a_2 s + a_3}{(s+1)^2+2^2} \\ &= A\left[\frac{4}{5}\frac{1}{s} - \frac{\frac{4}{5}s + \frac{8}{5}}{(s+1)^2+2^2}\right] \\ &= A\left[\frac{4}{5}\frac{1}{s} - \frac{\frac{4}{5}(s+1)}{(s+1)^2+2^2} - \frac{2}{5}\frac{2}{(s+1)^2+2^2}\right] \end{aligned}$$

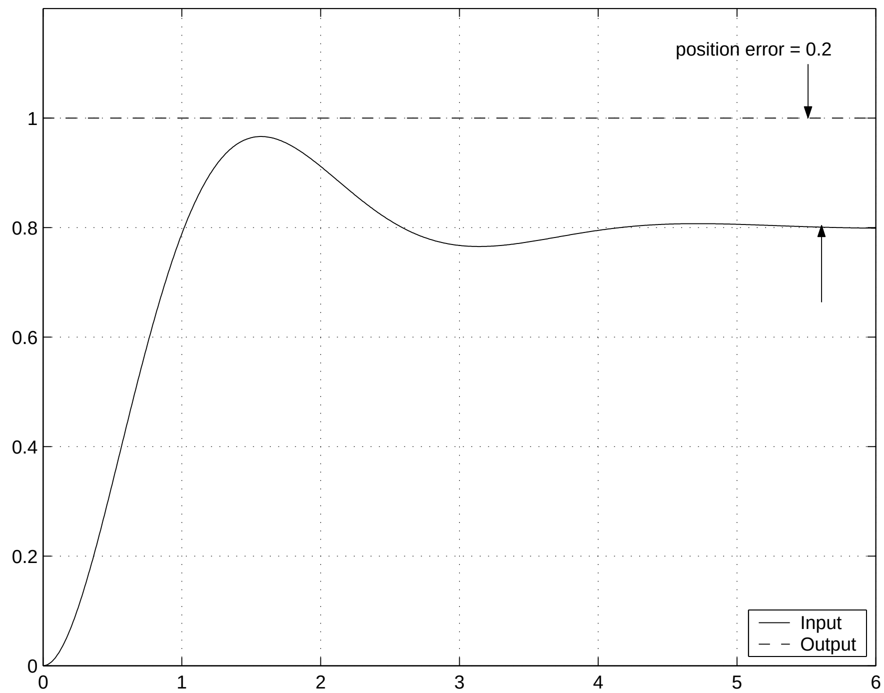


Figure 1: The unit step response and position error for the system in Example 1.

or

$$y(t) = A\left[\frac{4}{5}u(t) - \frac{4}{5}e^{-t}\cos(2t)u(t) - \frac{2}{5}e^{-t}\sin(2t)u(t)\right]$$

Then the position error is

$$\begin{aligned} e_p &= A - \lim_{t\to\infty} A\left[\frac{4}{5}u(t) - \frac{4}{5}e^{-t}\cos(2t)u(t) - \frac{2}{5}e^{-t}\sin(2t)u(t)\right] \\ &= A - \frac{4A}{5} \\ &= \frac{A}{5} \end{aligned}$$

The step response and position error of this system are shown in Figure 1 for a a unit step input.

Example 2. Consider the system with transfer function $H(s) = \frac{1}{(s+1)(s+3)}$. Determine step response and position error for this system.
First we find the step response

$$\begin{aligned} Y(s) &= \frac{1}{(s+1)(s+3)}\frac{A}{s} = a_1\frac{1}{s} + a_2\frac{1}{s+1} + a_3\frac{1}{s+3} \\ &= \frac{A}{3}\frac{1}{s} - \frac{A}{2}\frac{1}{s+1} + \frac{A}{6}\frac{1}{s+3} \end{aligned}$$

or

$$y(t) = A\left[\frac{1}{3}u(t) - \frac{1}{2}e^{-t}u(t) + \frac{1}{6}e^{-3t}u(t)\right]$$

Then the position error is

$$\begin{aligned} e_p &= A - \lim_{t\to\infty} A\left[\frac{1}{3}u(t) - \frac{1}{2}e^{-t}u(t) + \frac{1}{6}e^{-3t}u(t)\right] \\ &= A - \frac{A}{3} \\ &= \frac{2A}{3} \end{aligned}$$

Now as much as I'm sure you like completing the square and doing partial fractions, there is an easier way to do this. We already have learned that if $Y(s)$ has all of its poles in the open left half plane (with the possible exception of a single pole at the origin), we can use the final value theorem to find the steady state value of the step response. Specifically,

$$\begin{aligned} \lim_{t\to\infty} y(t) &= \lim_{s\to 0} sY(s) \\ &= \lim_{s\to 0} s\left[H(s)\frac{A}{s}\right] \\ &= \lim_{s\to 0} AH(s) \\ &= AH(0) \end{aligned}$$

and then, for stable $H(s)$ we have

$$e_p = A - AH(0)$$

where A is the amplitude of the step input (usually $A = 1$).

Example 3. From Example 1, we compute

$$\begin{aligned} e_p &= A - AH(0) \\ &= A - A\frac{4}{5} \\ &= \frac{A}{5} \end{aligned}$$

Example 4. From Example 2, we compute

$$\begin{aligned} e_p &= A - AH(0) \\ &= A - A\frac{1}{3} \\ &= \frac{2A}{3} \end{aligned}$$

There is yet another way to compute the position error, which is useful to know. Let's assume we write the transfer function as

$$H(s) = \frac{n_m s^m + n_{m-1}s^{m-1} + ... + n_2 s^2 + n_1 s + n_0}{s^n + d_{n-1}s^{n-1} + ... + d_2 s^2 + d_1 s + d_0}$$

We then need to compute

$$e_p = \lim_{s\to 0} A[1 - H(s)]$$

Let's write $1 - H(s)$ and put it all over a common denominator. Then we have

$$\begin{aligned} 1 - H(s) &= \frac{(s^n + d_{n-1}s^{n-1} + ... + d_2s^2 + d_1s + d_0) - (n_m s^m + n_{m-1}s^{m-1} + ... + n_2s^2 + n_1s + n_0)}{s^n + d_{n-1}s^{n-1} + ... + d_2s^2 + d_1s + d_0} \\ &= \frac{... + (d_2 - n_2)s^2 + (d_1 - n_1)s + (d_0 - n_0)}{s^n + d_{n-1}s^{n-1} + ... + d_2s^2 + d_1s + d_0} \end{aligned}$$

Then

$$\begin{aligned} e_p &= \lim_{s\to 0} A[1 - H(s)] \\ &= A\frac{d_0 - n_0}{d_0} \end{aligned}$$

Example 5. From Example 1, we have $n_0 = 4$ and $d_0 = 5$, so the position error is $e_p = A\frac{5-4}{5} = \frac{A}{5}$.

Example 6. From Example 2, we have $n_0 = 1$, $d_0 = 3$, so the position error is $e_p = A\frac{3-1}{3} = \frac{2A}{3}$.

4.2 Ramp Response and Velocity Error

The *ramp response* of a system is the response of the system to a ramp input. In the time domain, we compute the ramp response as

$$y(t) = h(t) \star Atu(t)$$

where A is the amplitude of the step and $u(t)$ is the unit step function. In the s domain, we compute the step response as

$$\begin{aligned} Y(s) &= H(s)\frac{A}{s^2} \\ y(t) &= \mathcal{L}^{-1}\{Y(s)\} \end{aligned}$$

The *velocity error*, e_v, is the difference between the input ramp and the resulting response as $t \to \infty$,

$$e_v = \lim_{t\to\infty} [Atu(t) - y(t)]$$

It should be clear that unless $y(t)$ has a term like $Atu(t)$, the ramp response will be infinite.

Example 7. Consider the system with transfer function $H(s) = \frac{1}{s+1}$. Determine the ramp response and velocity error for this system.

First we find the ramp response

$$\begin{aligned} Y(s) &= \frac{1}{s+1}\frac{A}{s^2} = a_1\frac{1}{s} + a_2\frac{1}{s^2} + a_3\frac{1}{s+1} \\ &= A\left[-\frac{1}{s} + \frac{1}{s^2} + \frac{1}{s+1}\right] \end{aligned}$$

or

$$y(t) \;=\; A\left[-u(t) + tu(t) + e^{-t}u(t)\right]$$

Then the velocity error is

$$\begin{aligned}
e_v &= Atu(t) - \lim_{t\to\infty} A\left[-u(t) + tu(t) + e^{-t}u(t)\right] \\
&= At - At + A \\
&= A
\end{aligned}$$

Example 8. Consider the system with transfer function $H(s) = \frac{s+2}{s^2+2s+2}$. Determine the ramp response and velocity error for this system.

First we find the ramp response

$$\begin{aligned}
Y(s) &= \frac{s+2}{s^2+2s+2}\frac{A}{s^2} \;=\; a_1\frac{1}{s} + a_2\frac{1}{s^2} + \frac{a_3 s + a_4}{s^2+2s+2} \\
&= A\left[-\frac{1}{2}\frac{1}{s} + \frac{1}{s^2} + \frac{1}{2}\frac{s}{(s+1)^2+1}\right] \\
&= A\left[-\frac{1}{2}\frac{1}{s} + \frac{1}{s^2} + \frac{1}{2}\frac{s+1}{(s+1)^2+1} - \frac{1}{2}\frac{1}{(s+1)^2+1}\right]
\end{aligned}$$

or

$$y(t) \;=\; A\left[-\frac{1}{2}u(t) + tu(t) + \frac{1}{2}e^{-t}\cos(t)u(t) - \frac{1}{2}e^{-t}\sin(t)u(t)\right]$$

Then the velocity error is

$$\begin{aligned}
e_v &= Atu(t) - \lim_{t\to\infty} A\left[-\frac{1}{2}u(t) + tu(t) + \frac{1}{2}e^{-t}\cos(t)u(t) - \frac{1}{2}e^{-t}\sin(t)u(t)\right] \\
&= At - At + \frac{1}{2}A \\
&= \frac{A}{2}
\end{aligned}$$

The ramp response and velocity error of this system are shown in Figure 2 for a a unit ramp input. We can try and use the final value Theorem again, but it becomes a bit more complicated. We want to find

$$\begin{aligned}
e_v &= \lim_{t\to\infty}\left[Atu(t) - y(t)\right] \\
&= \lim_{s\to 0} s\left[\frac{A}{s^2} - \frac{A}{s^2}H(s)\right] \\
&= \lim_{s\to 0} \frac{A}{s}\left[1 - H(s)\right]
\end{aligned}$$

Let's assume again we can write the transfer function as

$$H(s) \;=\; \frac{n_m s^m + n_{m-1}s^{m-1} + \ldots + n_2 s^2 + n_1 s + n_0}{s^n + d_{n-1}s^{n-1} + \ldots + d_2 s^2 + d_1 s + d_0}$$

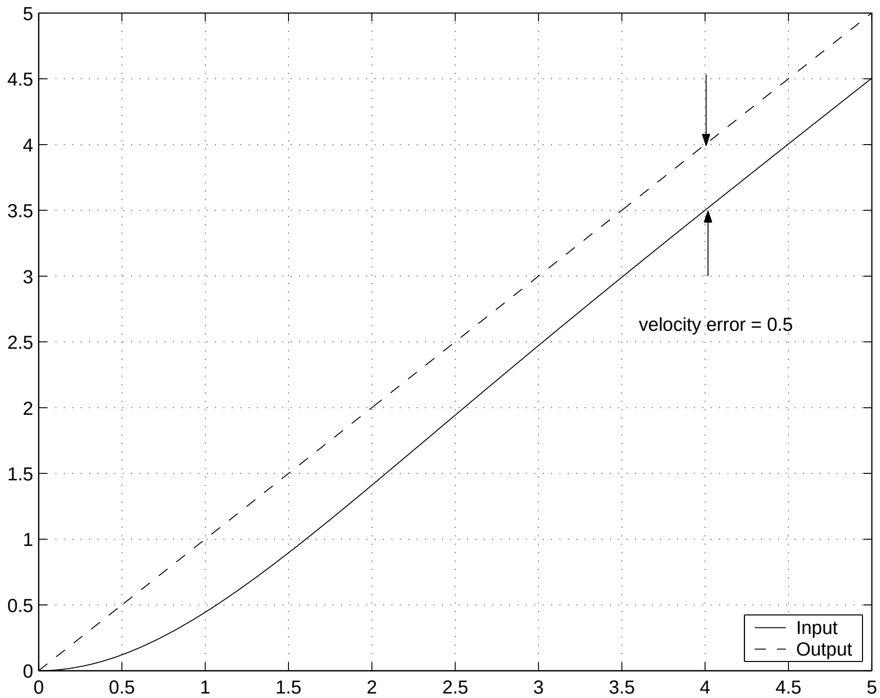


Figure 2: The unit ramp response and velocity error for the system in Example 8.

If we compute $1 - H(s)$ and put things over a common denominator, we have

$$
\begin{aligned}
1 - H(s) &= \frac{(s^n + d_{n-1}s^{n-1} + ... + d_2s^2 + d_1s + d_0) - (n_m s^m + n_{m-1}s^{m-1} + ... + n_2s^2 + n_1s + n_0)}{s^n + d_{n-1}s^{n-1} + ... + d_2s^2 + d_1s + d_0} \\
&= \frac{... + (d_2 - n_2)s^2 + (d_1 - n_1)s + (d_0 - n_0)}{s^n + d_{n-1}s^{n-1} + ... + d_2s^2 + d_1s + d_0}
\end{aligned}
$$

and

$$
\frac{1}{s}[1 - H(s)] = \frac{... + (d_2 - n_2)s^+(d_1 - n_1) + (d_0 - n_0)\frac{1}{s}}{s^n + d_{n-1}s^{n-1} + ... + d_2s^2 + d_1s + d_0}
$$

Now, in order to have e_v be finite we must get a finite value as $s \rightarrow 0$ in this expression. The value of the denominator will be d_0 as $s \rightarrow 0$, so the denominator will be OK. All of the terms in the numerator will be zero except the last two: $(d_1 - n_1) + (d_0 - n_0)\frac{1}{s}$ In order to get a finite value from these terms, we must have $n_0 = d_0$, that is, constant terms in the numerator and denominator must be the same. *This also means that the system must have zero position error !* If the system does not have zero position error, the velocity error will be infinite! If $n_0 = d_0$, then we have

$$
e_v = \lim_{s \rightarrow 0} \frac{A}{s}[1 - H(s)] = A\frac{d_1 - n_1}{d_0}
$$

Example 9. For the system in Example 7, $H(s) = \frac{1}{s+1}$. Here $n_0 = d_0 = 1$, so the system has zero position error, and $n_1 = 0$, $d_1 = 1$. Hence $e_v = A\frac{d_1 - n_1}{d_0} = A$.

Example 10. For the system in Example 8, $H(s) = \frac{s+2}{s^2+2s+2}$. Here $n_0 = d_0 = 2$, so the system has zero position error, and $n_1 = 1$, $d_1 = 2$. Hence $e_v = A\frac{d_1-n_1}{d_0} = \frac{A}{2}$.

4.3 Summary

Assume we write the transfer function of a system as

$$H(s) \quad = \quad \frac{n_m s^m + n_{m-1}s^{m-1} + ... + n_2 s^2 + n_1 s + n_0}{s^n + d_{n-1}s^{n-1} + ... + d_2 s^2 + d_1 s + d_0}$$

The *step response* of a system is the response of the system to a step input. The *position error*, e_p, is the difference between the input and the output of the system in steady state. We can compute the position error in a variety of ways:

$$\begin{aligned} e_p &= \lim_{t\to\infty} [Au(t) - y(t)] \\ &= A - \lim_{t\to\infty} y(t) \\ &= A(1 - H(0)) \\ &= A\frac{d_0 - n_0}{d_0} \end{aligned}$$

The *ramp response* of a system is the response of the system to a ramp input. The *velocity error*, e_v, is the difference between the input and output of the system in steady state. A system has infinite velocity error unless the position error is zero. We can compute the position error in a variety of ways:

$$\begin{aligned} e_v &= \lim_{t\to\infty} [At - y(t)] \\ &= A\frac{d_1 - n_1}{d_0} \end{aligned}$$

5 Response of a Ideal Second Order System

This is an important example, which you have probably seen before. Let's assume we have an ideal second order system with transfer function

$$H(s) = \frac{K_{static}}{\frac{1}{\omega_n}^2 s^2 + \frac{2\zeta}{\omega_n}s + 1} = \frac{K_{static}\ {\omega_n}^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

where ζ is the damping ratio, ω_n is the natural frequency, and K_{static} is the static gain. The poles of the transfer function are the roots of the denominator, which are given by the quadratic formula

$$\begin{aligned} roots &= \frac{-2\zeta\omega_n \pm \sqrt{(2\zeta\omega_n)^2 - 4\omega_n^2}}{2} \\ &= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \\ &= -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2} \\ &= -\zeta\omega_n \pm j\omega_d \\ &= -\sigma \pm j\omega_d \\ &= -1/\tau \pm j\omega_d \end{aligned}$$

where we have used the damped frequency $\omega_d = \omega_n\sqrt{1 - \zeta^2}$ and $\sigma = \frac{1}{\tau} = \zeta\omega_n$. As we start to talk about systems with more than two poles, it is easier to remember to use the form of the poles $-\sigma \pm \omega_d$ or $-1/\tau \pm \omega_d$.

5.1 Step Response of an Ideal Second Order System

To find the step response,

$$Y(s) = H(s)U(s) = \frac{K_{static}\ \omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\frac{1}{s}$$

We then look for a partial fraction expansion in the form

$$Y(s) = \frac{K_{static}\ \omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\frac{1}{s} = a_1\frac{1}{s} + \frac{a_2 s + a_3}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

From this, we can determine that $a_1 = K_{static}$, $a_2 = -K_{static}$, and $a_3 = -2\zeta\omega_n K_{static}$. Hence we have

$$Y(s) = K_{static}\frac{1}{s} - K_{static}\frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Completing the square in the denominator we have

$$Y(s) = K_{static}\frac{1}{s} - K_{static}\frac{s + 2\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2}$$

or

$$
\begin{aligned}
Y(s) &= K_{static}\frac{1}{s} - K_{static}\frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2} - K_{static}\frac{\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2} \\
&= K_{static}\frac{1}{s} - K_{static}\frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2} - K_{static}\frac{\zeta\omega_n}{\omega_d}\frac{\omega_d}{(s+\zeta\omega_n)^2+\omega_d^2}
\end{aligned}
$$

or in the time domain

$$
y(t) = K_{static}\left[1 - e^{-\zeta\omega_n t}\cos(\omega_d t) - \frac{\zeta\omega_n}{\omega_d}e^{-\zeta\omega_n t}\sin(\omega_d t)\right]u(t)
$$

We would now like to write the sine and cosine in terms of a sine and a phase angle. To do this, we use the identity

$$
r\sin(\omega_d + \theta) = r\cos(\omega_d)\sin(\theta) + r\sin(\omega_d)\cos(\theta)
$$

Hence we have

$$
\begin{aligned}
r\sin(\theta) &= 1 \\
r\cos(\theta) &= \frac{\zeta\omega_n}{\omega_d} = \frac{\zeta}{\sqrt{1-\zeta^2}}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\theta &= \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) \\
r &= \frac{1}{\sqrt{1-\zeta^2}}
\end{aligned}
$$

Note that

$$
\cos(\theta) = \frac{\zeta}{\sqrt{1-\zeta^2}}\frac{1}{r} = \frac{\zeta}{\sqrt{1-\zeta^2}}\sqrt{1-\zeta^2}
$$

or $\theta = \cos^{-1}(\zeta)$. Finally we have

$$
y(t) = K_{static}\left[1 - \frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin(\omega_d t + \theta)\right]u(t)
$$

5.2 Time to Peak, T_p

From our solution of the response of the ideal second order system to a unit step, we can compute the time to peak by taking the derivative of $y(t)$ and setting it equal to zero. This will give us the maximum value of $y(t)$ and the time that this occurs at is called the time to peak, T_p.

$$
\frac{dy(t)}{dt} = -\frac{K_{static}}{\sqrt{1-\zeta^2}}\left[-\zeta\omega_n e^{-\zeta\omega_n t}\sin(\omega_d t+\theta) + \omega_d e^{-\zeta\omega_n t}\cos(\omega_d t+\theta)\right] = 0
$$

or

$$\begin{aligned}
\zeta\omega_n \sin(\omega_d t + \theta) &= \omega_d \cos(\omega_d t + \theta) \\
\tan(\omega_d t + \theta) &= \frac{\sqrt{1-\zeta^2}}{\zeta} \\
\theta + \omega_d t &= \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)
\end{aligned}$$

but we already have $\theta = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$, hence $\omega_d t$ must be equal to one period of the tangent, which is π. Hence

$$\boxed{T_p = \frac{\pi}{\omega_d}}$$

Remember that ω_d is equal to the imaginary part of the complex poles.

5.3 Percent Overshoot, PO

Evaluating $y(t)$ at the peak time T_p we get the maximum value of $y(t)$,

$$\begin{aligned}
y(T_p) &= K_{static}\left[1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n T_p} \sin(\omega_d T_p + \theta)\right] \\
&= K_{static}\left[1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n \pi/\omega_d} \sin(\omega_d \frac{\pi}{\omega_d} + \theta)\right] \\
&= K_{static}\left[1 + \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\pi/\sqrt{1-\zeta^2}} \sin(\theta)\right]
\end{aligned}$$

since $\sin(\theta+\pi) = -\sin(\theta)$. Then $\sin(\theta) = \sqrt{1-\zeta^2}$, hence

$$y(t) = K_{static}\left[1 + e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}\right]$$

The percent overshoot is defined as

$$Percent\ Overshoot = P.O. = \frac{y(T_p) - y(\infty)}{y(\infty)} \times 100\%$$

For our second order system we have $y(\infty) = K_{static}$, so

$$P.O. = \frac{K_{static}\left[1 + e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}\right] - K_{static}}{K_{static}} \times 100\%$$

or

$$\boxed{P.O. = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100\%}$$

5.4 Settling Time, T_s

The settling time is defined as the time it takes for the output of a system with a step input to stay within a given percentage of its final value. In this course, we use the 2% settling time criteria, which is generally four time constants. For any exponential decay, the general form is written as $e^{-t/\tau}$, where τ is the time constant. For the ideal second order system response, we have $\tau = 1/\zeta\omega_n$ or $\sigma = \zeta\omega_n$. Hence for and ideal second order system we estimate the settling time as

$$T_s \;=\; 4\tau \;=\; \frac{4}{\sigma} \;=\; \frac{4}{\zeta\omega_n}$$

For systems other than second order system we will want to talk about the settling time, hence the use of the forms

$$\boxed{T_s = 4\tau \;=\; \frac{4}{\sigma}}$$

are often more appropriate to remember.

Example 1. Consider the system with transfer function given by

$$H(s) \;=\; \frac{9}{s^2 + \beta s + 9}$$

determine the range of β so that $T_s \leq 5$ seconds and $T_p \leq 1.2$ seconds.

For the transfer function, we see that $\omega_n = 3$ and $2\zeta\omega_n = \beta$, so $\zeta = \beta/(2\omega_n) = \beta/6$. For the settling time constraint we have

$$\begin{aligned}
T_s &= \frac{4}{\zeta\omega_n} \leq 5 \\
\frac{4}{\frac{\beta}{6}3} &\leq 5 \\
\frac{8}{5} &\leq \beta
\end{aligned}$$

so $\beta \geq 1.60$. For the time to peak constraint we have

$$\begin{aligned}
T_p &= \frac{\pi}{\omega_d} \leq 1.2 \\
\frac{\pi}{\omega_n\sqrt{1-\zeta^2}} &\leq 1.2 \\
\frac{\pi}{1.2\omega_n} &\leq \sqrt{1-\zeta^2} \\
\left(\frac{\pi}{1.2\omega_n}\right)^2 &\leq 1-\zeta^2 \\
\zeta^2 &\leq 1-\left(\frac{\pi}{1.2\omega_n}\right)^2 \\
\zeta &\leq \sqrt{1-\left(\frac{\pi}{1.2\omega_n}\right)^2} \\
\beta &\leq 6\sqrt{1-\left(\frac{\pi}{1.2\omega_n}\right)^2}
\end{aligned}$$

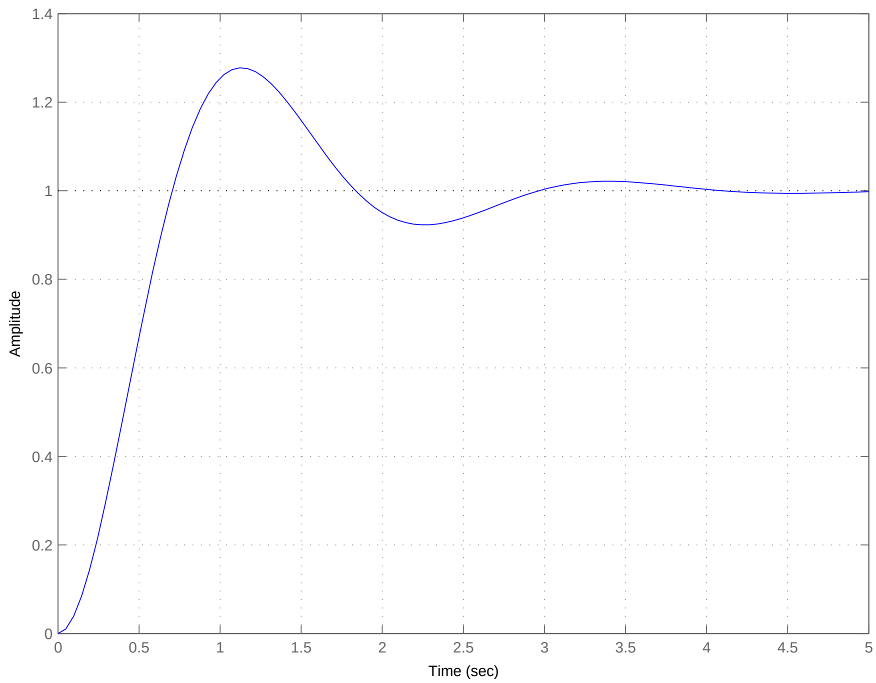


Figure 3: Step response for the system $H(s) = \frac{9}{s^2+2.265s+9}$. The settling time should be less than 5 seconds, the time to peak should be less than 1.2 seconds, and the percent overshoot should be 27.8%.

or $\beta \leq 2.93$. To meet both constraints we need $1.60 \leq \beta \leq 2.93$. Let's choose the average, so $\beta = 2.265$. Then $\zeta = 0.3775$ and the percent overshoot is 27.8%. The step response of this system is shown in Figure 3.

Example 2. Consider the system with transfer function given by

$$H(s) \quad = \quad \frac{K}{s^2 + 2s + K}$$

determine the range of K so that $PO \leq 20^o$. Is there any value of K so that $T_s \leq 2$ seconds?

For the transfer function, we see that $\omega_n = \sqrt{K}$ and $2\zeta\omega_n = 2$, so $\zeta\omega_n = 1$ and $\zeta = \frac{1}{\sqrt{K}}$. For the percent overshoot we have $b = 20/100 = 0.2$ and

$$\begin{aligned} e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} &\leq b \\ -\frac{\zeta\pi}{\sqrt{1-\zeta^2}} &\leq \ln(b) \\ -\frac{\pi}{\sqrt{K}}\frac{1}{\sqrt{1-\frac{1}{K}}} &\leq \ln(b) \\ -\frac{\pi}{\sqrt{K-1}} &\leq \ln(b) \end{aligned}$$

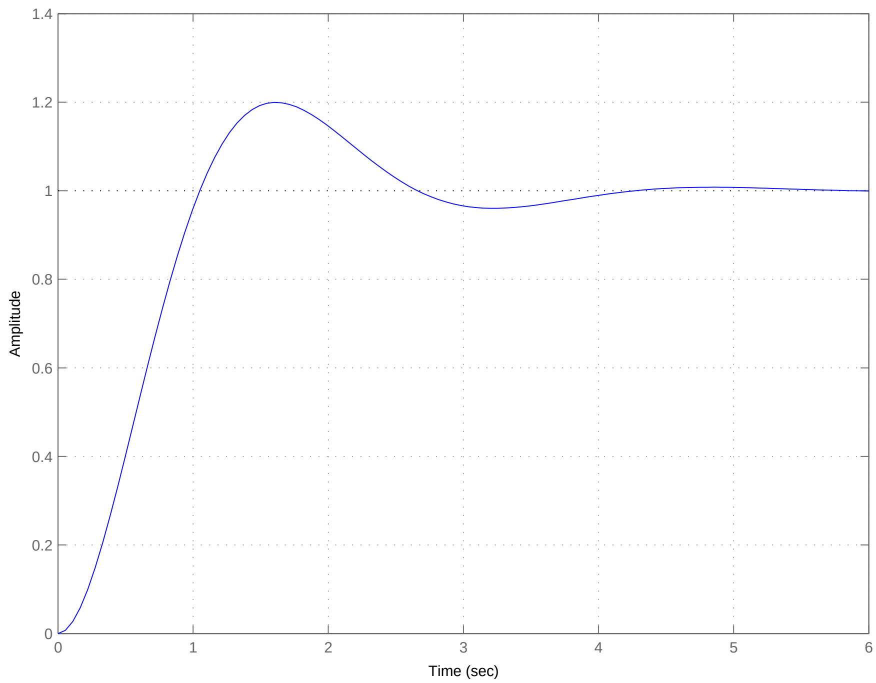


Figure 4: Step response for the system $H(s) = \frac{K}{s^2+2s+K}$. The percent overshoot should be less than or equal to 20% and the settling time should be 4 seconds.

$$\begin{aligned} -\frac{\pi}{\ln(b)} &\leq \sqrt{K-1} \\ \left(\frac{\pi}{\ln(b)}\right)^2 &\leq K-1 \\ 1+\left(\frac{\pi}{\ln(b)}\right)^2 &\leq K \end{aligned}$$

Hence we need $K \geq 4.8$ to meet the percent overshoot requirement. Now we try to meet the settling time requirment

$$T_s = \frac{4}{\zeta\omega_n} \leq 2$$

but $\frac{4}{\zeta\omega_n} = \frac{4}{1} = 4$. Thus we cannot meet the settling time constraint for any value of K. The step response of this system for $K = 2.8$ is shown in Figure 4.

5.5 Constraint Regions in the s-Plane

Sometimes, instead of looking at a transfer function and trying to determine the percent overshoot, settling time, or time to peak, we can take the opposite approach and try to determine the region in the s-plane the poles of the system should be located in to achieve a given criteria. Each one of the three criteria will determine a region of space in the s-plane.

Time to Peak (T_p) Let's assume we have a minimum time to peak given, T_p^{max}, and we want to know where to find all of the poles that will meet this constraint. We have

$$T_p \;=\; \frac{\pi}{\omega_d} \leq T_p^{max}$$

we can rearrange this as

$$\boxed{\frac{\pi}{T_p^{max}} \leq \omega_d}$$

Since we can write the complex poles as $-\sigma \pm j\omega_d$ this means that the imaginary part of the poles must be greater than $\frac{\pi}{T_p^{max}}$.

Example 3. Determine all acceptable pole location so the time to peak will be less than 2 seconds. We have $T_p^{max} = 2$, so $\omega_d \geq \frac{\pi}{2} = 1.57$. The acceptable pole locations are shown in the shaded region of Figure 5.

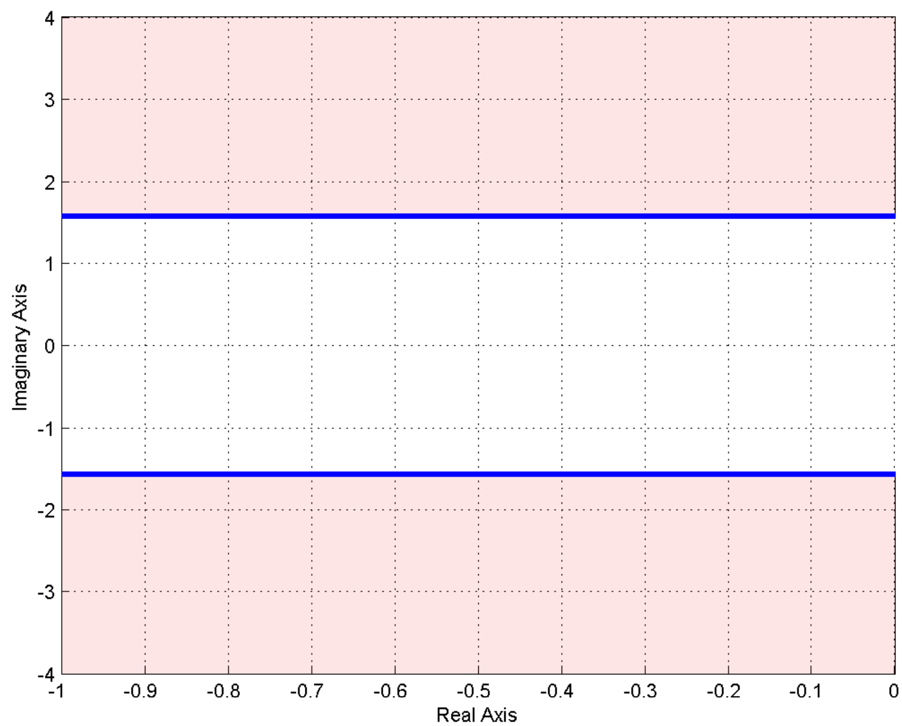


Figure 5: Acceptable pole locations for $T_p \leq 2$ seconds are shown in the shaded region.

Percent Overshoot (P.O.) Let's assume we have a maximum percent overshoot given, PO^{max}, and we want to know where to find all of the poles that will meet this constraint. We have

$$P.O. \;\; = \;\; e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100\% \;\; \leq PO^{max}$$

or

$$e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \;\; \leq \;\; \frac{PO^{max}}{100} \;\; = b$$

where we have defined the parameter $b = PO^{max}/100$ for notational convenience. We need to first solve the above expression for ζ.

$$\begin{aligned}
-\frac{\zeta\pi}{\sqrt{1-\zeta^2}} \;\; &\leq \;\; \ln(b) \\
\frac{\zeta}{\sqrt{1-\zeta^2}} \;\; &\geq \;\; \frac{-\ln(b)}{\pi} \\
\frac{\zeta^2}{1-\zeta^2} \;\; &\geq \;\; \left(\frac{-\ln(b)}{\pi}\right)^2 \\
\zeta^2 \;\; &\geq \;\; \left(\frac{-\ln(b)}{\pi}\right)^2 - \zeta^2\left(\frac{-\ln(b)}{\pi}\right)^2 \\
\zeta^2\left[1+\left(\frac{-\ln(b)}{\pi}\right)^2\right] \;\; &\geq \;\; \left(\frac{-\ln(b)}{\pi}\right)^2 \\
\zeta \;\; &\geq \;\; \frac{\frac{-\ln(b)}{\pi}}{\sqrt{1+\left(\frac{-\ln(b)}{\pi}\right)^2}}
\end{aligned}$$

Now we use the relationship

$$\theta \;\; = \;\; \cos^{-1}(\zeta)$$

In summary, we have

$$\boxed{\theta \leq \cos^{-1}(\zeta)\,, \;\; \zeta \geq \frac{\frac{-\ln(b)}{\pi}}{\sqrt{1+\left(\frac{-\ln(b)}{\pi}\right)^2}}, \;\; b = \frac{PO^{max}}{100}}$$

This angle θ is measured from the negative real axis. Hence an angle of 90 degrees indicates $\zeta = 0$ and there is no damping (the poles are on the $j\omega$ axis), while an angle of 0 degrees means the system has a damping ratio of 1, and the poles are purely real.

Example 4. Determine all acceptable pole location so the percent overshoot will be less than 10%. We have $b = 0.1$, so $\zeta \geq 0.59$ and $\theta \leq 53.8^o$ The acceptable pole locations are shown in the shaded region of Figure 6.

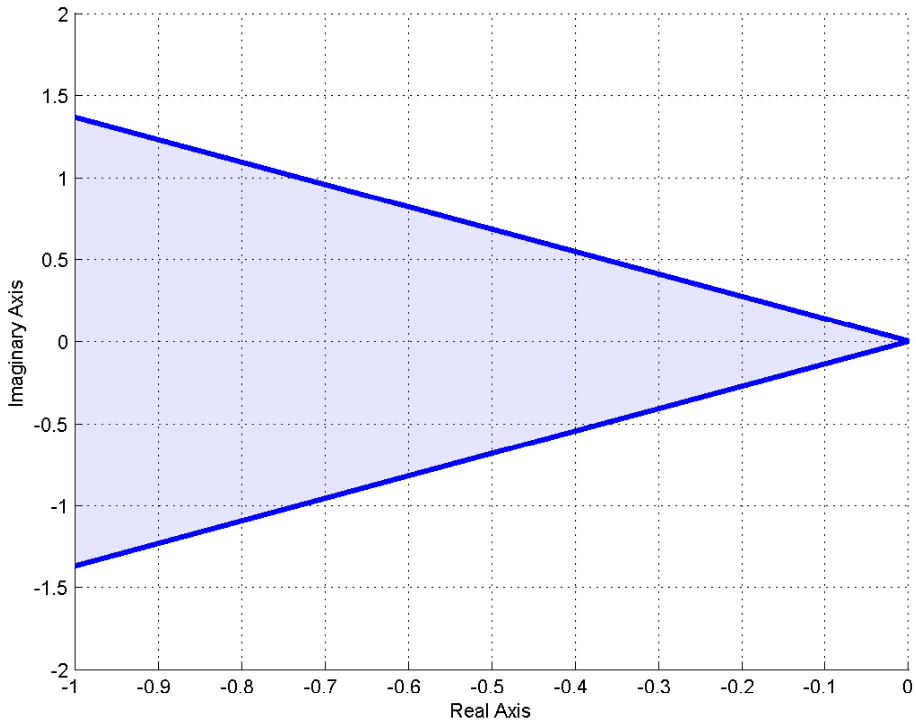


Figure 6: Acceptable pole locations for Percent Overshoot less than or equal to 10%. The acceptable pole locations are shown in the shaded region.

Example 5. Determine all acceptable pole location so the percent overshoot will be less than 20% and the time to peak will be less than 3 seconds. We have $b = 0.2$, so $\zeta \geq 0.46$ and $\theta \leq 62.9^o$. We also have $T_p^{max} = 3$, so $\omega_d \geq \frac{\pi}{3} = 1.04$ The acceptable pole locations for each constraint are shown in Figure 7. The overlapping regions are the acceptable pole locations to meet both the percent overshoot and time to peak constraints.

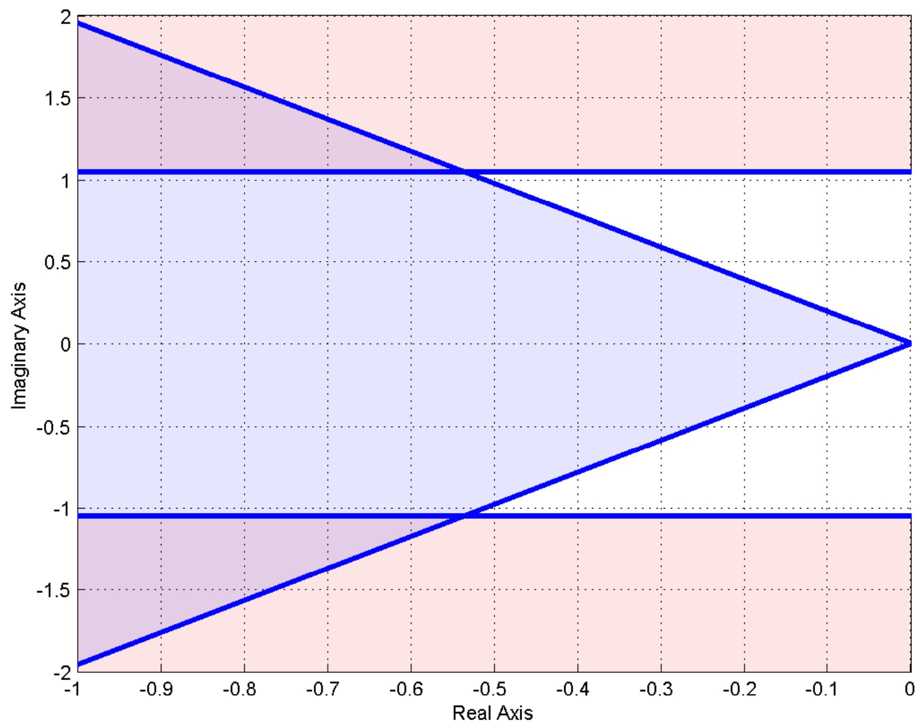


Figure 7: Acceptable pole locations for Percent Overshoot less than or equal to 20% and time to peak less than or equal to 3 seconds. The acceptable pole locations for each constraint are shown in the shaded regions. The overlapping regions are those pole locations that will meet both constraints.

Settling Time (T_s) Let's assume we have a maximum settling time T_s^{max}, and we want to know where to find all of the poles that will meet this constraint. We have

$$T_s = \frac{4}{\sigma} \leq T_s^{max}$$

or

$$\boxed{\frac{4}{T_s^{max}} \leq \sigma}$$

Since we can write the complex poles as $-\sigma \pm j\omega_d$ this means that the real part of the poles must be greater (in magnitude) than $\frac{4}{T_s^{max}}$. In other words, the poles must have real parts less than $-\frac{4}{T_s^{max}}$

Example 6. Determine all acceptable pole location so the settling time will be less than 3 seconds. We have $T_s^{max} = 3$, so $\sigma \geq \frac{4}{T_s^{max}} = \frac{4}{3} = 1.333$. The acceptable pole locations for each constraint are shown in Figure 8. The overlapping regions are the acceptable pole locations to meet both the percent overshoot and time to peak constraints.

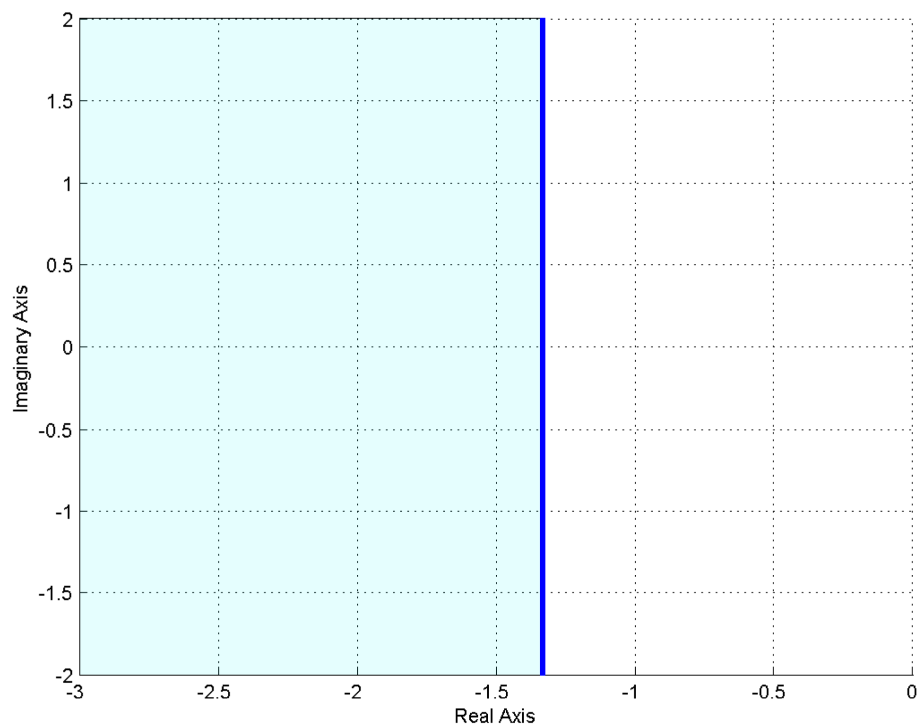


Figure 8: Acceptable pole locations for settling time less than or equal to 3 seconds. The acceptable pole locations are shown in the shaded region.

Example 7. Determine all acceptable pole location so the settling time will be less than 1 second and the time to peak will be less than or equal to 0.5 seconds. We have $T_s^{max} = 1$, so $\sigma \geq \frac{4}{T_s^{max}} = \frac{4}{1} = 4$. We also have $T_p^{max} = 0.5$ so $\omega_d \geq \frac{\pi}{T_p^{max}} = \frac{\pi}{0.5} = 6.28$. The acceptable pole locations for each constraint are shown in Figure 9. The overlapping regions (upper left corner, lower left corner) are the acceptable pole locations to meet both the settling time and time to peak constraints.

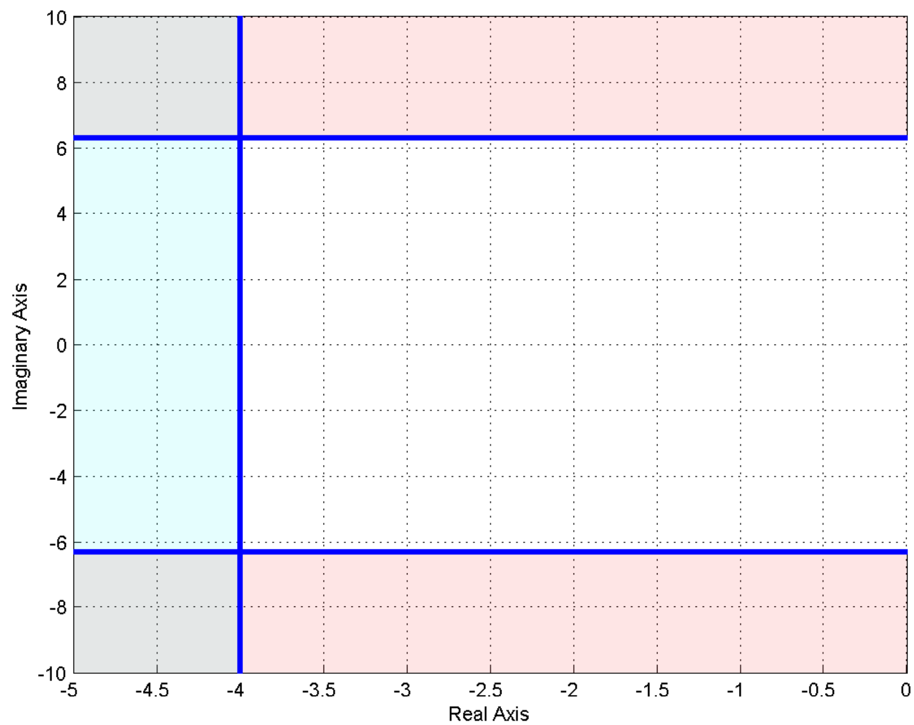


Figure 9: Acceptable pole locations for settling time less than or equal to 1 second and time to peak less than 0.5 seconds. The acceptable pole locations for each constraint are shown in the shaded regions. The overlapping regions are those pole locations that will meet both constraints.

Example 8. Determine all acceptable pole location so the settling time will be less than 5 seconds, the time to peak will be less than or equal to 2 seconds, and the percent overshoot will be less than 5%. We have $T_s^{max} = 5$, so $\sigma \geq \frac{4}{T_s^{max}} = \frac{4}{5} = 0.8$. We also have $T_p^{max} = 2$ so $\omega_d \geq \frac{\pi}{T_p^{max}} = \frac{\pi}{2} = 1.57$. Finally, $b = 0.05$, $\zeta \geq 0.69$ or $\theta < 46.4^o$. The acceptable pole locations for each constraint are shown in Figure 10. The overlapping regions (two triangular wedges) are the acceptable pole locations to meet all three constraints.

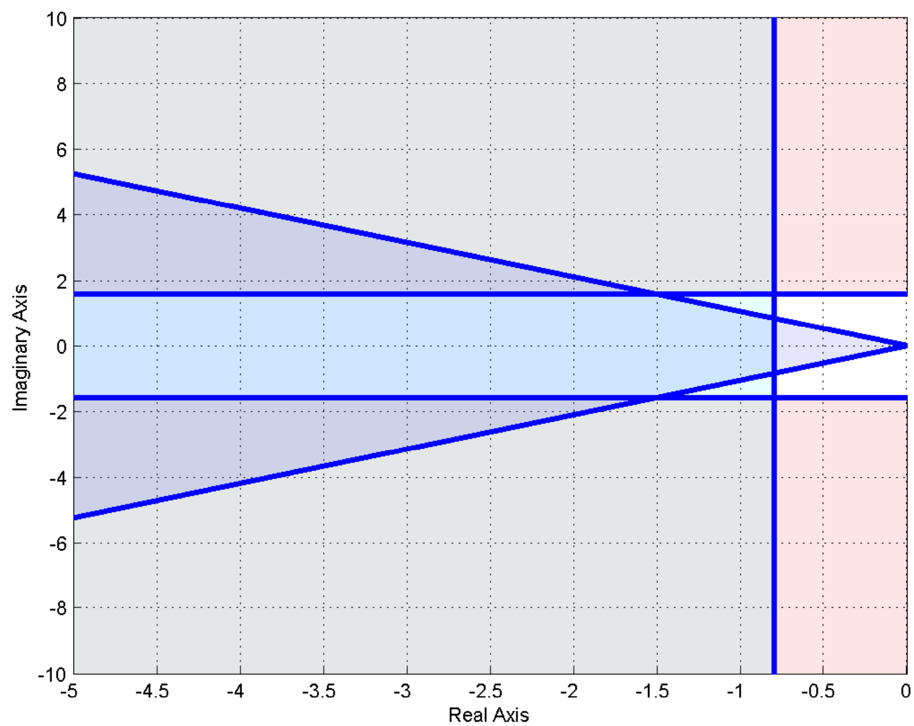


Figure 10: Acceptable pole locations for settling time less than 5 seconds, time to peak less than or equal to 2 seconds, and the percent overshoot less than 5%. The acceptable pole locations for each constraint are shown in the shaded regions. The overlapping regions (two triangular wedges) are those pole locations that will meet all three constraints.

5.6 Summary

For an *ideal* second order system with transfer function

$$H(s) = \frac{K_{static}}{\frac{1}{\omega_n}^2 s^2 + \frac{2\zeta}{\omega_n} s + 1} = \frac{K_{static}\,\omega_n{}^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

the poles are located at $-\zeta\omega_n \pm j\omega_d$, which is commonly written as either $-\sigma \pm j\omega_d$ or $-\frac{1}{\tau} \pm j\omega_d$. We can compute the percent overshoot (PO), the settling time (T_s), and the time to peak (T_p)

$$PO = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100\%$$

$$
\begin{aligned}
T_s &= \frac{4}{\zeta\omega_n} = 4\tau = \frac{4}{\sigma} \\
T_p &= \frac{\pi}{\omega_d}
\end{aligned}
$$

It is important to remember that *these relationships are only valid for ideal second order systems!*

What is generally more useful to us is to use these relationships to determine acceptable pole locations to meet the various design criteria. If the maximum desired settling time is T_s^{max}, then all poles must have real parts less than $-4/T_s^{max}$. If the maximum desired time to peak is T_p^{max}, then the imaginary parts of the dominant poles must have imaginary parts larger than π/T_p^{max}, or less than $-\pi/T_p^{max}$ (since poles come in complex conjugate pairs). If the maximum percent overshoot is PO^{max}, then the poles must lie in a wedge determined by $\theta = \cos^{-1}(\zeta)$ where θ is measured from the negative real axis and

$$
\zeta \geq \frac{\frac{-\ln(b)}{\pi}}{\sqrt{1+\left(\frac{-\ln(b)}{\pi}\right)^2}}, \quad b = \frac{PO^{max}}{100}
$$

Each of these constraints can be used to define a region of acceptable pole locations for an ideal second order system. However, they are often used as a guide (or starting point) for higher order systems, and systems with zeros.

6 Characteristic Polynomial, Modes, and Stability

In this section we first introduce the concepts of characteristic polynomial, characteristic equation, and characteristic modes. You'll obviously note the word *characteristic* is used quite a lot here. Then we utilize these concepts to define stability of our systems.

6.1 Characteristic Polynomial, Equation, and Modes

Consider a transfer function

$$H(s) = \frac{N(s)}{D(s)}$$

where $N(s)$ and $D(s)$ are polynomials in s with no common factors. $D(s)$ is called the *characteristic polynomial* of the system, and the equation $D(s) = 0$ is called the *characteristic equation.* The time functions associated with the roots of the characteristic equation (the poles of the system) are called the *characteristic modes*. Some examples will probably help. To determine the characteristic modes, it is often easiest to think of doing partial fraction expansion and looking at the resulting time functions.

Example 1. Consider the transfer function

$$H(s) = \frac{s+2}{s^2(s+1)(s+3)} = a_1\frac{1}{s} + a_2\frac{1}{s^2} + a_3\frac{1}{s+1} + a_4\frac{1}{s+3}$$

Then we have:

Characteristic Polynomial:	$s^2(s+1)(s+3)$
Characteristic Equation:	$s^2(s+1)(s+3) = 0$
Characteristic Modes:	$u(t),\ tu(t),\ e^{-t}u(t),\ e^{-3t}u(t)$

The impulse response is a linear combination of characteristic modes:

$$h(t) = a_1u(t) + a_2tu(t) + a_3e^{-t}u(t) + a_4e^{-3t}u(t)$$

Example 2. Consider the transfer function

$$H(s) = \frac{s-3}{s(s+1)^2(s+3)} = a_1\frac{1}{s} + a_2\frac{1}{s+1} + a_3\frac{1}{(s+1)^2} + a_4\frac{1}{s+3}$$

Then we have:

Characteristic Polynomial:	$s(s+1)^2(s+3)$
Characteristic Equation:	$s(s+1)^2(s+3) = 0$
Characteristic Modes:	$u(t),\ e^{-t}u(t),\ te^{-t}u(t),\ e^{-3t}u(t)$

The impulse response is a linear combination of characteristic modes:

$$h(t) = a_1u(t) + a_2e^{-t}u(t) + a_3te^{-t}u(t) + a_4e^{-3t}u(t)$$

Example 3. Consider the transfer function

$$H(s) = \frac{1}{s^2+s+1} = \frac{1}{(s+\frac{1}{2})^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$
$$= a_1 \frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2 + \left(\frac{\sqrt{3}}{2}\right)^2} + a_2 \frac{\frac{\sqrt{3}}{2}}{(s+\frac{1}{2})^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

Then we have:

Characteristic Polynomial:	s^2+s+1
Characteristic Equation:	$s^2+s+1=0$
Characteristic Modes:	$e^{-t/2}\cos(\frac{\sqrt{3}}{2}t)u(t),\ e^{-t/2}\sin(\frac{\sqrt{3}}{2}t)u(t)$

The impulse response is going to be a linear combination of characteristic modes:

$$h(t) = a_1 e^{-t/2}\cos(\frac{\sqrt{3}}{2}t)u(t) + a_2 e^{-t/2}\sin(\frac{\sqrt{3}}{2}t)u(t)$$

6.2 Characteristic Mode Reminders

There are are few things to keep in mind when finding characteristic modes

- There are as many characteristic modes as there are poles of the transfer function. Each characteristic mode must be different from the others.
- For any complex poles $-\sigma \pm j\omega_d$, the characteristic mode will be of the form $e^{-\sigma t}\cos(\omega_d t)u(t)$, and $e^{-\sigma t}\sin(\omega_d t)u(t)$.
- Assume pole p_i corresponds to characteristic mode $\phi_i(t)$. If there are two poles at p_i, the characteristic modes associated with pole p_i will be $\phi_i(t)$ and $t\phi_i(t)$. If there are three poles at p_i, the characteristic modes associated with p_i will be $\phi_i(t)$, $t\phi_i(t)$, and $t^2\phi_i(t)$. If pole p_i is repeated n times, the characteristic modes associated with pole p_i will be $\phi_i(t),\ t\phi_i(t),\ t^2\phi_i(t),\ \ldots\ t^{n-1}\phi_i(t)$
- The impulse response is a linear combination of the characteristic modes of a system.

Example 4. If a transfer function has poles at $-1, -1, -2 \pm 3j, -5 \pm 2j$, the characteristic modes are given by $e^{-t}u(t)$, $te^{-t}u(t)$, $e^{-2t}\cos(3t)u(t)$, $e^{-2t}\sin(3t)u(t)$, $e^{-5t}\cos(2t)u(t)$, and $e^{-5t}\sin(2t)u(t)$.

Example 5. If a transfer function has poles at $-2, -2, -2, -3 \pm 2j$, $-3 \pm 2j$, the characteristic modes are at $e^{-2t}u(t)$, $te^{-2t}u(t)$, $t^2e^{-2t}u(t)$, $e^{-3t}\cos(2t)u(t)$, $e^{-3t}\sin(2t)u(t)$, $te^{-3t}\cos(2t)u(t)$, and $te^{-3t}\sin(2t)u(t)$.

6.3 Stability

A system is defined to be stable if **all** of its characteristic modes go to zero as $t \to \infty$. A system is defined to be marginally stable if **all** of it's characteristic modes are bounded as $t \to \infty$. A system is unstable if **any** of it's characteristic modes is unbounded as $t \to \infty$. There are other definitions of stability, each with their own purpose. For the systems we will be studying in this course (generally linear time invariant systems) these are the most appropriate. Note that the stability of a system is independent of the input.

In determining stability, the following mathematical truths should be remembered

$$\begin{aligned}
\lim_{t\to\infty} t^n e^{-at} &= 0 \ \text{ for all positive } a \text{ and } n \\
\lim_{t\to\infty} e^{-at}\cos(\omega_d t + \phi) &= 0 \text{ for all positive } a \\
\lim_{t\to\infty} e^{-at}\sin(\omega_d t + \phi) &= 0 \text{ for all positive } a \\
u(t) & \ \textit{is bounded} \\
\cos(\omega_d t + \phi) & \ \textit{is bounded} \\
\sin(\omega_d t + \phi) & \ \textit{is bounded}
\end{aligned}$$

Example 6. Assume a system has poles at $-1, 0, -2$. Is the system stable?

The characteristic modes of the system are $e^{-t}u(t)$, $u(t)$, and $e^{-2t}u(t)$. Both $e^{-t}u(t)$ and $e^{-2t}u(t)$ go to zero as $t \to \infty$. $u(t)$ does not go to zero, but it is bounded. Hence the system is *marginally stable*.

Example 7. Assume a system has poles at $-1, 1, -2 \pm 3j$. Is the system stable?

The characteristic modes of the system are $e^{-t}u(t)$, $e^{t}u(t)$, $e^{-2t}u(t)$, $e^{-2t}\cos(3t)u(t)$, and $e^{-2t}\sin(3t)$. All of these modes go to zero as t goes to infinity, except the mode $e^{t}u(t)$. This mode is unbounded as $t \to \infty$. Hence the system is *unstable*.

Example 8. Assume a system has poles at $-1, -1, -2 \pm j, -2 \pm j$. Is the system stable?

The characteristic modes of the system are $e^{-t}u(t)$, $te^{-t}u(t)$, $e^{-2t}\cos(t)u(t)$, $e^{-2t}\sin(t)u(t)$, $te^{-2t}\cos(t)u(t)$, and $te^{-2t}\sin(t)u(t)$. All of the characteristic modes go to zero as t goes to infinity, so the system is *stable*.

6.4 Settling Time and Dominant Poles

For an ideal second order system, we have already shown that the (2%) settling time is given by

$$T_s = \frac{1}{\zeta\omega_n}$$

We need to be able to deal with systems with more than two poles. To do this, we first make the following observations:

- We normally write decaying exponentials in the form $e^{-t/\tau}$, where τ is the *time constant*. Using the 2 % settling time, we set the settling time equal to four time constants, $T_s = 4\tau$.
- If a system has a real pole at $-\sigma$, the corresponding mode is $e^{-\sigma t}u(t)$. Hence the time constant τ is equal to $\frac{1}{\sigma}$. The settling time for this pole is then $T_s = 4\tau = 4\frac{1}{\sigma}$.
- If a system has complex conjugate poles at $-\sigma \pm j\omega_d$, the corresponding modes are $e^{-\sigma t}\cos(\omega_d t)u(t)$ and $e^{-\sigma t}\sin(\omega_d t)u(t)$. Although these modes oscillate, the settling time depends on the time constants, which again leads to $\tau = \frac{1}{\sigma}$, and the settling time for this type of mode is given by $T_s = 4\frac{1}{\sigma}$

Hence, to determine the settling time associated with the i^{th} pole of the system, p_i, we compute

$$T_s^i \;=\; 4\frac{1}{Re\{-p_i\}} = \frac{4}{\sigma}$$

where we have written the real part of the pole, $Re\{-p_i\}$, is equal to σ.

To determine the settling time of a system with multiple poles, determine the characteristic mode associated with each pole, and the compute the settling time corresponding to that mode. The largest such settling time is the setting time of the system. The poles associated with the largest settling time are the **dominant** *poles of the system.*

Example 9. Assume we have a system with poles at $-5, -4, -3 \pm 2j$. Determine the settling time and the dominant poles of the system.

We have the settling times $T_s^1 = \frac{4}{5}$, $T_s^2 = \frac{4}{4}$, and $T_s^3 = \frac{4}{3}$. The largest of these is $T_s = \frac{4}{3}$, so this is the estimated settling time of the system. This settling time is associated with the poles at $-3 \pm 2j$, so these are the dominant poles.

Example 10. Assume we have a system with poles at $-2 \pm 3j, -1, -5 \pm 2j$. Determine the settling time and the dominant poles of the system.

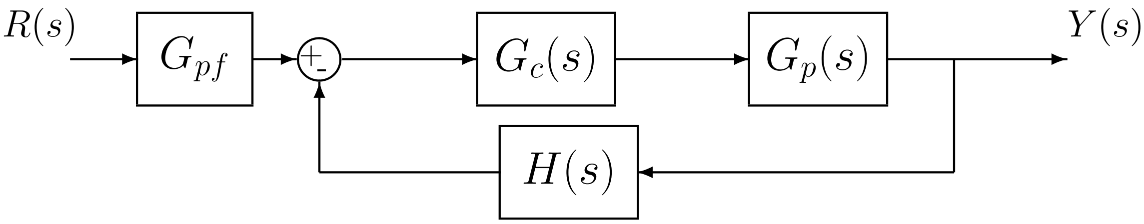
We have the settling times $T_s^1 = \frac{4}{2}$, $T_s^2 = \frac{4}{1}$, and $T_s^3 = \frac{4}{5}$. The largest of these is $T_s = \frac{4}{1}$, so this is the estimated settling time of the system. This settling time is associated with the pole at -1, so this is the dominant pole.

While the poles of the system determine the characteristic modes of the system, the amplitudes that multiply these modes (the a_i in the partial fraction expansion) are determined by both the poles and zeros of the system. In addition, when a pole is repeated, the form of the characteristic mode is $t^n e^{-\sigma t}$ (multiplied by sine or cosine for complex poles). Neither of these affects, the zeros of a system and the effects of repeated poles, was considered in estimating the settling time for a system. However, the approximation we have made is usually fairly reasonable.

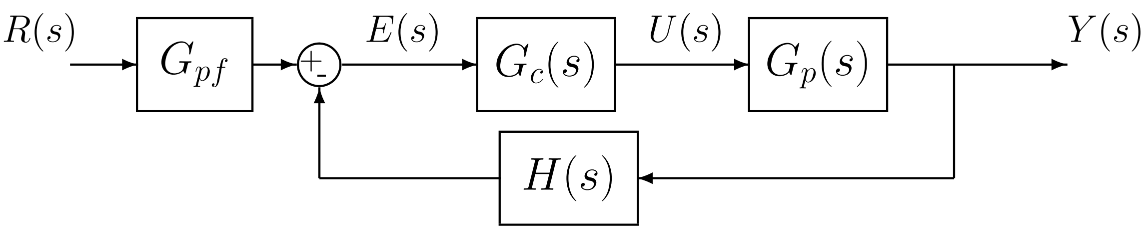
Dominant poles are the slowest responding poles in a system. If we want faster response, these are the poles we must move away from the $\jmath\omega$ axis.

7 Basic Feedback Configuration

The most basic feedback configuration we will deal with is shown below



Here $R(s)$ is the *reference input*. This is usually the signal we are trying to follow. $G_{pf}(s)$ is a *prefilter* which is usually used to condition the signal (change units) or to scale the input to fix the final value of the output. $G_p(s)$ is a model of the *plant* we are trying to control. $G_c(s)$ is a *controller* (or product of controllers) we have designed to improve performance. $Y(s)$ is the system output, and $H(s)$ is a signal conditioner often used to change the units of the output into more convenient units. There are usually two other variables identifies in the block diagram, which are shown below:



Here $U(s)$ is the input the the plant, so $Y(s) = G_p(s)U(s)$. Finally, $E(s)$ is the error signal, or actuating error.

To determine the overall transfer function, we find

$$\begin{aligned} Y(s) &= G_p(s)U(s) \\ &= G_p(s)G_c(s)E(s) \end{aligned}$$

and

$$E(s) = G_{pf}(s)R(s) - H(s)Y(s)$$

Combining these we get

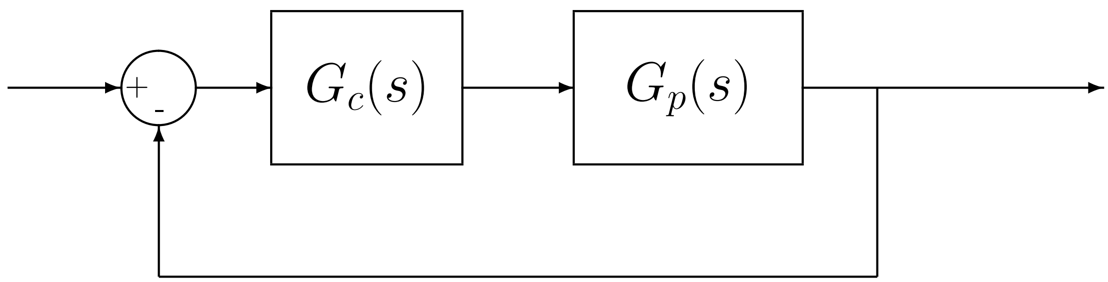
$$\begin{aligned} Y(s) &= G_p(s)G_c(s)\left[G_{pf}(s)R(s) - H(s)Y(s)\right] \\ &= G_{pf}(s)G_c(s)G_p(s)R(s) - G_c(s)G_p(s)H(s)Y(s) \\ Y(s) + G_c(s)G_p(s)H(s)Y(s) &= G_{pf}(s)G_c(s)G_p(s)R(s) \\ Y(s)\left[1 + G_c(s)G_p(s)H(s)\right] &= G_{pf}(s)G_c(s)G_p(s) \end{aligned}$$

or the closed loop transfer function is

$$G_0(s) = \frac{Y(s)}{R(s)} = \frac{G_{pf}(s)G_c(s)G_p(s)}{1 + G_c(s)G_p(s)H(s)}$$

8 Model Matching

The first type of control scheme we will discuss is that of model matching. Here we assume we have a plant $G_p(s)$ with a controller $G_c(s)$ in a untiy feedback scheme, as shown below.



For this closed loop feedback system, the close loop transfer function $G_0(s)$ is given by

$$G_0(s) = \frac{G_c(s)G_p(s)}{1 + G_c(s)G_p(s)}$$

The object of this course it to determine how to choose the controller $G_c(s)$ so the overall system meets some design criteria. The idea behind model matching is to assume we know what we want the closed loop transfer function $G_0(s)$ to be. Then, since $G_0(s)$ and $G_p(s)$ are known, we can determine the controller $G_c(s)$ as

$$\begin{aligned}
[1 + G_c(s)G_p(s)]\, G_0(s) &= G_c(s)G_p(s) \\
G_0(s) + G_c(s)G_p(s)G_0(s) &= G_c(s)G_p(s) \\
G_0(s) &= G_c(s)G_p(s) - G_c(s)G_p(s)G_0(s) \\
G_0(s) &= G_c(s)G_p(s)\, [1 - G_0(s)]
\end{aligned}$$

or

$$G_c(s) = \frac{G_0(s)}{G_p(s)\, [1 - G_0(s)]}$$

While this looks simple, there are certain restrictions on when this will work. The closed loop transfer function $G_0(s)$ is said to be implementable if[1]

1. The controller $G_c(s)$ is a proper rational transfer function
2. The controller $G_c(s)$ is stable

Consider a plant with proper transfer function $G_p(s) = \frac{N(s)}{D(s)}$ where we want the closed loop transfer function to be $G_0(s) = \frac{N_0(s)}{D_0(s)}$. We can find a $G_c(s)$ so that $G_0(s)$ is implementable only under the following conditions

[1]There are other restrictions, but they are not important in this course.

1. The degree of $D_0(s)$ - the degree of $N_0(s) \geq$ the degree $D(s)$ - the degree of $N(s)$
2. All right half plane zeros of $N(s)$ are retained in $N_0(s)$ (the RHP zeros of the plant must also be in the closed loop transfer function)
3. $G_0(s)$ is stable, i.e., all poles of $G_0(s)$ are in the left half plane (none on the axes)

Example 1. Consider the system with plant

$$G_p(s) = \frac{(s+2)(s-1)}{s(s^2-2s+2)}$$

Are the following closed loop transfer functions implementable?

1. $G_0(s) = 1$. (*No, it violates (1) and (2)*)
2. $G_0(s) = \frac{(s+2)}{(s+3)(s+1)}$. (*No, violates (2)*)
3. $G_0(s) = \frac{(s-1)}{(s+3)(s+1)}$. (*Yes*)
4. $G_0(s) = \frac{(s-1)}{s(s+2)}$. (*No, violates (3)*)
5. $G_0(s) = \frac{(s-1)}{(s+3)(s+1)^2}$. (*Yes*)
6. $G_0(s) = \frac{(s-1)(2s-3)}{(s+2)^3}$. (*Yes*)

Now that we know when we can use model matching, we need to find some good models. That is, how do we find a desirable $G_0(s)$? We will look at two possible choices, ITAE optimal systems, and quadratic optimal systems.

8.1 ITAE Optimal Systems

ITAE optimal systems minimize the **I**ntegral of **T**ime multiplied by the **A**bsolute **E**rror. These have been determined numerically. The second, third, and fourth order zero position error ITAE systems have the following closed loop transfer functions

$$G_0(s) = \frac{\omega_0^2}{s^2+1.4\omega_0 s+\omega_0^2}$$

$$G_0(s) = \frac{\omega_0^3}{s^3+1.75\omega_0 s^2+2.15\omega_0^2 s+\omega_0^3}$$

$$G_0(s) = \frac{\omega_0^4}{s^4+2.1\omega_0 s^3+3.4\omega_0^2 s^2+2.7\omega_0^3 s+\omega_0^4}$$

The second, third, and fourth order velocity position error ITAE systems have the following closed loop transfer functions

$$G_0(s) = \frac{3.2\omega_0 s+\omega_0^2}{s^2+3.2\omega_0 s+\omega_0^2}$$

$$G_0(s) = \frac{3.25\omega_0^2 s + \omega_0^3}{s^3 + 1.75\omega_0 s^2 + 3.25\omega_0^2 s + \omega_0^3}$$

$$G_0(s) = \frac{5.14\omega_0^3 s + \omega_0^4}{s^4 + 2.41\omega_0 s^3 + 4.93\omega_0^2 s^2 + 5.14\omega_0^3 s + \omega_0^4}$$

You, the designer, need to choose the value of ω_0. The larger the ω_0, the faster the system responds (good) and the larger the control effort (bad).

8.2 Quadratic Optimal Systems

For a quadratic optimal system, we want to find the closed loop transfer function $G_0(s)$ to minimize the quadratic performance index

$$J = \int_0^\infty \left\{ q\left[y(t) - r(t)\right]^2 + u^2(t) \right\} dt$$

where $y(t)$ is the output of the system, $r(t)$ is the input to the system, q is a positive constant that weighs the difference between the input to the system and the output to the system, and $u(t)$ is the actuating signal (the input to the plant we are trying to control.) In general, for this type of controller, we want the output of our system to match (or track) the input to the system. To determine $G_0(s)$ to solve this problem, we need to first discuss spectral factorization.

Consider first the plant with proper transfer function $G_p(s) = \frac{N(s)}{D(s)}$ where $D(s)$ and $N(s)$ have no common factors. Next, consider the polynomial

$$Q(s) = D(s)D(-s) + qN(s)N(-s)$$

Clearly $Q(-s) = Q(s)$, hence if s_1 is a root of $Q(s)$, then so is $-s_1$. Since all of the coefficients of $Q(s)$ are real by assumption (we assume $G_p(s)$ is real), if s_1 is a root of $Q(s)$ than so is its complex conjugate s_1^*. Hence all of the roots of $Q(s)$ are symmetric with respect to

- the real axis
- the imaginary axis
- the origin of the s-plane

Now consider

$$\begin{aligned} Q(j\omega) &= D(j\omega)D(-j\omega) + qN(j\omega)N(-j\omega) \\ &= |D(j\omega)|^2 + q|N(j\omega)|^2 \end{aligned}$$

Since $D(s)$ and $N(s)$ have no common factors, there exists no ω_0 so that both $D(j\omega_0) = 0$ and $N(j\omega_0) = 0$. Since $q \neq 0$, there exists no ω_0 so that $Q(j\omega_0) = 0$. Hence $Q(s)$ has no roots on the $j\omega$ axis.

Since the roots of $Q(s)$ are either in the left half plane or the right half plane (none on the $j\omega$ axis), and since by symmetry there will be an equal number in each plane, we will divide up the roots of $Q(s)$ into those in the open LHP and those in the open RHP. Let's denote the polynomial whose roots are those of the open LHP roots of $Q(s)$ as $D_0(s)$. Then, by symmetry, $D_0(-s)$ is a polynomial whose roots are the open RHP roots of $Q(s)$. Thus

$$Q(s) \quad = \quad D(s)D(-s) + qN(s)N(-s) = D_0(s)D_0(-s)$$

This is called the spectral factorization of $Q(s)$. Now we can give the result.

Consider a plant with proper transfer function $G_p(s) = N(s)/D(s)$ where

- $N(s)$ and $D(s)$ have no common factors
- The leading coefficient of $D(s)$ (the coefficient of the highest power of s in $D(s)$) is 1. That is, $D(s)$ is monic.

An implementable transfer function $G_0(s)$ that minimizes the performance index

$$J \quad = \quad \int_0^\infty \left\{ q\left[y(t) - r(t)\right]^2 + u^2(t) \right\} dt$$

where $r(t) = 1$ (a unit step) and $q > 0$ is given by

$$G_0(s) \quad = \quad \frac{qN(0)N(s)}{D_0(0)D_0(s)}$$

where

$$Q(s) \quad = \quad D(s)D(-s) + qN(s)N(-s) = D_0(s)D_0(-s)$$

Note that we are not guaranteed a zero position error with this method. For a zero position error we should have $G_0(0) = 1$. In addition, if $N(s) = s$ this will not work (since $N(0) = 0$.)

Example 2. Suppose we have the plant with transfer function

$$G_p(s) \quad = \quad \frac{1}{s^2 + 1}$$

and we want to find $G_0(s)$ to minimize

$$J \quad = \quad \int_0^\infty \left\{ 10\left[y(t) - r(t)\right]^2 + u^2(t) \right\} dt$$

Clearly $q = 10$, $G_p(s)$ is a proper transfer function, and $N(s)$ and $D(s)$ have no common factors and $D(s)$ is a monic polynomial. Now $N(s) = 1$ and $D(s) = s^2 + 1$, and $D(s)$ is monic. So we have

$$D(s) = s^2 + 1 \qquad N(s) = 1$$
$$D(-s) = s^2 + 1 \qquad N(-s) = 1$$

and

$$\begin{aligned} Q(s) &= D(s)D(-s) + qN(s)N(-s) \\ &= \left[s^2 + 1\right]\left[s^2 + 1\right] + 10\,[1]\,[1] \\ &= \left[s^4 + 2s^2 + 1\right] + 10\,[1] \\ &= s^4 + 2s^2 + 11 \end{aligned}$$

Note that $Q(s)$ is an even function of s. If it is not, you screwed up! Now we need to find the roots of $Q(s)$. These roots are $-1.0762 \pm 1.4691j$ and $1.0762 \pm 1.4969j$. To construct $D_0(s)$ we use only those roots in the LHP, i.e. the roots at -1.0762 ± 1.4691.

$$\begin{aligned} D_0(s) &= (s + 1.0762 - 1.4691j)(s + 1.0762 + 1.4691j) \\ D_0(s) &= s^2 + 2.1525s + 3.3166 \end{aligned}$$

Now we can compute the optimal $G_0(s)$ as

$$\begin{aligned} G_0(s) &= \frac{qN(0)N(s)}{D_0(0)D_0(s)} = \frac{(10)(1)(1))}{(3.3166)(s^2 + 2.1525s + 3.3166)} \\ &= \frac{3.01514}{s^2 + 2.1525s + 3.3166} \end{aligned}$$

Note that $G_0(0) = 0.909$, which yields a position error of $e_p = 0.091$. Finally, to determine the controller, we use the formula

$$G_c(s) = \frac{G_0(s)}{G_p(s)\left(1 - G_0(s)\right)}$$

which produces the controller

$$\begin{aligned} G_c(s) &= \frac{1.401s^2 + 1.401}{0.4646s^2 + s + 0.1401} \\ &= \frac{1.401[s^2 + 1]}{0.4646s^2 + s + 0.1401} \end{aligned}$$

Note that the controller has been scaled, and there will be a pole/zero cancellation between the plant and the controller. Since these are marginally stable poles this may not be a good idea.

Example 3. Suppose we have the plant with transfer function

$$G_p(s) = \frac{0.3(s+2)}{0.01s^2 + 0.2s + 1}$$

and we want to find $G_0(s)$ to minimize

$$J = \int_0^\infty \left\{15\left[y(t) - r(t)\right]^2 + u^2(t)\right\} dt$$

Clearly $q = 15$, $G_p(s)$ is a proper transfer function, and $N(s)$ and $D(s)$ have no common factors. However, before we use the algorithm we must be sure $D(s)$ is a monic polynomial. To do this, we multiply both the top and bottom by 100

$$G_p(s) = \frac{100}{100}\frac{0.3(s+2)}{0.01s^2 + 0.2s + 1} = \frac{30(s+2)}{s^2 + 20s + 100}$$

Now $N(s) = 30(s+2)$ and $D(s) = s^2 + 20s + 100$, and $D(s)$ is monic. So we have

$$\begin{aligned} D(s) &= s^2 + 20s + 100 \qquad N(s) = 30(s+2) \\ D(-s) &= s^2 - 20s + 100 \qquad N(-s) = 30(-s+2) \end{aligned}$$

and

$$\begin{aligned}
Q(s) &= D(s)D(-s) + qN(s)N(-s) \\
&= \left[s^2 + 20s + 100\right]\left[s^2 - 20s + 100\right] + 15\,[30(s+2)]\,[30(-s+2)] \\
&= \left[s^4 - 200s^2 + 10000\right] + 15\left[-900s^2 + 3600\right] \\
&= s^4 - 13,700s^2 + 64,000
\end{aligned}$$

Note that $Q(s)$ is an even function of s. If it is not, you screwed up! Now we need to find the roots of $Q(s)$. These roots are ± 117.027 and $\pm - 2.161$. To construct $D_0(s)$ we use only those roots in the LHP, i.e. the roots at -117.027 and -2.161

$$\begin{aligned}
D_0(s) &= (s + 117.027)(s + 2.161) \\
D_0(s) &= s^2 + 119.2s + 252.9
\end{aligned}$$

Now we can compute the optimal $G_0(s)$ as

$$\begin{aligned}
G_0(s) &= \frac{qN(0)N(s)}{D_0(0)D_0(s)} = \frac{(15)(60)30(s+2)}{(252.9)(s^2 + 119.2s + 252.9)} \\
&= \frac{106.7(s+2)}{s^2 + 119.2s + 252.9}
\end{aligned}$$

Note that $G_0(0) = 0.8438$, which yields a position error of $e_p = 0.156$. Finally, to determine the controller, we use the formula

$$G_c(s) = \frac{G_0(s)}{G_p(s)\left(1 - G_0(s)\right)}$$

which produces the controller

$$\begin{aligned}
G_c(s) &= \frac{0.09s^2 + 1.8s + 9}{0.0253s^2 + 0.3153s + 1} \\
&= \frac{0.09[s^2 + 20s + 100]}{0.0253s^2 + 0.3153s + 1}
\end{aligned}$$

Note that this controller has been scaled and there is a pole/zero cancellation between the controller and the plant. The poles being cancelled are stable poles, so this is probably acceptable.

8.3 Summary and Caveates

In the first part of this section, the conditions under which it is possible to to obtain an implementable closed loop transfer function $G_0(s)$ have been given, it may not be possible to find such a $G_0(s)$ using the ITAE and quadratic optimal criteria. Specifically, these two methods may not work with a unity feedback system if

- the plant has open RHP zeros
- the plant has two or more zeros at the origin

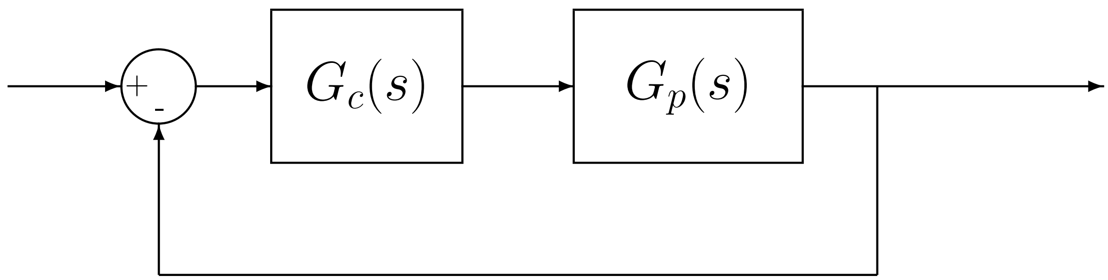
In these cases different approaches must often be taken to find a good closed loop transfer function, or a different approach must be taken to try and control the system.

The model matching methods we have discussed often utilize pole-zero cancellations between the controller $G_c(s)$ and the plant $G_p(s)$ to achieve the desired closed loop transfer function. As long as a stable pole is being cancelled, this is usually OK. However cancelling an unstable pole is not acceptable. However, the plant may change over time, and we are dealing with models of the plant in the first place. Hence the pole-zero cancellations may not be very effective for some systems.

9 System Type and Steady State Errors

9.1 Review

Let's assume we have a control system in a unity feedback configuration, as shown below:



where $G_p(s)$ is the plant transfer function and $G_c(s)$ is a controller. The closed-loop transfer function is given by

$$G_0(s) \quad = \quad \frac{G_c(s)G_p(s)}{1 + G_c(s)G_p(s)}$$

We already know that if we write

$$G_0(s) \quad = \quad \frac{n_m s^m + n_{m-1}s^{m-1} + ... + n_2 s^2 + b_1 s + b_0}{s^n + d_{n-1}s^{n-1} + ... + d_2 s^2 + d_1 s + d_0}$$

that the position error for an input step of amplitude A is given by

$$e_p \quad = \quad A\frac{d_0 - n_0}{d_0}$$

If $G_0(0) = 1$ (the constant terms in the numerator and denominator are the same) then $e_p = 0$. The velocity error for an input of tA is given by

$$e_v = A\frac{d_1 - n_1}{d_0}$$

If the coefficients of s^1 and s^0 are the same, then the velocity error is zero.

9.2 System Type For a Unity Feedback Configuration

Unity feedback configurations are very common, and we would like to be able to analyze this type of system very quickly without computing the closed loop transfer function.

Let's assume we group the all of the transfer functions together into one transfer function, which we will call $G(s)$, so $G(s) = G_c(s)G_p(s)$. Assume we write $G(s)$ as[2]

$$G(s) \quad = \quad \frac{K(T_a s + 1)(T_b s + 1)...(T_m s + 1)}{s^N(T_1 s + 1)(T_2 s + 1)...(T_n s + 1)}$$

[2]We do not actually want to rewrite $G(s)$, this is just used for illustrative purposes.

This is said to be a type N system, where N is the number of poles at the origin of $G(s)$. (These poles at the origin are also called "free integrators".) If the system output is $Y(s)$ and the system input is $R(s)$, then the system transfer function is

$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

Let's define the error $E(s)$ to be the difference between the input $R(s)$ and the output $Y(s)$,

$$\begin{aligned} E(s) &= R(s) - Y(s) \\ &= R(s) - \frac{G(s)}{1+G(s)} R(s) \\ &= \frac{R(s)\left\{[1+G(s)] - G(s)\right\}}{1+G(s)} \\ &= \frac{R(s)}{1+G(s)} \end{aligned}$$

The steady state error is then

$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{sR(s)}{1+G(s)}$$

We will use this expression to determine expressions for the position and velocity errors for unity feedback systems.

9.3 Position and Velocity Errors

As we have previously defined, the *position error* is the difference between a step input $r(t)$ and the corresponding output $y(t)$ as we let $t \to \infty$. Hence $e_p = \lim_{s\to 0} sE(s)$ for $R(s) = \frac{A}{s}$, or

$$\begin{aligned} e_p &= \lim_{s\to 0} sE(s) \\ &= \lim_{s\to 0} \frac{s\frac{A}{s}}{1+G(s)} \\ &= \lim_{s\to 0} \frac{A}{1+G(s)} \\ &= \frac{A}{1+G(0)} \\ &= \frac{A}{1+K_p} \end{aligned}$$

The <u>*position error constant*</u> K_p is defined to be $G(0)$. For a type 0 system $K_p = K$ and $e_p = \frac{A}{1+K}$, while for a type 1 or higher system, $K_p = \infty$ and $e_p = 0$.

The *velocity error* is the difference between a ramp input $r(t)$ and the corresponding output $y(t)$ as we let $t \to \infty$. Hence $e_v = lim_{s\to 0} sE(s)$ for $R(s) = \frac{A}{s^2}$, or

$$e_v = \lim_{s\to 0} sE(s)$$

$$
\begin{aligned}
&= \lim_{s\to 0} \frac{s\frac{A}{s^2}}{1+G(s)} \\
&= \lim_{s\to 0} \frac{A}{s+sG(s)} \\
&= \lim_{s\to 0} \frac{A}{sG(s)} \\
&= \frac{A}{K_v}
\end{aligned}
$$

The *velocity error constant* K_v is defined to be $\lim_{s\to 0} sG(s)$. For a type 0 system $K_v = 0$ and $e_v = \infty$. For a type 1 system $K_v = K$ and $e_v = \frac{A}{K}$. For a type 2 or higher system, $K_v = \infty$ and $e_v = 0$.

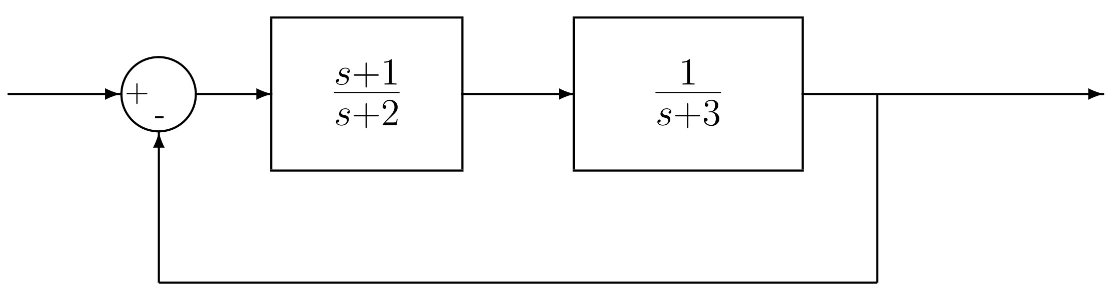
We can summarize these results in table 1 below.

System Type	e_p	e_v
0	$\frac{A}{1+K_p}$	∞
1	0	$\frac{A}{K_v}$
2	0	0
3	0	0

Table 1: Summary of system type (number of poles at the origin), position error e_p for an input $Au(t)$, and velocity error e_v for an input $Atu(t)$ for a unity feedback system.

9.4 Examples

Example 1. For the unity feedback system shown below, determine the system type, the position error e_p and the velocity error e_v.



Here

$$G(s) = \frac{(s+1)}{(s+2)(s+3)}$$

there are no poles at zero so this is a type 0 system. The position error constant is then

$$
\begin{aligned}
K_p &= \lim_{s\to 0} G(s) \\
&= \frac{1}{(2)(3)}
\end{aligned}
$$

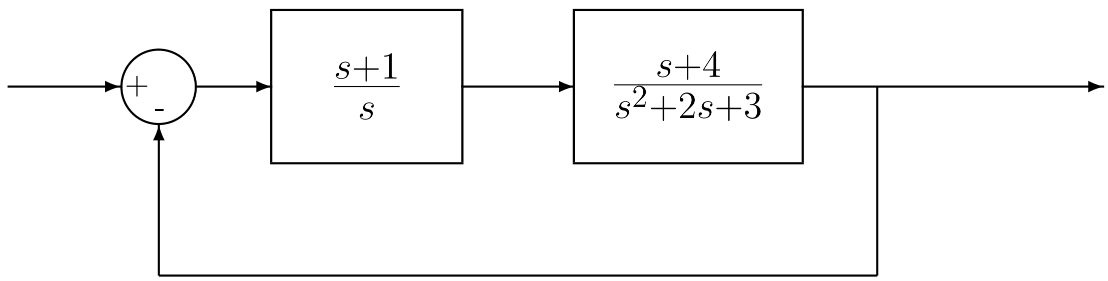
$$= \frac{1}{6}$$

so

$$\begin{aligned} e_p &= \frac{A}{1+K_p} \\ &= \frac{A}{1+0.1667} \\ &= 0.857A \end{aligned}$$

Since e_p is not zero, $e_v = \infty$. ($e_v = \infty$ since this is a type 0 system).

Example 2. For the unity feedback system shown below, determine the system type, the position error e_p and the velocity error e_v.



Here

$$G(s) = \frac{(s+1)(s+4)}{s(s^2+2s+3)}$$

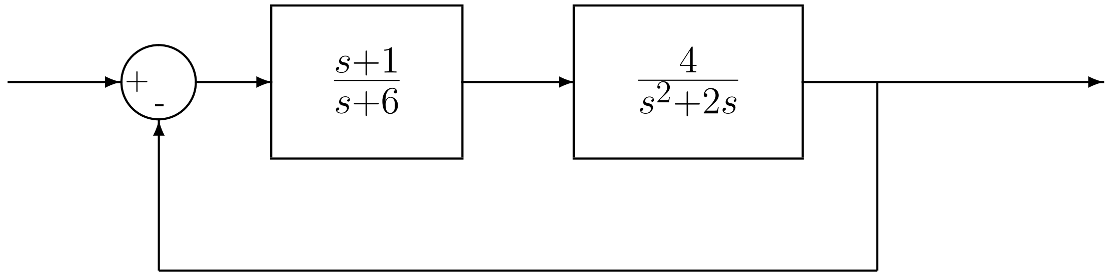
there is one pole at zero so this is a type 1 system. The position error is then $e_p = 0$. Note that we do note need to do any computation for this once we recognize this as a type 1 system!. The velocity error constant is then

$$\begin{aligned} K_v &= \lim_{s\to 0} sG(s) \\ &= \frac{(1)(4)}{3} \\ &= \frac{4}{3} \end{aligned}$$

so

$$\begin{aligned} e_v &= \frac{A}{K_v} \\ &= \frac{A}{\left(\frac{4}{3}\right)} \\ &= 0.75A \end{aligned}$$

Example 3. For the unity feedback system shown below, determine the system type, the position error e_p and the velocity error e_v.



Here

$$G(s) = \frac{(s+1)(4)}{(s+6)s(s+2)}$$

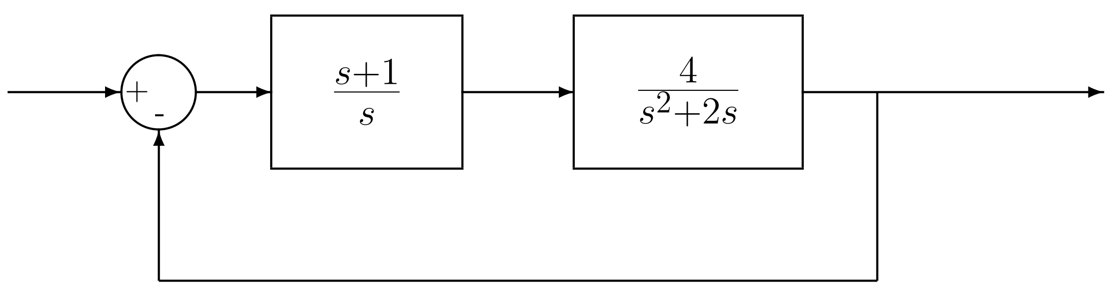
there is one pole at zero so this is a type 1 system. The position error is then $e_p = 0$. The velocity error constant is then

$$\begin{aligned} K_v &= \lim_{s\to 0} sG(s) \\ &= \frac{(1)(4)}{(6)(2)} \\ &= \frac{1}{3} \end{aligned}$$

so

$$\begin{aligned} e_v &= \frac{A}{K_v} \\ &= \frac{A}{\left(\frac{1}{3}\right)} \\ &= 3A \end{aligned}$$

Example 4. For the unity feedback system shown below, determine the system type, the position error e_p and the velocity error e_v.



Here

$$G(s) = \frac{(s+1)(4)}{s^2(s+2)}$$

there are two poles at zero so this is a type 2 system. Hence both e_p and e_v are zero.

10 Controller Design Using the Root Locus

This section has not been written yet.

1. Loci Branches

$$\textbf{poles}\ (k=0) \rightarrow \textbf{zeros}\ (k=\infty)$$

Continuous curves, which comprise the locus, start at each of the n poles of $G(s)H(s)$ for which $k = 0$. As k approaches ∞, the branches of the locus approach the m zeros of $G(s)H(s)$. Locus branches for excess poles extend to infinity.

The root locus is symmetric about the real axis.

2. Real Axis Segments

The root locus includes all points along the real axis to the left of an odd number of poles plus zeros of $G(s)H(s)$.

3. Asymptotic Angles

As $k \rightarrow \infty$, the branches of the locus become asymptotic to straight lines with angles

$$\theta = \frac{180^o + i360^o}{n-m}, \quad i = 0, \pm 1, \pm 2, ...$$

until all $(n-m)$ angles not differing by multiples of 360^o are obtained. n is the number of poles of $G(s)H(s)$ and m is the number of zeros of $G(s)H(s)$.

4. Centroid of the Asymptotes

The starting point on the real axis from the the asymptotic lines radiate is given by

$$\sigma = \frac{\sum_i p_i - \sum_j z_j}{n-m}$$

where p_i is the i^{th} pole of $G(s)H(s)$, z_j is the j^{th} zero of $G(s)H(s)$, n is the number of poles of $G(s)H(s)$ and m is the number of zeros of $G(s)H(s)$. This point is terms the centroid of the asymptotes.

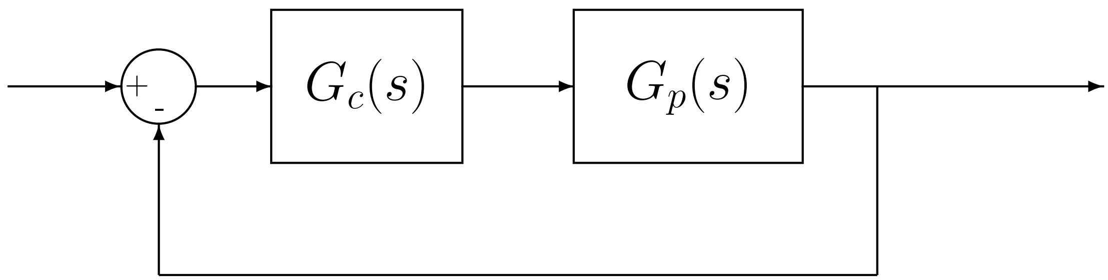
5. Leaving/Entering the Real Axis

When two branches of the root locus leave or enter the real axis, they usually do so at angles of ± 90 degrees.

11 Pole Placement By Matching Coefficients: Diophantine Equations

An alternative approach to controller design is to use a controller $G_c(s)$ to put the closed loop poles of a system in desired locations. We will start this section with an example, then explain the conditions under which this approach will work, and then do some more examples.

Consider the following unity feedback system



with plant

$$G_p(s) = \frac{s+1}{s^2+s+1}$$

Assume we want to place the closed loop poles at $-2 \pm j$ and -8, so we want the denominator of the closed loop system to be

$$\begin{aligned} D_0(s) &= (s+2+j)(s+2-j)(s+8) \\ &= s^3 + 12s^2 + 37s + 40 \end{aligned}$$

Let's assume the controller has the form

$$G_c(s) = \frac{B_0 + B_1 s}{A_0 + A_1 s}$$

where $A_1 \neq 0$ (so the controller is proper). Now the closed loop transfer function $G_0(s)$ is given by

$$\begin{aligned} G_0(s) &= \frac{G_c(s)G_p(s)}{1 + G_c(s)G_p(s)} \\ &= \frac{\left(\frac{B_0+B_1 s}{A_0+A_1 s}\right)\left(\frac{s+1}{s^2+s+1}\right)}{1 + \left(\frac{B_0+B_1 s}{A_0+A_1 s}\right)\left(\frac{s+1}{s^2+s+1}\right)} \\ &= \frac{(B_0+B_1 s)(s+1)}{(A_0+A_1 s)(s^2+s+1) + (B_0+B_1 s)(s+1)} \end{aligned}$$

Since we know where we want the closed loop poles, we equate denominators:

$$D_0(s) = s^3 + 12s^2 + 37s + 40 = (A_0 + A_1 s)(s^2+s+1) + (B_0 + B_1 s)(s+1)$$

and then equate powers of s:

$$\begin{array}{ll} s^3: & 1 = A_1 \\ s^2: & 12 = A_1 + A_0 + B_1 \\ s^1: & 37 = A_0 + A_1 + B_0 + B_1 \\ s^0: & 40 = A_0 + B_0 \end{array}$$

We then have the system of equations

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} A_0 \\ B_0 \\ A_1 \\ B_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 12 \\ 37 \\ 40 \end{bmatrix}$$

The solution to this system of equations is $A_0 = 15$, $B_0 = 25$, $A_1 = 1$, and $B_1 = -4$. The controller is then

$$G_c(s) = \frac{25 - 4s}{15 + s}$$

and the closed loop transfer function is

$$\begin{aligned} G_0(s) &= \frac{G_c(s)G_p(s)}{1 + G_c(s)G_p(s)} \\ &= \frac{\left(\frac{25-4s}{15+s}\right)\left(\frac{s+1}{s^2+s+1}\right)}{1 + \left(\frac{25-4s}{15+s}\right)\left(\frac{s+1}{s^2+s+1}\right)} \\ &= \frac{(25 - 4s)(s + 1)}{(15 + s)(s^2 + s + 1) + (25 - 4s)(s + 1)} \\ &= \frac{(25 - 4s)(s + 1)}{s^3 + 12s^2 + 39s + 40} \end{aligned}$$

We have achieved the desired closed loop poles. However, we have introduced a new zero into the system at $\frac{25}{4}$. As you will see, this is the major drawback to this kind of controller. While we can force the closed loop poles to be anything we want, we will be introducing zeros into the system. If these zeros are acceptable, then we are done. If they are not acceptable, then we need to try and do something (such as changing where we want the closed loop poles to be) or try a different type of controller. For this example, the position error is $e_p = 1 - \frac{5}{8} = \frac{3}{8}$. One method of obtaining zero position error is with a prefilter (with gain $\frac{8}{5}$). A better way is to design the controller so that the resulting system is a type 1 system. We will show how to do the latter in a subsequent section.

11.1 Theoretical Background

The results we need to know are stated in the following Theorem. There are two parts to the Theorem. The first part states the results for a strictly proper plant, while the second part states the results for a plant where the numerator and denominator polynomials have the same degree. The important information from the Theorem is knowing the minimum order of the required controller m and the order of the closed loop transfer function $n+m$.

<u>Theorem</u> *Strictly Proper Plant* Assume we have a strictly proper n^{th} order plant transfer function, $G_p(s) = N(s)/D(s)$. Since $G_p(s)$ is strictly proper we have the degree of $N(s) <$ the degree of $D(s)$. Since $G_p(s)$ is n^{th} order the degree of $D(s) = n$. Assume also that $N(s)$ and $D(s)$ have no common factors. Then for any polynomial $D_0(s)$ of degree $n+m$ a strictly proper controller $G_c(s) = B(s)/A(s)$ of degree m exists so that the characteristic equation of the resulting closed loop system is equal to $D_0(s)$. If $m = n-1$, the controller is unique. If $m \geq n$, the controller is not unique and some of the coefficients can be used to achieve other design objectives.

<u>Theorem</u> *Special case: degree $N(s)$ = degree $D(s)$.* Assume we have a proper n^{th} order plant transfer function, $G_p(s) = N(s)/D(s)$, where the degree of $D(s)$ = degree $N(s) = n$ Assume also that $N(s)$ and $D(s)$ have no common factors. Then for any polynomial $D_0(s)$ of degree $n+m$ a strictly proper controller $G_c(s) = B(s)/A(s)$ of degree m exists so that the characteristic equation of the resulting closed loop system is equal to $D_0(s)$. If $m = n$, the controller is unique. If $m \geq n+1$, the controller is not unique and some of the coefficients can be used to achieve other design objectives.

How do we do this? For plant $G_p(s) = N(s)/D(s)$, controller $G_c(s) = B(s)/A(s)$, and desired characteristic equation $D_0(s)$ we will have to solve the equation

$$A(s)D(s) + B(s)N(s) \;\; = \;\; D_0(s)$$

This is called the *Diophantine* equation. We solve this equation by equating powers of s, setting up a system of equations, and then solving. The closed loop transfer function will be

$$G_0(s) \;\; = \;\; \frac{B(s)N(s)}{D_0(s)}$$

where $B(s)$ contains the zeros we have added to the system.

<u>Example 1</u>. Assume we are trying to control the plant

$$G_p(s) \;\; = \;\; \frac{10}{s^2+1}$$

Since $n = 2$ we need the order of the controller $m \geq n-1$ or $m \geq 1$. We'll choose $m = 1$. Hence we will be looking at a controller of the form

$$G_c(s) \;\; = \;\; \frac{B_0 + B_1 s}{A_0 + A_1 s}$$

where $A_1 \neq 0$ (we need a proper controller transfer function). Next, we need to know the desired characteristic equation, $D_0(s)$. We need to have $n + m = 3$ poles. Let's assume we want the closed loop poles to be at $-10 \pm 5j$ and -20. Then

$$\begin{aligned} D_0(s) &= (s+10+5j)(s+10-5j)(s+20) \\ &= s^3 + 40s^2 + 525s + 2500 \end{aligned}$$

Now we need to solve the Diophantine equations

$$\begin{aligned} A(s)D(s) + B(s)N(s) &= D_0(s) \\ (A_0 + A_1 s)\left(s^2+1\right) + (B_0 + B_1 s)(10) &= s^3 + 40s^2 + 525s + 2500 \end{aligned}$$

Now we equate powers of s

$$\begin{aligned} s^3 &: \quad 1 = A_1 \\ s^2 &: \quad 40 = A_0 \\ s^1 &: \quad 525 = A_1 + 10B_1 \\ s^0 &: \quad 2500 = A_0 + 10B_0 \end{aligned}$$

In this case we can solve directly to get $A_0 = 40$, $B_0 = 246$, $A_1 = 1$ ad $B_1 = 52.4$. Hence our controller is

$$G_c(s) = \frac{246 + 52.4s}{40 + s}$$

and the closed loop transfer function is

$$G_0(s) = \frac{(246 + 52.4s)10}{s^3 + 40s^2 + 525s + 2500}$$

We have introduced a zero at -4.69. The position error is $e_p = 1 - G_0(0) = 0.016$.

Example 2. Assume we are trying to control the plant

$$G_p(s) = \frac{(s+1)}{s^3 + 4s^2 + 3s + 6}$$

Since $n = 3$ we need the order of the controller $m \geq n-1$ or $m \geq 2$. We'll choose $m = 2$. Hence we will be looking at a controller of the form

$$G_c(s) = \frac{B_0 + B_1 s + B_2 s^2}{A_0 + A_1 s + A_2 s^2}$$

where $A_2 \neq 0$ (we need a proper controller transfer function). Next, we need to know the desired characteristic equation, $D_0(s)$. We need to have $n + m = 5$ poles. Let's assume we want all the closed loop poles to be at -5. Then

$$\begin{aligned} D_0(s) &= (s+5)^5 \\ &= s^5 + 25s^4 + 250s^3 + 1250s^2 + 3125s + 3125 \end{aligned}$$

Now we need to solve the Diophantine equations

$$\begin{aligned} A(s)D(s) + B(s)N(s) &= D_0(s) \\ \left(A_0 + A_1 s + A_2 s^2\right)\left(s^3 + 4s^2 + 3s + 6\right) + \left(B_0 + B_1 s + B_2 s^2\right)(s+1) &= D_0(s) \end{aligned}$$

Now we equate powers of s

$$
\begin{aligned}
s^5: &\quad 1 = A_2 \\
s^4: &\quad 25 = A_1 + 4A_2 \\
s^3: &\quad 250 = A_0 + 4A_1 + 3A_2 + B_2 \\
s^2: &\quad 1250 = 4A_0 + 3A_1 + 6A_2 + B_1 + B_2 \\
s^1: &\quad 3125 = 3A_0 + 6A_1 + B_0 + B_1 \\
s^0: &\quad 3125 = 6A_0 + B_0
\end{aligned}
$$

Hence we have to solve the system of equations

$$
\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 0 & 0 & 0 \\ 1 & 4 & 3 & 0 & 0 & 1 \\ 4 & 3 & 6 & 0 & 1 & 1 \\ 3 & 6 & 0 & 1 & 1 & 0 \\ 6 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}
\begin{bmatrix} A_0 \\ A_1 \\ A_2 \\ B_0 \\ B_1 \\ B_2 \end{bmatrix}
=
\begin{bmatrix} 1 \\ 25 \\ 250 \\ 1250 \\ 3125 \\ 3125 \end{bmatrix}
$$

Solving this system we get $A_0 = 190.6$, $B_0 = 1981.0$, $A_1 = 21.0$, $B_1 = 446.0$, $A_2 = 1$, and $B_2 = -27.7$. Hence our controller is

$$
G_c(s) \quad = \quad \frac{1981.0 + 446.0s - 27.7s^2}{190.6 + 21.0s + s^2}
$$

and the closed loop transfer function is

$$
G_0(s) \quad = \quad \frac{(1981.0 + 446.0s - 27.7s^2)(s+1)}{s^5 + 25s^4 + 250s^3 + 1250s^2 + 3125s + 3125}
$$

We have introduced two zeros at 19.7 and -3.62. The position error is $e_p = 1 - G_0(0) = 0.366$, which is quite poor. In this case we would probably use a prefilter with amplitude $\frac{1}{G_0(0)} = 1.581$.

11.2 Pole Placement with Robust Tracking

We would like to avoid the prefilter approach to achieving zero position error, since the system may change over time. If we can make the plant-controller combination a type 1 system, then the closed loop system will have zero position error even if the plant changes over time (or our model is not exact). To do this, we will insert an integrator in the controller. Do do this, we increase the degree of the controller we need by 1, and use the extra parameter to create a type 1 system. To create the type one system, we will increase the order of the controller by one and set $A_0 = 0$.

Example 3. Assume we are trying to control the plant

$$
G_p(s) \quad = \quad \frac{3}{s^2 + 3s + 2}
$$

Since $n = 2$ we need the order of the controller $m \geq n-1$ or $m \geq 1$. We'll choose $m = 1$. Hence we will be looking at a controller of the form

$$G_c(s) = \frac{B_0 + B_1 s}{A_0 + A_1 s}$$

where $A_1 \neq 0$ (we need a proper controller transfer function). Next, we need to know the desired characteristic equation, $D_0(s)$. We need to have $n + m = 3$ poles. Let's assume we want the closed loop poles to be at $-5 \pm j$ and -20. Then

$$\begin{aligned} D_0(s) &= (s+5+j)(s+5-j)(s+20) \\ &= s^3 + 30s^2 + 226s + 520 \end{aligned}$$

Now we need to solve the Diophantine equations

$$\begin{aligned} A(s)D(s) + B(s)N(s) &= D_0(s) \\ (A_0 + A_1 s)\left(s^2 + 3s + 2\right) + (B_0 + B_1 s)(3) &= s^3 + 30s^2 + 226s + 520 \end{aligned}$$

Now we equate powers of s

$$\begin{aligned} s^3 &: \quad 1 = A_1 \\ s^2 &: \quad 30 = A_0 + 3A_1 \\ s^1 &: \quad 226 = 3A_0 + 2A_1 + 3B_1 \\ s^0 &: \quad 520 = 2A_0 + 3B_0 \end{aligned}$$

In this case we can solve directly to get $A_0 = 27.00$, $B_0 = 155.33$, $A_1 = 1.00$ ad $B_1 = 47.67$. Hence our controller is

$$G_c(s) = \frac{155.33 + 47.67s}{27.00 + s}$$

and the closed loop transfer function is

$$G_0(s) = \frac{3(155.33 + 47.67s)}{s^3 + 30s^2 + 226s + 520}$$

We have introduced a zero at -3.26. The position error is $e_p = 1 - G_0(0) = 0.104$.

Now let's assume we want zero position error, but don't want to use a prefilter. To do this, we increase the order of the controller by one, (so $m = 2$) and to be sure we have a type one system we set $A_0 = 0$. Hence we assume a controller of the form

$$G_c(s) = \frac{B_0 + B_1 s + B_2 s^2}{A_1 s + A_2 s^2}$$

where $A_2 \neq 0$. We now need a characteristic polynomial with $n + m = 4$ roots, so there are four closed loop poles to assign. Let's assume we want to keep the poles we have, and put the new pole at -30. Hence the closed loop poles are at $-5 \pm j$, -20, and -30. Then

$$\begin{aligned} D_0(s) &= (s+5+j)(s+5-j)(s+20)(s+30) \\ &= s^4 + 60s^3 + 1126s^2 + 7300s + 15600 \end{aligned}$$

Now we need to solve the Diophantine equations

$$\begin{aligned} A(s)D(s) + B(s)N(s) &= D_0(s) \\ \left(A_1 s + A_2 s^2\right)\left(s^2 + 3s + 2\right) + \left(B_0 + B_1 s + B_2 s^2\right)(3) &= s^4 + 60s^3 + 1126s^2 + 7300s + 15600 \end{aligned}$$

Now we equate powers of s

$$\begin{aligned} s^4 : &\quad 1 = A_2 \\ s^3 : &\quad 60 = A_1 + 3A_2 \\ s^2 : &\quad 1126 = 3A_1 + 2A_2 + 3B_2 \\ s^1 : &\quad 7300 = 2A_1 + 3B_1 \\ s^0 : &\quad 15600 = 3B_0 \end{aligned}$$

We can easily solve these equations to give $B_0 = 5200$, $A_1 = 57.0$, $B_1 = 2395.3$, $A_2 = 1$, and $B_2 = 317.6$.

$$G_c(s) = \frac{5200 + 2395.3s + 317.7s^2}{57s + s^2}$$

and the closed loop transfer function is

$$G_0(s) = \frac{3(5200 + 2395.3s + 317.7s^2)}{s^4 + 60s^3 + 1126s^2 + 7300s + 15600}$$

We have introduced zeros at -3.7 $\pm 1.49j$. Since we have a type one system, the position error is zero.

11.3 Summary

We have shown that by utilizing the Diophantine equations, we can place the closed loop poles wherever we want. In addition, by choosing the order of the controller larger than is necessary to place the poles, we can also force the system to be a type 1 system (or even a type two system). However, in utilizing this method, we introduce zeros into the system. The only way to determine if the added zeros are detrimental to acceptable transient behavior is to simulate the system. By appropriate choice of desired closed loop poles we can sometimes change the locations of these zeros so the system response is acceptable.

12 System Sensitivity

There are generally two kinds of sensitivity used in control systems. The first type of sensitivity refers to the sensitivity of a system to variations in a parameter or transfer function. This type of sensitivity is important to study since we need to be able to determine how to design a control system to reduce the sensitivity of the system to changes in the plant, since we often have to estimate the plant and this estimation will contain some errors. The other type of sensitivity usually refers to how sensitive the system is to outside disturbances. Again, this is important to understand so we can design a control system to reduce the effects of external disturbances. Finally, it is important to understand that sensitivity is a function of frequency, and you need to understand the range of frequencies you expect to be operation your system under. For example, a system may be very sensitive to a parameter at frequencies near 100 Hz, but if your system is typically operating in the 1-10 Hz range this sensitivity is not very important.

12.1 Sensitivity to Parameter Variations

The system sensitivity to changes in a parameter α is defined as the ratio of the percentage change in the system transfer function $G_0(s)$ to the percentage change in the parameter α to its nominal value α_0. Note that α may itself be a transfer function or a block in the block diagram representation of a system.

To mathematically define the sensitivity, let's denote the system transfer function as

$$G_0(s) = \frac{N_0(s)}{D_0(s)}$$

Then the sensitivity of G_0 with respect to changes in α is

$$\begin{aligned}
S_\alpha^{G_0}(s) &= \left.\frac{\Delta G_0(s)/G_0(s)}{\Delta\alpha/\alpha}\right|_{\alpha_0} \\
&= \left.\frac{\alpha}{G_0(s)}\frac{\Delta G_0(s)}{\Delta\alpha}\right|_{\alpha_0} \\
&= \left.\frac{\alpha}{G_0(s)}\frac{\partial G_0(s)}{\partial\alpha}\right|_{\alpha_0}
\end{aligned}$$

A simpler formula for this can be derived as follows:

$$\begin{aligned}
\frac{\partial G_0(s)}{\partial\alpha} &= \frac{\partial}{\partial\alpha}\frac{N_0(s)}{D_0(s)} \\
&= \frac{D_0(s)\frac{\partial N_0(s)}{\partial\alpha} - N_0(s)\frac{\partial D_0(s)}{\partial\alpha}}{D_0(s)^2} \\
&= \frac{1}{D_0(s)}\frac{\partial N_0(s)}{\partial\alpha} - \frac{N_0(s)}{D_0(s)^2}\frac{\partial D_0(s)}{\partial\alpha} \\
&= \frac{N_0(s)}{D_0(s)}\left(\frac{1}{N_0(s)}\frac{\partial N_0(s)}{\partial\alpha} - \frac{1}{D_0(s)}\frac{\partial D_0(s)}{\partial\alpha}\right)
\end{aligned}$$

Hence

$$
\begin{aligned}
S_{\alpha}^{G_0}(s) &= \left.\frac{\alpha}{G_0(s)}\frac{\partial G_0(s)}{\partial \alpha}\right|_{\alpha_0} \\
&= \left.\frac{\alpha}{G_0(s)}G_0(s)\left(\frac{1}{N_0(s)}\frac{\partial N_0(s)}{\partial \alpha} - \frac{1}{D_0(s)}\frac{\partial D_0(s)}{\partial \alpha}\right)\right|_{\alpha_0}
\end{aligned}
$$

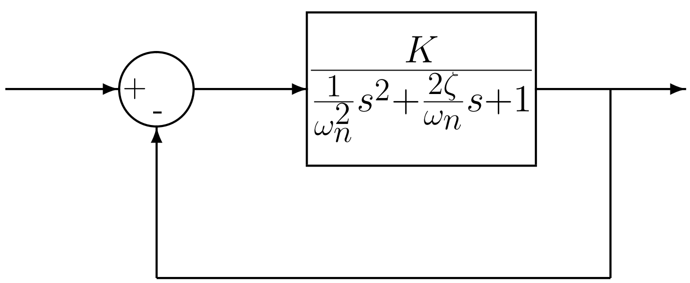
or

$$
\boxed{S_{\alpha}^{G_0}(s) = \left.\left(\frac{\alpha}{N_0(s)}\frac{\partial N_0(s)}{\partial \alpha} - \frac{\alpha}{D_0(s)}\frac{\partial D_0(s)}{\partial \alpha}\right)\right|_{\alpha_0}}
$$

It is important to note that:

- The sensitivity is really a function of frequency $s = j\omega$, and we normally look at the magnitude as a function of frequency, $\left|S_{\alpha_0}^{G_0}(j\omega)\right|$
- We are looking at variations from the nominal values of α_0

Example 1. Consider the closed loop system shown below:



where the nominal values of the parameters are $\omega_n = 20$, $\zeta = 0.1$, and $K = 0.1$. To compute the sensitivity of the closed loop system to variations in ω_n (from the nominal value) we first determine the close loop transfer function

$$
\begin{aligned}
G_0(s) &= \frac{K}{\frac{1}{\omega_n^2}s^2 + \frac{2\zeta}{\omega_n}s + 1 + K} \\
&= \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2(K+1)}
\end{aligned}
$$

Hence

$$
\begin{aligned}
N_0(s) &= K\omega_n^2 \\
D_0(s) &= s^2 + 2\zeta\omega_n s + \omega_n^2(K+1)
\end{aligned}
$$

We then compute

$$
\begin{aligned}
\frac{\partial N_0(s)}{\partial \omega_n} &= 2\omega_n K \\
\frac{\partial D_0(S)}{\partial \omega_n} &= 2\zeta s + 2\omega_n(K+1) \\
S_{\omega_n}^{G_0}(s) &= \left(\frac{\omega_n}{N_0(s)}\right)(2\omega_n K) + \left(\frac{\omega_n}{D_0(s)}\right)[2\zeta s + 2\omega_n(K+1)] \\
&= \frac{2\omega_n^2 K}{\omega_n^2 K} - \frac{2\zeta\omega_n s + 2\omega_n^2(K+1)}{s^2 + 2\zeta\omega_n s + \omega_n^2(K+1)} \\
&= 2 - \frac{2\zeta\omega_n s + 2\omega_n^2(K+1)}{s^2 + 2\zeta\omega_n s + \omega_n^2(K+1)} \\
&= \frac{[2s^2 + 4\zeta\omega_n s + 2\omega_n^2(K+1)] - [2\zeta\omega_n s + 2\omega_n^2(K+1)]}{s^2 + 2\zeta\omega_n s + \omega_n^2(K+1)} \\
&= \frac{2s^2 + 2\zeta\omega_n s}{s^2 + 2\zeta\omega_n s + \omega_n^2(K+1)}
\end{aligned}
$$

In terms of frequency this is

$$
S_{\omega_n}^{G_0}(j\omega) = \frac{-2\omega^2 + 2j\zeta\omega_n\omega}{-\omega^2 + j2\zeta\omega_n\omega + \omega_n^2(K+1)}
$$

In terms of the magnitude this is

$$
\left|S_{\omega_n}^{G_0}(j\omega)\right| = \frac{\sqrt{(2\omega^2)^2 + (2\zeta\omega_n\omega)^2}}{\sqrt{(\omega_n^2(K+1) - \omega^2)^2 + (2\zeta\omega_n\omega)^2}}
$$

Figure 11 shows a graph of the sensitivity function $|S_{\omega_n}^{G_0}(j\omega)|$ as a function of frequency, for the nominal values $K = 0.1$, $\omega_n = 20$, and $\zeta = 0.1$. As the figure shows, the system is not very sensitive to changes in ω_n until ω is around 10 rad/sec.

Example 2. Consider the following two systems, the first is an open loop system with a prefilter (G_{pf}) and controller ($G_c(s)$) before the plant ($G_p(s)$), and the second is a closed loop system with a prefilter outside of the closed loop and a controller inside the loop before the plant. Let's

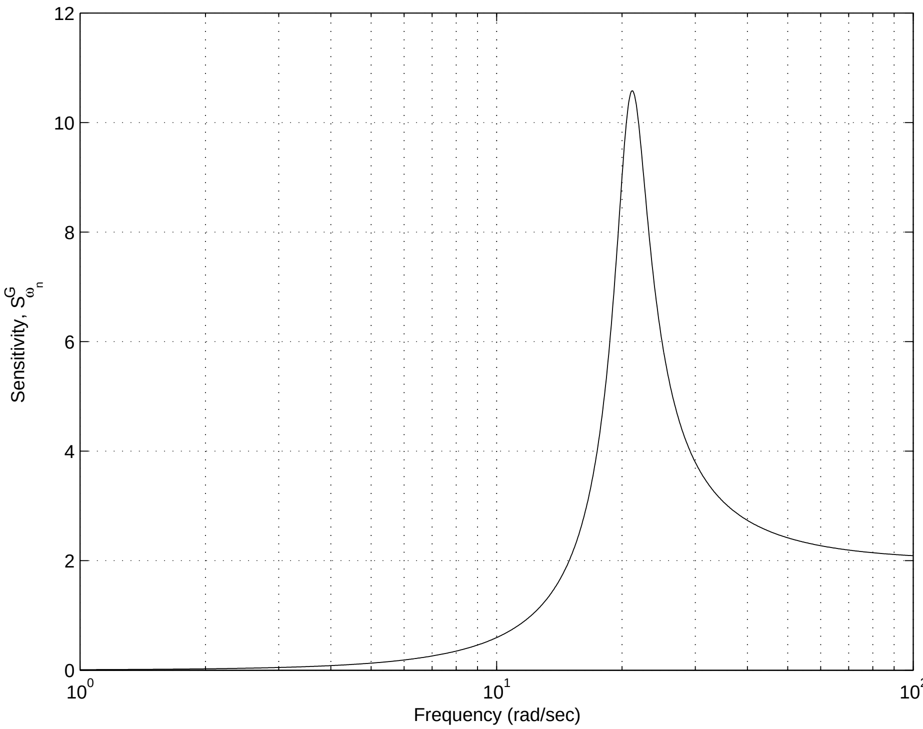
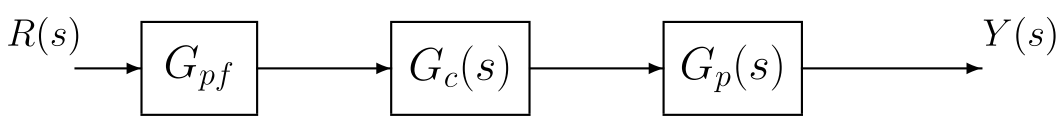
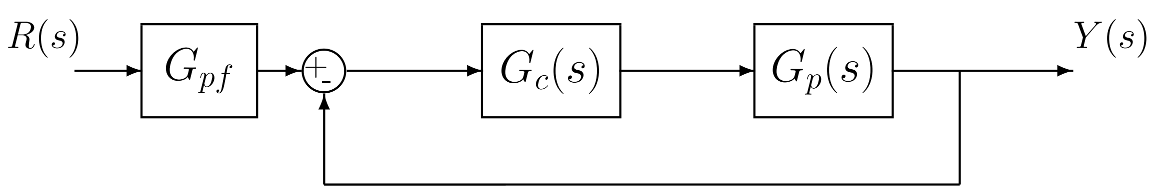


Figure 11: The sensitivity function of Example 1, $S_{\omega_n}^{G_0}(j\omega)|$, as a function of frequency for the nominal values $K = 0.1$, $\omega_n = 20$, and $\zeta = 0.1$.

examine the sensitivity of each system to variations in the prefilter and controller.





First we need to determine expressions for the transfer function between the input $R(s)$ and output $Y(s)$ for the two systems. For the open loop system we have

$$G_0^{open}(s) = G_{pf}(s)G_c(s)G_p(s)$$

while for the closed loop system we have

$$G_0^{closed}(s) = \frac{G_{pf}(s)G_c(s)G_p(s)}{1 + G_c(s)G_p(s)}$$

Let's first compute the sensitivity to variations in the prefilter, $G_{pf}(s)$. For the open loop system

$$\begin{aligned} S_{G_{pf}}^{G_0^{open}} &= \frac{G_{pf}(s)}{N_0(s)}\frac{\partial N_0(s)}{\partial G_{pf}(s)} - \frac{G_{pf}(s)}{D_0(s)}\frac{\partial D_0(s)}{\partial G_{pf}(s)} \\ &= \frac{G_{pf}(s)}{G_{pf}(s)G_c(s)G_p(s)}G_c(s)G_p(s) - 0 \\ &= 1 \end{aligned}$$

For the close loop system

$$\begin{aligned} S_{G_{pf}}^{G_0^{closed}} &= \frac{G_{pf}(s)}{N_0(s)}\frac{\partial N_0(s)}{\partial G_{pf}(s)} - \frac{G_{pf}(s)}{D_0(s)}\frac{\partial D_0(s)}{\partial G_{pf}(s)} \\ &= \frac{G_{pf}(s)}{G_{pf}(s)G_c(s)G_p(s)}G_c(s)G_p(s) - 0 \\ &= 1 \end{aligned}$$

Hence both the open and closed loop systems are equally sensitive to variations in the prefilter $G_{pf}(s)$. *This is because the prefilter is outside of the close loop. Feedback cannot help compensate for variations outside of the closed loop!*
Now let's compute the sensitivity to variations in the plant, $G_p(s)$. For the open loop system

$$\begin{aligned} S_{G_p}^{G_0^{open}} &= \frac{G_p(s)}{N_0(s)}\frac{\partial N_0(s)}{\partial G_p(s)} - \frac{G_p(s)}{D_0(s)}\frac{\partial D_0(s)}{\partial G_p(s)} \\ &= \frac{G_p(s)}{G_{pf}(s)G_c(s)G_p(s)}G_{pf}(s)G_c(s) - 0 \\ &= 1 \end{aligned}$$

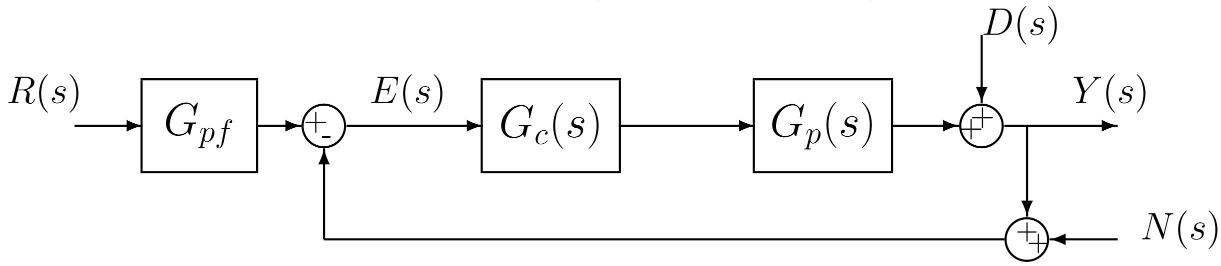
For the close loop system

$$\begin{aligned} S_{G_p}^{G_0^{closed}} &= \frac{G_p(s)}{N_0(s)}\frac{\partial N_0(s)}{\partial G_p(s)} - \frac{G_p(s)}{D_0(s)}\frac{\partial D_0(s)}{\partial G_p(s)} \\ &= \frac{G_p(s)}{G_{pf}(s)G_c(s)G_p(s)}G_{pf}(s)G_c(s) - \frac{G_p(s)}{1+G_c(s)G_p(s)}G_c(s) \\ &= 1 - \frac{G_c(s)G_p(s)}{1+G_c(s)G_p(s)} \\ &= \frac{[1+G_c(s)G_p(s)] - [G_c(s)G_p(s)]}{1+G_c(s)G_p(s)} \\ &= \frac{1}{1+G_c(s)G_p(s)} \end{aligned}$$

In order to reduce the sensitivity of the system to variations in the plant, we want $|1 + G_c(j\omega)G_p(j\omega)|$ to be large. In this case, the closed loop system can be made much less sensitive to variations in the plant than the open loop systems. *This is because the plant is inside of the close loop. Feedback can help compensate for parameter/plant variations inside of the closed loop!*

12.2 Sensitivity to External Disturbances

In addition to the sensitivity of a system to variation in a parameter, we need to also look at the sensitivity of a system to external disturbances. The two most common models of external disturbances are (1) a disturbance that changes the controlled variable, and (2) additive noise in a sensor. Consider the system shown below, with additive disturbances $D(s)$, which models an *output disturbance*, and $N(s)$, which models a *noise disturbance*. When analyzing each of these disturbances we assume there is only one input to the system at a time.



For the output disturbance, we compute the transfer function from $D(s)$ to $Y(s)$ (assuming $N(s)$ and $R(s)$ are zero) as

$$\begin{aligned} E(s) &= 0 - Y(s) \\ Y(s) &= E(s)G_c(s)G_p(s) + D(s) \\ &= -G_c(s)G_p(s)Y(s) + D(s) \end{aligned}$$

or the closed loop transfer function from $D(s)$ to $Y(s)$ is

$$G_0^D(s) = \frac{1}{1 + G_c(s)G_p(s)}$$

Hence to reduce the sensitivity of the system to output disturbances we need $|1+G_c(j\omega)G_p(j\omega)|$ to be large. This is the same condition we had to reduce the system sensitivity to variations in $G_p(s)$.

For the noise disturbance, we compute the transfer function from $N(s)$ to $Y(s)$ (assuming $D(s)$ and $R(s)$ are zero) as

$$\begin{aligned} E(s) &= 0 - [N(s) + Y(s)] \\ Y(s) &= E(s)G_c(s)G_p(s) \\ &= -G_c(s)G_p(s)Y(s) - G_c(s)G_p(s)N(s) \end{aligned}$$

or the closed loop transfer function from $N(s)$ to $Y(s)$ is

$$G_0^N(s) = \frac{-G_c(s)G_p(s)}{1 + G_c(s)G_p(s)}$$

Hence to reduce the sensitivity of the system to noise disturbances we need $|G_c(j\omega)G_p(j\omega)|$ to be small. This is essentially the opposite of the condition we need to reduce the system sensitivity to variations in $G_p(s)$ or to output disturbances.

12.3 Summary

There are generally two kinds of sensitivity used in control systems. The first type of sensitivity refers to the sensitivity of a system to variations in a parameter or transfer function. We compute this sensitivity as

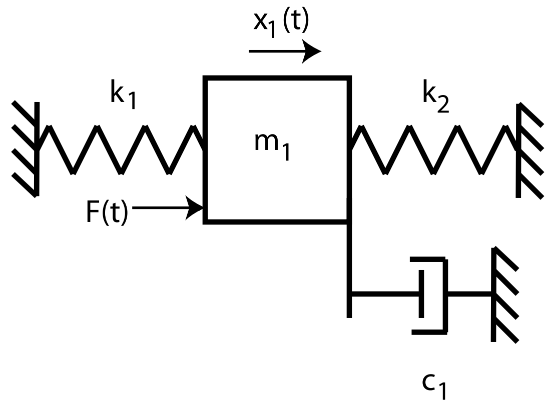
$$\boxed{S_{\alpha}^{G_0}(s) = \left(\frac{\alpha}{N_0(s)} \frac{\partial N_0(s)}{\partial \alpha} - \frac{\alpha}{D_0(s)} \frac{\partial D_0(s)}{\partial \alpha} \right) \Bigg|_{\alpha_0}}$$

We usually compute the sensitivity as a function of frequency, ω, $\left| S_{\alpha_0}^{G_0}(j\omega) \right|$. We are generally only concerned with the sensitivity within a range of frequencies that our system will be operating in. From the examples we see that, from a system sensitivity view, a closed loop system has no advantages over an open loop system for parameters or transfer functions outside the feedback loop. For a closed loop system with plant $G_p(s)$, to minimize the sensitivity of the closed loop system to variations in the plant we want $|1 + G_c(j\omega)G_p(j\omega)|$ to be *large*.

The other type of sensitivity usually refers to how sensitive the system is to *output disturbances* or *noise disturbances*. To reduce the effects of output disturbances, we again want $|1 + G_c(j\omega)G_p(j\omega)|$ to be *large*. To reduce the effects of noise disturbances we want $|G_c(j\omega)G_p(j\omega)|$ to be *small*. These are contradictory conditions. The relative importance of the different disturbances depends on the particular system being analyzed.

13 State Variables and State Variable Feedback

Consider the model of the rectilinear spring-mass-damper system we have been using in lab.



The equations of motion can be written

$$m_1\ddot{x}_1(t) + c_1\dot{x}_1(t) + (k_1+k_2)x_1(t) = F(t)$$

or

$$\frac{1}{\omega_n^2}\ddot{x}_1(t) + \frac{2\zeta}{\omega_n}\dot{x}_1(t) + x_1(t) = \frac{1}{k_1+k_2}F(t) \equiv K_{static}u(t)$$

where $u(t)$ is the motor input in volts, and K_{static} is the static gain for the system. Note that this gain also includes the open loop motor gain. We can also write this as

$$\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^2 x(t) = \omega_n^2 K_{static}u(t)$$

We can take then take Laplace transforms to get the transfer function

$$G_p(s) = \frac{X_1(s)}{U(s)} = \frac{K_{static}}{\frac{1}{\omega_n^2}s^2 + \frac{2\zeta}{\omega_n}s + 1}$$

We can also write the model in *state variable form.* For linear, time-invariant models, a state variable model has the general form

$$\begin{aligned}\underline{\dot{x}}(t) &= A\underline{x}(t) + B\underline{u}(t)\\ \underline{y}(t) &= C\underline{x}(t) + D\underline{u}(t)\end{aligned}$$

where $\underline{x}(t)$ is the state vector, $\underline{u}(t)$ is the input vector, $\underline{y}(t)$ is the output vector, and A, B, C, and D are constant matrices.

For our system, let's let $q_1(t) = x(t)$ and $q_2(t) = \dot{x}(t)$. Then we can write

$$\begin{aligned}\dot{q}_1(t) &= q_2(t)\\ \dot{q}_2(t) &= -2\zeta\omega_n\dot{x}(t) - \omega_n^2 x(t) + \omega_n^2 K_{static}u(t)\\ &= -2\zeta\omega_n q_2(t) - \omega_n^2 q_1(t) + \omega_n^2 K_{static}u(t)\\ &= -\omega_n^2 q_1(t) - 2\zeta\omega_n q_2(t) + \omega_n^2 K_{static}u(t)\end{aligned}$$

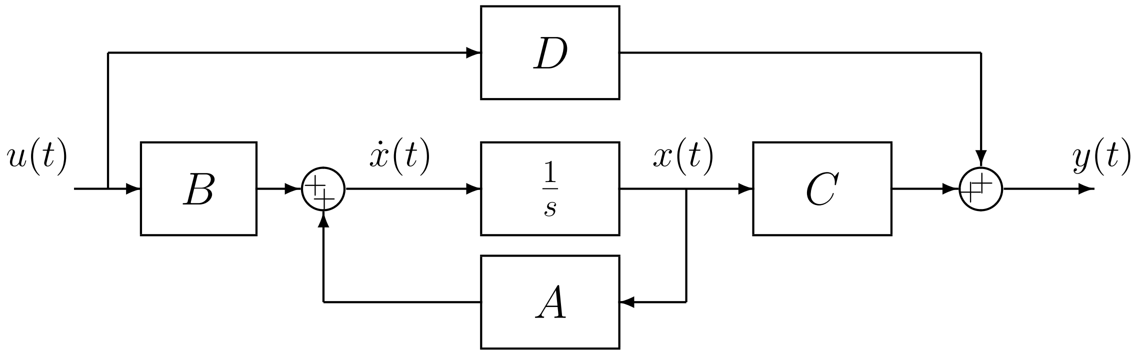


Figure 12: General state variable form for an open loop plant

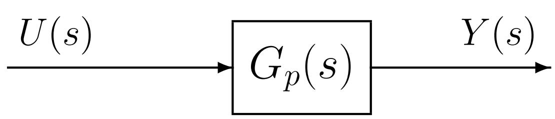


Figure 13: General transfer function form for an open loop plant

If the output is considered to be the position of the cart, the correct state variable form is

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix}\begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \omega_n^2 K_{static} \end{bmatrix} u(t) \\ y(t) &= [1\ 0]\begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} \end{aligned}$$

If the output was considered to be the velocity of the cart, the output equation would be

$$y(t) = [0\ 1]\begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix}$$

while if both the position of the cart and the velocity of the cart were the desired outputs, the output equation would be

$$\underline{y}(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix}$$

We would like to be able to go between a state variable model of a system to a transfer function model. Each type of model has its benefits. Figure 12 shows the general form for an open loop state variable model of a plant, while Figure 13 shows the equivalent transfer function form.

13.1 State Variable to Transfer Function Model

Assume we have the state variable description written in scalar form:

$$\begin{aligned}\dot{x}_1(t) &= a_{11}x_1(t) + a_{12}x_2(t) + b_1u(t)\\ \dot{x}_2(t) &= a_{21}x_1(t) + a_{22}x_2(t) + b_2u(t)\\ y(t) &= c_1x_1(t) + c_2x_2(t) + du(t)\end{aligned}$$

In matrix/vector form, this is

$$\begin{aligned}\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} &= \begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}b_1\\ b_2\end{bmatrix}u(t)\\ y(t) &= \begin{bmatrix}c_1 & c_2\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + [d]u(t)\end{aligned}$$

or

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ y(t) &= Cx(t) + Du(t)\end{aligned}$$

Taking the Laplace transform of the scalar equations (assuming zero initial conditions) we get

$$\begin{aligned}\begin{bmatrix}sX_1(s)\\ sX_2(s)\end{bmatrix} &= \begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}X_1(s)\\ X_2(s)\end{bmatrix} + \begin{bmatrix}b_1\\ b_2\end{bmatrix}U(s)\\ Y(s) &= \begin{bmatrix}c_1 & c_2\end{bmatrix}\begin{bmatrix}X_1(s)\\ X_2(s)\end{bmatrix} + [d]U(s)\end{aligned}$$

We can write this new system of equations in matrix form as

$$\begin{aligned}sX(s) &= AX(s) + BU(s)\\ Y(s) &= CX(s) + DU(s)\end{aligned}$$

We can rewrite the first equation as

$$(sI - A)\,X(s) = BU(s)$$

or

$$X(s) = (sI - A)^{-1}BU(s)$$

We can then solve for $Y(s)$ as

$$Y(s) = \left[C(sI - A)^{-1}B + D\right]U(s)$$

or

$$Y(s) = G(s)U(s)$$

Hence, the transfer function (or transfer matrix, if there is more than one input or output), is given by

$$G(s) = \left[C(sI - A)^{-1}B + D\right]$$

In going from a state variable model to a transfer function model, you need to be able to compute the inverse of a matrix. You are expected to be able to compute the inverse of a 2x2 matrix without a computer (or calculator). If matrix P is given as

$$P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then

$$P^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

and the *determinant* of P is given by $ad - bc$.

<u>**Example 1**</u>. Assume we have the state variable model

$$\begin{aligned} \dot{\underline{x}} &= \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \underline{x} + \begin{bmatrix} 5 \\ 0 \end{bmatrix} u \\ y &= [1\ 2]\ \underline{x} \end{aligned}$$

and we want to find the transfer function model. We need to compute

$$G(s) = \left[C(sI - A)^{-1}B + D\right]$$

First we compute $sI - A$ as

$$sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} s-1 & 0 \\ -2 & s-3 \end{bmatrix}$$

Next we compute

$$(sI - A)^{-1} = \frac{1}{(s-1)(s-3) - (-2)(0)} \begin{bmatrix} s-3 & 0 \\ 2 & s-1 \end{bmatrix}$$

then

$$\begin{aligned} C\,(sI - A)^{-1} &= [1\ 2]\ \frac{1}{(s-1)(s-3)} \begin{bmatrix} s-3 & 0 \\ 2 & s-1 \end{bmatrix} \\ &= \frac{1}{(s-1)(s-3)} \left[(1)(s-3) + (2)(2) \quad (1)(0) + (2)(s-1)\right] \\ &= \frac{1}{(s-1)(s-3)} \left[s+1 \quad 2s-2\right] \end{aligned}$$

and finally

$$
\begin{aligned}
G(s) &= C\,(sI-A)^{-1}\,B \\
&= \frac{1}{(s-1)(s-3)}\left[s+1 \quad 2s-2\right]\begin{bmatrix}5\\0\end{bmatrix} \\
&= \frac{1}{(s-1)(s-3)}\left[5(s+1)+0(2s-2)\right] \\
&= \frac{5(s+1)}{(s-1)(s-3)}
\end{aligned}
$$

The poles of the transfer function are at 1 and 3, and there is a zero at -1. The system is clearly unstable.

Example 2. Assume we have the state variable model

$$
\begin{aligned}
\dot{\underline{x}} &= \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\underline{x}+\begin{bmatrix}1\\1\end{bmatrix}u \\
y &= [1\ 2]\ \underline{x}
\end{aligned}
$$

and we want to find the transfer function model. We need to compute

$$
G(s) = \left[C(sI-A)^{-1}B+D\right]
$$

First we compute $sI-A$ as

$$
sI-A = \begin{bmatrix}s & 0\\0 & s\end{bmatrix}-\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} = \begin{bmatrix}s-1 & 0\\0 & s\end{bmatrix}
$$

Next we compute

$$
(sI-A)^{-1} = \frac{1}{(s-1)(s)-(0)(0)}\begin{bmatrix}s & 0\\0 & s-1\end{bmatrix}
$$

then

$$
\begin{aligned}
C\,(sI-A)^{-1} &= [1\ 2]\ \frac{1}{s(s-1)}\begin{bmatrix}s & 0\\0 & s-1\end{bmatrix} \\
&= \frac{1}{s(s-1)}\left[(1)(s) \quad (2)(s-1)\right] \\
&= \frac{1}{s(s-1)}\left[s \quad 2s-2\right]
\end{aligned}
$$

and finally

$$
\begin{aligned}
G(s) &= C\,(sI-A)^{-1}\,B \\
&= \frac{1}{s(s-1)}\left[s \quad 2s-2\right]\begin{bmatrix}1\\1\end{bmatrix} \\
&= \frac{1}{s(s-1)}\left[s+(2s-2)\right] \\
&= \frac{3s-2}{s(s-1)}
\end{aligned}
$$

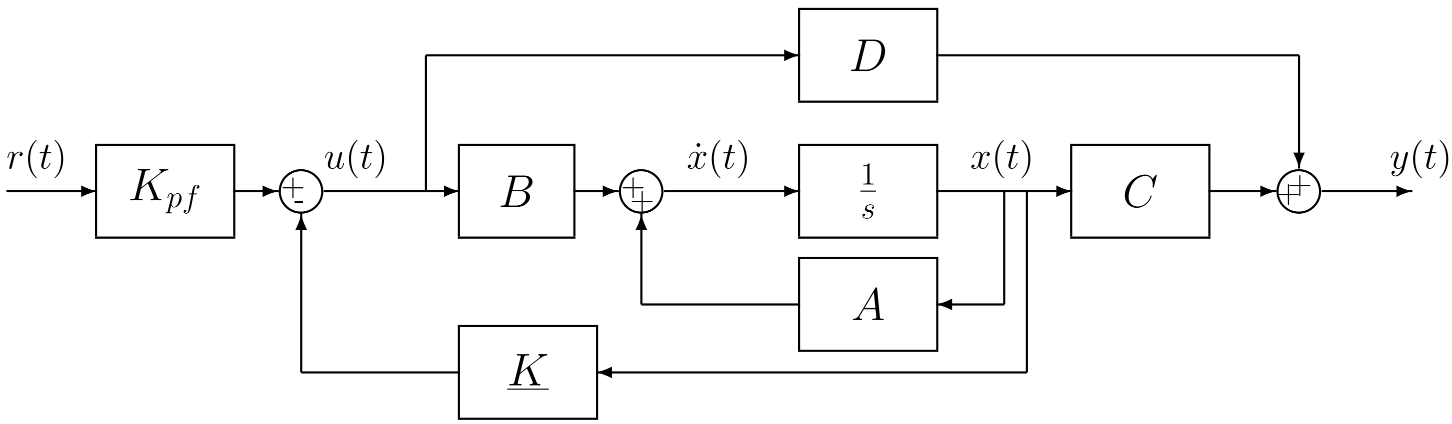


Figure 14: State variable model of a plant with state variable feedback.

The poles of the transfer function are at 0 and 1, and there is a zero at $-\frac{2}{3}$. The system is clearly unstable.

13.2 State Variable Feedback

Up to this point, we have shown how we can go from a state variable description of an open loop system to a transfer function model. In particular, we can model a plant using either a transfer function description or a state variable description. We can then implement any of the single-input single-output controllers we have been utilizing in this course. However, each of these methods assumes we are feeding back only one variable, usually the output. However, a state variable model allows us a much more powerful method of control, that of feeding back all of the states, which is called *state variable feedback*.

Let's assume the input to the plant, $u(t)$, is the difference between the scaled reference input, $K_{pf}r(t)$, and scaled states, $\underline{K}\underline{x}(t)$, or

$$u(t) = K_{pf}r(t) - \underline{K}\underline{x}(t)$$

Here K_{pf} is a prefilter, much like we used $G_{pf}(s)$ for the transfer function feedback systems. Figure 14 displays a state variable model of a plant with state variable feedback.

With the state variable feedback the state equations become

$$\begin{aligned}
\dot{\underline{x}}(t) &= A\underline{x}(t) + Bu(t) \\
&= A\underline{x}(t) + B\left[K_{pf}r(t) - \underline{K}\underline{x}(t)\right] \\
&= \left[A - B\underline{K}\right]\underline{x}(t) + \left[BK_{pf}\right]r(t) \\
&= \tilde{A}\underline{x}(t) + \tilde{B}r(t)
\end{aligned}$$

where

$$\begin{aligned}\tilde{A} &= [A - B\underline{K}] \\ \tilde{B} &= BK_{pf}\end{aligned}$$

The output equation is then

$$\begin{aligned}y(t) &= C\underline{x}(t) + Du(t) \\ &= C\underline{x}(t) + D\left[K_{pf}r(t) - \underline{K}\,\underline{x}(t)\right] \\ &= [C - D\underline{K}]\,\underline{x}(t) + [DK_{pf}]\,r(t) \\ &= \tilde{C}\underline{x}(t) + \tilde{D}r(t)\end{aligned}$$

where

$$\begin{aligned}\tilde{C} &= [C - D\underline{K}] \\ \tilde{D} &= DK_{pf}\end{aligned}$$

Under most circumstances $D = 0$ so $\tilde{C} = C$ and $\tilde{D} = 0$.

The new input to our system is $r(t)$. The transfer function between the input $R(s)$ and the output $Y(s)$ for the state variable model with state variable feedback is given by

$$G(s) = \tilde{C}\left(sI - \tilde{A}\right)^{-1}\tilde{B} + \tilde{D}$$

<u>**Example 3**</u>. Assume we again have the state variable model

$$\begin{aligned}\underline{\dot{x}} &= \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}\underline{x} + \begin{bmatrix} 5 \\ 0 \end{bmatrix}u \\ y &= [1\ 2]\ \underline{x}\end{aligned}$$

but now we have state variable feedback. We want to find the transfer function model for the system with the state variable feedback. We need to compute

$$G(s) = \left[\tilde{C}(sI - \tilde{A})^{-1}\tilde{B} + \tilde{D}\right]$$

First we compute

$$\begin{aligned}\tilde{A} &= A - B\underline{K} = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} 5 \\ 0 \end{bmatrix}[K_1\ K_2] \\ &= \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} 5K_1 & 5K_2 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 - 5K_1 & -5K_2 \\ 2 & 3 \end{bmatrix}\end{aligned}$$

and

$$\tilde{B} = BK_{pf} = \begin{bmatrix} 5K_{pf} \\ 0 \end{bmatrix}$$

Since $D = 0$ we have $\tilde{C} = C$ and $\tilde{D} = 0$.

Next we compute

$$\begin{aligned} sI - \tilde{A} &= \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 1 - 5K_1 & -5K_2 \\ 2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} s - 1 + 5K_1 & 5K_2 \\ -2 & s - 3 \end{bmatrix} \end{aligned}$$

and

$$\left(sI - \tilde{A}\right)^{-1} = \frac{1}{(s - 1 + 5K_1)(s - 3) - (-2)(5K_2)} \begin{bmatrix} s - 3 & -5K_2 \\ 2 & s - 1 + 5K_1 \end{bmatrix}$$

At this point it is probably easiest to postmultiply by $\tilde{B}$ first

$$\begin{aligned} \left(sI - \tilde{A}\right)^{-1} \tilde{B} &= \frac{1}{(s - 1 + 5K_1)(s - 3) - (-2)(5K_2)} \begin{bmatrix} s - 3 & -5K_2 \\ 2 & s - 1 + 5K_1 \end{bmatrix} \begin{bmatrix} 5K_{pf} \\ 0 \end{bmatrix} \\ &= \frac{1}{(s - 1 + 5K_1)(s - 3) + 10K_2} \begin{bmatrix} 5K_{pf}(s - 3) \\ 10K_{pf} \end{bmatrix} \end{aligned}$$

Finally, premultiplying by C we get

$$\begin{aligned} G(s) &= [1\ 2]\, \frac{1}{(s - 1 + 5K_1)(s - 3) + 10K_2} \begin{bmatrix} 5K_{pf}(s - 3) \\ 10K_{pf} \end{bmatrix} \\ &= \frac{5K_{pf}(s - 3) + (2)(10K_{pf})}{(s - 1 + 5K_1)(s - 3) + 10K_2} \\ &= \frac{K_{pf} 5(s + 1)}{s^2 + (5K_1 - 4)s + (10K_2 - 15K_1 + 3)} \end{aligned}$$

You should note

- the state variable feedback *did not change the zeros of the system*
- K_{pf} is just a scaling factor
- For $K_1 = K_2 = 0$ (open loop) and $K_{pf} = 1$ (no prefilter), we get

  $$G(s) = \frac{5(s + 1)}{(s - 1)(s - 3)}$$

  as before.

Example 4. Assume we again have the state variable model

$$\begin{aligned} \dot{\underline{x}} &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u \\ y &= [1\ 2]\ \underline{x} \end{aligned}$$

but now we have state variable feedback. We want to find the transfer function model for the system with the state variable feedback. We need to compute

$$G(s) \;=\; \left[\tilde{C}(sI-\tilde{A})^{-1}\tilde{B}+\tilde{D}\right]$$

First we compute

$$\begin{aligned}
\tilde{A} &= A - B\underline{K} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} [K_1 \; K_2] \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} K_1 & K_2 \\ K_1 & K_2 \end{bmatrix} \\
&= \begin{bmatrix} 1-K_1 & -K_2 \\ -K_1 & -K_2 \end{bmatrix}
\end{aligned}$$

and

$$\tilde{B} \;=\; BK_{pf} \;=\; \begin{bmatrix} K_{pf} \\ K_{pf} \end{bmatrix}$$

Since $D=0$ we have $\tilde{C}=C$ and $\tilde{D}=0$.

Next we compute

$$\begin{aligned}
sI-\tilde{A} &= \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 1-K_1 & -K_2 \\ -K_1 & -K_2 \end{bmatrix} \\
&= \begin{bmatrix} s-1+K_1 & K_2 \\ K_1 & s+K_2 \end{bmatrix}
\end{aligned}$$

and

$$\left(sI-\tilde{A}\right)^{-1} \;=\; \frac{1}{(s-1+K_1)(s+K_2)-(K_1)(K_2)} \begin{bmatrix} s+K_2 & -K_2 \\ -K_1 & s-1+K_1 \end{bmatrix}$$

At this point it is probably easiest to postmultiply by $\tilde{B}$ first

$$\begin{aligned}
\left(sI-\tilde{A}\right)^{-1}\tilde{B} &= \frac{1}{(s-1+K_1)(s+K_2)-K_1K_2} \begin{bmatrix} s+K_2 & -K_2 \\ -K_1 & s-1+K_1 \end{bmatrix} \begin{bmatrix} K_{pf} \\ K_{pf} \end{bmatrix} \\
&= \frac{K_{pf}}{(s-1+K_1)(s+K_2)-K_1K_2} \begin{bmatrix} s \\ s-1 \end{bmatrix}
\end{aligned}$$

Finally, premultiplying by C we get

$$\begin{aligned}
G(s) &= [1 \; 2] \, \frac{K_{pf}}{(s-1+K_1)(s+K_2)-K_1K_2} \begin{bmatrix} s \\ s-1 \end{bmatrix} \\
&= \frac{K_{pf}(3s-2)}{(s-1+K_1)(s+K_2)-K_1K_2} \\
&= \frac{K_{pf}(3s-2)}{s^2+(K_1+K_2-1)s-K_2}
\end{aligned}$$

You should note

- the state variable feedback *did not change the zeros of the system*
- K_{pf} is just a scaling factor
- For $K_1 = K_2 = 0$ (open loop) and $K_{pf} = 1$ (no prefilter), we get

$$G(s) = \frac{3s-2}{s(s-1)}$$

as before.

Example 5. Assume we have the state variable model

$$\begin{aligned} \dot{\underline{x}} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \underline{x} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u \\ y &= [3\ 4]\ \underline{x} \end{aligned}$$

We want to find the transfer function model for the system with the state variable feedback. We need to compute

$$G(s) = \left[\tilde{C}(sI - \tilde{A})^{-1}\tilde{B} + \tilde{D}\right]$$

First we compute

$$\begin{aligned} \tilde{A} &= A - B\underline{K} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} [K_1\ K_2] \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} K_1 & K_2 \\ 2K_1 & 2K_2 \end{bmatrix} \\ &= \begin{bmatrix} 1-K_1 & -K_2 \\ -2K_1 & 1-2K_2 \end{bmatrix} \end{aligned}$$

and

$$\tilde{B} = BK_{pf} = \begin{bmatrix} K_{pf} \\ 2K_{pf} \end{bmatrix}$$

Since $D = 0$ we have $\tilde{C} = C$ and $\tilde{D} = 0$.

Next we compute

$$\begin{aligned} sI - \tilde{A} &= \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 1-K_1 & -K_2 \\ -2K1 & 1-2K_2 \end{bmatrix} \\ &= \begin{bmatrix} s-1+K_1 & K_2 \\ 2K_1 & s-1+2K_2 \end{bmatrix} \end{aligned}$$

and

$$\left(sI - \tilde{A}\right)^{-1} = \frac{1}{(s-1+K_1)(s-1+2K_2) - (2K_1)(K_2)} \begin{bmatrix} s-1+2K_2 & -K_2 \\ -2K_1 & s-1+K_1 \end{bmatrix}$$

Let's postmultiply by $\tilde{B}$ first

$$\begin{aligned}
\left(sI - \tilde{A}\right)^{-1}\tilde{B} &= \frac{1}{(s-1+K_1)(s-1+2K_2)-(2K_1)(K_2)}\left[\begin{array}{cc} s-1+2K_2 & -K_2 \\ -2K_1 & s-1+K_1 \end{array}\right]\left[\begin{array}{c} K_{pf} \\ 2K_{pf} \end{array}\right] \\
&= \frac{K_{pf}}{(s-1+K_1)(s-1+2K_2)-2K_1K_2}\left[\begin{array}{c} s-1 \\ 2s-2 \end{array}\right]
\end{aligned}$$

Finally, premultiplying by C we get

$$\begin{aligned}
G(s) &= [3\ 4]\ \frac{K_{pf}}{(s-1+K_1)(s-1+2K_2)-2K_1K_2}\left[\begin{array}{c} s-1 \\ 2s-2 \end{array}\right] \\
&= \frac{K_{pf}[3(s-1)+4(2s-2)]}{(s-1+K_1)(s-1+2K_2)-2K_1K_2} \\
&= \frac{11K_{pf}(s-1)}{(s-1+K_1)(s-1+2K_2)-2K_1K_2} \\
&= \frac{11K_{pf}(s-1)}{[(s-1)+K_1][(s-1)+2K_2]-2K_1K_2} \\
&= \frac{11K_{pf}(s-1)}{(s-1)^2+(K_1+2K_2)(s-1)+2K_1K_2-2K_1K_2} \\
&= \frac{11K_{pf}}{s-1+K_1+2K_2}
\end{aligned}$$

Note that this transfer function has only one pole.

13.3 Controllability for State Variable Systems

A single-input single-output state variable system is said to be *controllable* [3] if we can place as many poles of the closed loop transfer function as there are states of the state variable model. For example, if there are two states in the state variable model we assume we want the closed loop characteristic equation to be $s^2 + a_1 s + a_0$ and see if we can find K_1 and K_2 to achieve any possible values for a_1 and a_0. If, when when the transfer function is simplified as much as possible, the order of the characteristic equation (the denominator of the transfer function) is less than the number of states of the system the system is *not controllable* or *uncontrollable.*

Example 6. For the state variable system in Example 3, we set the characteristic polynomial (after all pole/zero cancellations) to an arbitrary second order polynomial (since there are two states)

$$s^2 + (5K_1 - 4)s + (10K_2 - 15K_1 + 3) = s^2 + a_1 s + a_0$$

from which we get

$$\begin{aligned}
5K_1 - 4 &= a_1 \\
5K_1 &= a_1 + 4 \\
K_1 &= \frac{a_1+4}{5}
\end{aligned}$$

[3] This is one of many possible (and equivalent) definitions.

and

$$\begin{aligned}
10K_2 - 15K_1 + 3 &= a_0 \\
10K_2 &= a_0 + 15K_1 - 3 \\
10K_2 &= a_0 + 3(a_1 + 4) - 3 \\
10K_2 &= a_0 + 3a_1 - 9 \\
K_2 &= \frac{a_0 + 3a_1 - 9}{10}
\end{aligned}$$

Hence we can determine a K_1 and K_2 to achieve any possible values of a_0 and a_1. This system is *controllable.*

Example 7. For the state variable system in Example 4, we set the characteristic polynomial (after all pole/zero cancellations) to an arbitrary second order polynomial (since there are two states)

$$s^2 + (K_1 + K_2 - 1)s - K_2 = s^2 + a_1 s + a_0$$

from which we get

$$K_2 = -a_0$$

and

$$\begin{aligned}
K_1 + K_2 - 1 &= a_1 \\
K_1 &= a_1 - K_2 + 1 \\
K_1 &= a_1 + a_0 + 1
\end{aligned}$$

Hence we can determine a K_1 and K_2 to achieve any possible values of a_0 and a_1. This system is *controllable.*

Example 8. For the state variable system in Example 5, we set the characteristic polynomial (after all pole/zero cancellations) to an arbitrary second order polynomial (since there are two states)

$$s - 1 + K_1 + 2K_2 = s^2 + a_1 s + a_0$$

Clearly it is not possible to find constant values of K_1 and K_2 so these two equations to be equal. Hence the system is not controllable.

13.4 Summary

State variable models are an alternative method of modelling a system. However, we can derive transfer function models from state variable models and state variable models from transfer function models. State variable models have an advantage over transfer function models in that we can utilize state variable feedback to place all of the poles of the system if the system is controllable. Unlike the coefficient matching (Diophantine equation) transfer function methods, state variable feedback does not add zeros to the closed loop system.

14 Controller Design Using Bode Plots

This section has not been written yet. Guidelines for phase lead and phase lag compensators have been included.

"Guidelines" for Phase Lead Compensator Design Using Bode Plots

The primary function of the lead compensator is to reshape the frequency response curve by adding phase to the system. The phase lead compensator also adds gain to the system.

[1] Assume the compensator has the form

$$G_c(s) \;=\; K_c \frac{s+\frac{1}{T}}{s+\frac{1}{\alpha T}} = K_c \alpha \frac{Ts+1}{\alpha Ts+1} = K \frac{Ts+1}{\alpha Ts+1}$$

Determine K to satisfy the static error constant requirements (for e_p and e_v, etc.)

[2] Using this value of K, draw the Bode diagram of $KG(s)H(s)$. Determine the phase margin.

[3] Determine the necessary phase-lead angle to be added to the system. Add an additional 5^o to 12^o to the phase lead required, because the phase lead compensator shifts the phase crossover frequency to the right and decreases the phase margin. ϕ_m is then the total phase our compensator needs to add to the system.

[4] Determine α using

$$\alpha \;=\; \frac{1-\sin(\phi_m)}{1+\sin(\phi_m)}$$

Determine the magnitude where $KG(j\omega)H(j\omega)$ is equal to $-20\log_{10}(\frac{1}{\sqrt{\alpha}}) = 10\log_{10}(\alpha)$. This is the new gain crossover frequency $\omega_m = \frac{1}{T\sqrt{\alpha}}$, or $T = \frac{1}{\omega_m\sqrt{\alpha}}$.

Note: If $\alpha < 0.05$, you will probably need two compensators. Choose a phase angle ϕ_m that produces an acceptable α. Finish the design, then treat $KG_c(s)G(s)H(s)$ as the system and go back to step [2].

[5] Determine the corner frequencies of the compensator as $z = \frac{1}{T}$ and $p = \frac{1}{\alpha T}$.

[6] Determine $K_c = \frac{K}{\alpha}$.

[7] Check the gain and phase margins to be sure they are satisfactory.

“Guidelines” for Phase Lag Compensator Design Using Bode Plots

The primary function of the lag compensator is to reshape the frequency response curve by removing gain of the system. A phase lag is used when the system would have a large enough phase margin if the gain crossover frequency was in a different place, but the gain is too large in that place. The phase lag compensator also removes phase from a the system.

[1] Assume the compensator has the form

$$G_c(s) \quad = \quad K_c\beta\frac{Ts+1}{\beta Ts+1} = K_c\frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}}$$

Determine $K(= K_c\beta)$ to satisfy the static error constant requirements (for e_p and e_v, etc.)

[2] Using this value of K, draw the Bode diagram of $KG(s)H(s)$. Determine the phase margin.

[3] Find the frequency point where the phase angle of $KG(j\omega)H(j\omega)$ is equal to -180^o plus the required phase margin. The required phase margin is the specified phase margin plus 5^o to 12^o. (The additional phase compensates for the phase the lag compensator will remove from the system.) Choose this frequency as the new phase crossover frequency.

[4] Choose the corner frequency $\omega = \frac{1}{T}$ (the zero of the lag compensator) 1 octave (a factor of 2) to 1 decade (a factor of 10) below the new gain crossover frequency.

[5] Determine the current magnitude at the new gain crossover frequency. This magnitude is equal to $20\log_{10}(\beta)$. This is the amount of gain the phase lag compensator must remove from the system. Determine the value of β. (β must be greater than 1, or you have either screwed up or you cannot use a phase lag compensator.) The pole of the compensator is at $\omega = \frac{1}{\beta T}$.

[6] Check the resulting phase and gain margins to be sure they are satisfactory.

15 Linearization

Up to this point we have assumed that we have a transfer function model of the system we are trying to control. However, a transfer function model only exists if the system has a linear model. If a model is not linear, then we need to determine a linear model of the system in order to use the techniques we have developed in this class. However, this model is likely to be valid only over a limited range of values. Before we go into how to get a linear model, we need to be clear on what we mean by a linear system and review Taylor series.

15.1 Linear Systems

In general, if we have input $u(t)$ and output $y(t)$ we can represent the input output relationship of a system, whether it is linear or not, as

$$u(t) \quad \rightarrow \quad y(t)$$

Assume input $u_1(t)$ produces output $y_1(t)$ and input $u_2(t)$ produced $y_2(t)$,

$$\begin{aligned} u_1(t) \quad &\rightarrow \quad y_1(t) \\ u_2(t) \quad &\rightarrow \quad y_2(t) \end{aligned}$$

The system is said to be *linear* if and only if

$$\alpha_1 u_1(t) + \alpha_2 u_2(t) \quad \rightarrow \quad \alpha_1 y_1(t) + \alpha_2 y_2(t)$$

for all α_1, α_2, $u_1(t)$, and $u_2(t)$. If a system is not linear, we cannot take its Laplace transform, and thus cannot use transfer functions. However, we can often produce a linear model of a system if we assume it does not deviate too much from a fixed (nominal) value. Hence we are looking for a linear model near a fixed point. Usually we will assume the fixed point is an equilibrium point. This is very similar to first biasing a transistor circuit, and then using small signal analysis about this biasing point.

15.2 Taylor Series

Assume we have a function $f(z)$ and we want to approximate the function near $z = 0$. The Taylor series approximation near $z = 0$ is

$$f(z) \quad \approx \quad f(0) + f'(0)z + \text{higher order terms}$$

You should be able to derive all of the entries in Table 15.2. This approximation is only valid for z near 0. The further away from zero we go, the worse the approximation is likely to be.

$f(z)$	Linear Approximation
$(1+z)^a$	$1+az$
e^{az}	$1+az$
$\cos(az)$	1
$\sin(az)$	az
$\ln(1+z)$	z
$\cos(\alpha+z)$	$\cos(\alpha) - z\sin(\alpha)$
$\sin(\alpha+z)$	$\sin(\alpha) + z\cos(\alpha)$

Table 2: Functions and their linear approximation near $z = 0$.

15.3 Linearization Procedure

Our goal here is to find a linear model that we can use to determine the transfer function of a system. The procedure we will go through is listed below, and will be followed with a few examples.

Step 1 Determine the nominal *operating point* of the system and the equation that these operating points solve. We will assume the operating points are the static equilibrium points. At the static equilibrium points, all derivatives are zero. For the linearization to be valid, the system must not stray very far from this operating point. Label these points x_0, y_0, u_0, etc. *These points are assumed to be constants.*

Step 2 Look at variations from these operation points. For example, we assume

$$\begin{aligned} x(t) &= x_0 + \Delta x(t) \\ y(t) &= y_0 + \Delta y(t) \\ u(t) &= u_0 + \Delta u(t) \end{aligned}$$

Note that only $\Delta x(t)$, $\Delta y(t)$, etc. vary with time. x_0, y_0, etc. are constants. Now we have two cases to consider:

Step 2a If our functions are *arguments* to other standard functions, we leave this approximation as it is. For example, $\cos(x(t))$ would be rewritten $\cos(x_0 + \Delta x(t))$. Similarly for all other trigonometric functions and exponentials.

Step 2b If out functions are not arguments to standard functions, we rewrite the functions as

$$\begin{aligned} x(t) &= x_0 + \Delta x(t) = x_0\left(1 + \frac{\Delta x(t)}{x_0}\right) \\ y(t) &= y_0 + \Delta y(t) = y_0\left(1 + \frac{\Delta y(t)}{y_0}\right) \\ u(t) &= u_0 + \Delta u(t) = u_0\left(1 + \frac{\Delta u(t)}{u_0}\right) \end{aligned}$$

We rewrite the functions in this way because this is the form will will use the Taylor series on. Here our *small* z will be $\frac{\Delta x(t)}{x_0}$, $\frac{\Delta y(t)}{y_0}$, etc.

Step 3 Substitute our expressions for $x(t)$, $y(t)$, etc. into the dynamics, and simplify where possible.

Step 4 Using Taylor series, expand out all nonlinear terms.

Step 5 Put the Taylor series expansion into the defining differential equation and multiply out all terms.

Step 6 Drop all second order (or higher) terms. Thus terms of the form $\left(\frac{\Delta x(t)}{x_0}\right)^2$, $\left(\frac{\Delta x(t)}{x_0}\right)\left(\frac{\Delta y(t)}{y_0}\right)$, etc. will be dropped.

Step 7 Using the relationships found in step 1, try and remove all constant terms in the model. If there are any constant terms left over, you have made an error. All of the remaining terms should be Δ terms.

Step 8 Find the resulting transfer function.

Example 1. Assume we have the model of a system with input $u(t)$ and output $x(t)$

$$\dot{x}(t) + 3x^2(t) \quad = \quad u(t) + 3$$

and we want to find a linearized model about the static equilibrium point.

Step 1 At equilibrium we have the equation $3x_0^2 = u_0 + 3$.

Step 2 Assume $x(t) = x_0\left(1 + \frac{\Delta x(t)}{x_0}\right)$ and $u(t) = \left(1 + \frac{\Delta u(t)}{u_0}\right)$

Step 3 Now substitute into the dynamics and do some simplification

$$\begin{aligned}
\frac{d}{dt}\left[x_0\left(1 + \frac{\Delta x(t)}{x_0}\right)\right] + 3\left[x_0\left(1 + \frac{\Delta x(t)}{x_0}\right)\right]^2 &= u_0\left(1 + \frac{\Delta u(t)}{u_0}\right) + 3 \\
\Delta\dot{x}(t) + 3x_0^2\left(1 + \frac{\Delta x(t)}{x_0}\right)^2 &= u_0 + \Delta u(t) + 3
\end{aligned}$$

Step 4 Expand out the only nonlinear term we have

$$\left(1 + \frac{\Delta x(t)}{x_0}\right)^2 \quad \approx \quad 1 + 2\frac{\Delta x(t)}{x_0}$$

Step 5 We now substitute the expanded term into the equation, and simplify as much as possible

$$\begin{aligned}
\Delta\dot{x}(t) + 3x_0^2\left[1 + 2\frac{\Delta x(t)}{x_0}\right] &\approx u_0 + \Delta u(t) + 3 \\
\Delta\dot{x}(t) + 3x_0^2 + 6x_0\Delta x(t) &\approx u_0 + \Delta u(t) + 3
\end{aligned}$$

Step 6 We have no higher order terms.

Step 7 From step 1, we have $3x_0^2 = u_0 + 3$. Substituting this into our equation from step 5 we have

$$\begin{aligned}\Delta\dot{x}(t) + [u_0 + 3] + 6x_0\Delta x(t) &\approx u_0 + \Delta u(t) + 3 \\ \Delta\dot{x}(t) + 6x_0\Delta x(t) &\approx \Delta u(t)\end{aligned}$$

Step 8 Taking Laplace transforms we have

$$s\Delta X(s) + 6x_0\Delta X(s) \approx \Delta U(s)$$

or

$$\frac{\Delta X(s)}{\Delta U(s)} \approx \frac{1}{s + 6x_0}$$

Example 2. Assume we have the model of a system with input $u(t)$ and output $x(t)$

$$2\dot{x}(t) + \sqrt{x(t)} = \cos(u(t))$$

and we want to find a linearized model about the static equilibrium point.

Step 1 At equilibrium we have the equation $\sqrt{x_0} = \cos(u_0)$.

Step 2 For the square root term we will assume the form $x(t) = x_0\left(1 + \frac{\Delta x(t)}{x_0}\right)$ while for the cosine term we will assume the form $u(t) = u_0 + \Delta u(t)$

Step 3 Now substitute into the dynamics and do some simplification

$$\begin{aligned}2\frac{d}{dt}\left[x_0\left(1 + \frac{\Delta x(t)}{x_0}\right)\right] + \sqrt{x_0\left(1 + \frac{\Delta x(t)}{x_0}\right)} &= \cos(u_0 + \Delta u(t)) \\ 2\Delta\dot{x}(t) + \sqrt{x_0}\sqrt{1 + \frac{\Delta x(t)}{x_0}} &= \cos(u_0 + \Delta u(t))\end{aligned}$$

Step 4 Expand out the nonlinear terms we have

$$\begin{aligned}\sqrt{1 + \frac{\Delta x(t)}{x_0}} = \left(1 + \frac{\Delta x(t)}{x_0}\right)^{\frac{1}{2}} &\approx 1 + \frac{1}{2}\frac{\Delta x(t)}{x_0} \\ \cos(u_0 + \Delta u(t)) &\approx \cos(u_0) - \Delta u(t)\sin(u_0)\end{aligned}$$

Step 5 We now substitute the expanded term into the equation, and simplify as much as possible

$$\begin{aligned}2\Delta\dot{x}(t) + \sqrt{x_0}\left[1 + \frac{1}{2}\frac{\Delta x(t)}{x_0}\right] &\approx \cos(u_0) - \Delta u(t)\sin(u_0) \\ 2\Delta\dot{x}(t) + \sqrt{x_0} + \frac{1}{2\sqrt{x_0}}\Delta x(t) &\approx \cos(u_0) - \Delta u(t)\sin(u_0)\end{aligned}$$

Step 6 We have no higher order terms.

Step 7 From step 1, we have $\sqrt{x_0} = \cos(u_0)$. Substituting this into our equation from step 5 we have

$$\begin{aligned} 2\Delta\dot{x}(t) + [\cos(u_0)] + \frac{1}{2\sqrt{x_0}}\Delta x(t) &\approx \cos(u_0) - \Delta u(t)\sin(u_0) \\ 2\Delta\dot{x}(t) + \frac{1}{2\sqrt{x_0}}\Delta x(t) &\approx -\Delta u(t)\sin(u_0) \end{aligned}$$

Step 8 Taking Laplace transforms we have

$$2s\Delta X(s) + \frac{1}{2\sqrt{x_0}}\Delta X(s) \approx -\Delta U(s)\sin(u_0)$$

or

$$\frac{\Delta X(s)}{\Delta U(s)} \approx \frac{-\sin(u_0)}{2s + \frac{1}{2\sqrt{x_0}}}$$

Example 3. Assume we have the model of a system with input $u(t)$ and output $x(t)$

$$\dot{x}(t) + \frac{1}{\sqrt{x(t)}}e^{-au(t)} = 1$$

and we want to find a linearized model about the static equilibrium point.

Step 1 At equilibrium we have the equation $\frac{1}{\sqrt{x_0}}e^{-au_0} = 1$.

Step 2 For the square root term we will assume the form $x(t) = x_0\left(1 + \frac{\Delta x(t)}{x_0}\right)$ while for the exponential term we will assume the form $u(t) = u_0 + \Delta u(t)$

Step 3 Now substitute into the dynamics and do some simplification

$$\begin{aligned} \frac{d}{dt}\left[x_0\left(1 + \frac{\Delta x(t)}{x_0}\right)\right] + \frac{1}{\sqrt{x_0\left(1 + \frac{\Delta x(t)}{x_0}\right)}}e^{-au_0 - a\Delta u(t)} &= 1 \\ \Delta\dot{x}(t) + \frac{e^{-au_0}}{\sqrt{x_0}}\frac{e^{-a\Delta u(t)}}{\sqrt{1 + \frac{\Delta x(t)}{x_0}}} &= 1 \end{aligned}$$

Step 4 Expand out the nonlinear terms we have

$$\begin{aligned} \frac{1}{\sqrt{1 + \frac{\Delta x(t)}{x_0}}} = \left(1 + \frac{\Delta x(t)}{x_0}\right)^{-\frac{1}{2}} &\approx 1 - \frac{1}{2}\frac{\Delta x(t)}{x_0} \\ e^{-a\Delta u(t)} &\approx 1 - a\Delta u(t) \end{aligned}$$

Step 5 We now substitute the expanded term into the equation, and simplify as much as possible

$$\begin{aligned} \Delta\dot{x}(t) + \frac{e^{-au_0}}{\sqrt{x_0}}\left(1 - \frac{1}{2}\frac{\Delta x(t)}{x_0}\right)(1 - a\Delta u(t)) &\approx 1 \\ \Delta\dot{x}(t) + \frac{e^{-au_0}}{\sqrt{x_0}}\left(1 - \frac{1}{2}\frac{\Delta x(t)}{x_0} - a\Delta u(t) + \frac{a}{2}\frac{\Delta x(t)}{x_0}\Delta u(t)\right) &\approx 1 \end{aligned}$$

Step 6 We drop the product $\Delta x(t)\Delta u(t)$ (i.e., we assume it is zero), so we have

$$\begin{aligned} \Delta\dot{x}(t) + \frac{e^{-au_0}}{\sqrt{x_0}}\left(1 - \frac{1}{2}\frac{\Delta x(t)}{x_0} - a\Delta u(t)\right) &\approx 1 \\ \Delta\dot{x}(t) + \frac{e^{-au_0}}{\sqrt{x_0}} - \frac{e^{-au_0}}{\sqrt{x_0}}\frac{1}{2}\frac{\Delta x(t)}{x_0} - \frac{e^{-au_0}}{\sqrt{x_0}}a\Delta u(t) &\approx 1 \end{aligned}$$

Step 7 From step 1, we have $\frac{e^{-au_0}}{\sqrt{x_0}} = 1$. Substituting this into our equation from step 6 we have

$$\Delta\dot{x}(t) - \frac{1}{2}\frac{\Delta x(t)}{x_0} - a\Delta u(t) \approx 0$$

Step 8 Taking Laplace transforms we have

$$s\Delta X(s) - \frac{1}{2x_0}\Delta X(s) - a\Delta U(s) \approx 0$$

or

$$\frac{\Delta X(s)}{\Delta U(s)} \approx \frac{a}{s - \frac{1}{2x_0}}$$

A Matlab Commands

In this section I have listed some common Matlab commands and sections of code that you will be using on the homework problems. You will probably want to use the **help**, **doc**, and **lookfor** commands to learn more about theses various functions and commands as you go on though this course. We will only go over some very simple uses of the commands here.

A.1 Figures

The first time you tell Matlab to plot something, it opens a new window and produces a graph. Matlab's default is to plot each graph in the same window, overwriting the previous graph. The **figure** command is given before plotting a new graph to tell Matlab to open a new window for a new graph.

A.2 Transfer Functions

We will make extensive use of transfer functions in this course, so we need to know how to enter them into Matlab. In general, to enter a polynomial such as

$$as^4 + bs^3 + cs^2 + ds + e$$

into Matlab, type

```
poly = [a b c d e];
```

where the powers are implied, only the coefficients are entered. (The semicolon at the end tell Matlab not to regurgitate what you just told it.) Hence, if we have a rational transfer function, such as

$$H(s) = \frac{s^3 + 2s}{s^4 + 3s^3 + s + 5}$$

we can enter the numerator and denominator polynomials separately, as

```
num = [1 0 2 0]; den = [1 3 0 1 5];
```

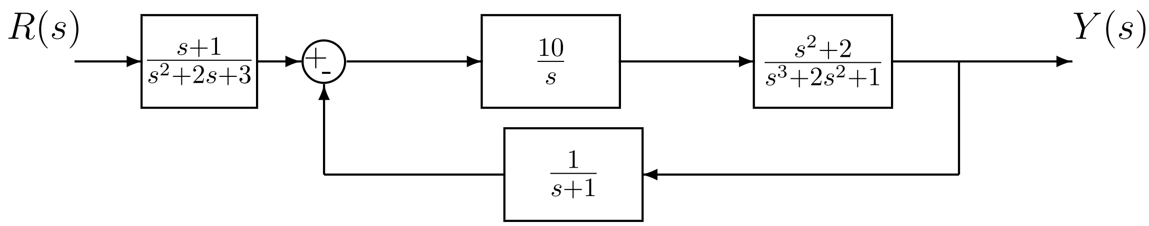
We will usually need to construct the transfer functions explicitly. To do this, type

```
H = tf(num,den)
```

This, without the semicolons, should display the transfer function, so you can check that you entered the correct function. In fact, at any time you can just type H to have Matlab display what the transfer function is.

A.3 Feedback Systems

Let's assume we want to find the closed loop transfer function for the following system using Matlab,



We first need to define all of the transfer functions

```
Gpre = tf([1 1],[1 2 3]);
Gc = tf(10,[1 0]);
Gp = tf([1 0 2],[1 2 0 1]);
H = tf(1,[1 1]);
```

Next, we compute the transfer function for the feedback block using the **feedback** command

```
T = feedback(Gc*Gp,H);
```

Finally we add the prefilter to get the close loop transfer function

```
G0 = Gpre*T;
```

A.4 System Response to Arbitrary Inputs

We will make extensive use both the unit step response and the unit ramp response of a system in this course. For the unit step response, we assume the system is at rest and the input is $u(t) = 1$ (a constant) for all $t \geq 0$, while for the unit ramp response, we assume the system is at rest and the input is $u(t) = t$ for all $t \geq 0$.

The simplest way to determine the step response to a system is

```
step(H);
```

A figure will appear on the screen, with the step response of the system. Note that the system will determine what it thinks are appropriate parameters. Sometimes, we want more control and want different inputs other than a step. In that case we use the command **lsim**. There are many forms for this command. In its basic form, you need to tell it a transfer function, the input function 'u', and the sample times 't'. For example, the following sequence of commands plots the response of the system

$$H(s) = \frac{1}{s^2 + 2s + 1}$$

which is initially at rest (the initial conditions are 0) to an input of $\cos(3t)$ from 0 to 100 seconds in increments of 0.05 seconds and then plots the output.

```
num=[1]; den=[1 2 1];
H = tf(num,den);   % get the transfer function
t=[0:0.05:100];    % times from 0 to 100 seconds by increments of 0.05
u = cos(3*t);      % input is cos(3t) at the sampe times
y=lsim(H,u,t);     % system output is y
plot(t,y);         % plot the output
```

We can (obviously) use the **lsim** command to determine the step response,

```
num=[1]; den=[1 2 1];
H = tf(num,den);   % get the transfer function
t=[0:0.05:100];    % times from 0 to 100 seconds by increments of 0.05
nt = length(t);    % get the length of the t array
u = ones(1,nt);    % input is a sequence of 1's
y=lsim(H,u,t);     % system output is y
plot(t,y);         % plot the output
```

The following piece of code will plot the step response of system H, showing both the system response and the input (we generally want the system to track the input), with neat labelling.

```
%
%  The Step Response
%
t = [0:0.1:10];              % time from 0 to 10 in increments of 0.1
u = ones(1,length(t));       % the input is a sequence of 1's
y = lsim(H,u,t);            % sumulate the friggin system
figure;                      % set up a new figure (window)
plot(t,y,'-',t,u,'.-');      % plot the system response/input on one graph
grid;                        % put on a grid;
title('Step Response of H');       % put on a title
xlabel('Time (Seconds)');             % put on an x axis label
legend('Step Response','Unit Step'); % put on a legend
```

A.5 Changing the Line Thickness

As you hopefully have figured out, Matlab allows you to chose the colors for your graphs. However, sometimes you do not have access to a color printer, or just want to do something different. The following section of code allows you to plot using different line thicknesses.

```
%
%  Now do line thickness
%
figure;
hold on                         % this basically means everything else is on one graph
plot(t,y,'-','Linewidth',4);    % make the linewidth 4 (really quite large)
```

```
plot(t,u,'-','Linewidth',0.2);  % make the linewidth 0.2 (really quite small)
legend('output','input'); grid;
hold off                         % we are done with this graph
%
```

You should note that even though you are changing the line width, you can still chose both the type of line to draw (dashed, dotted, etc) and the color. Also, this may not look so good on the screen, but usually prints out much better with a reasonable quality printer. Also, sometimes **hold on** and **hold off** can act really weird when you are doing many graphs. This is particularly true if you forgot the **hold off**.

A.6 Poles and Zeros

For any transfer function, the **poles** of the system are the roots of the denominator polynomial, while the **zeros** of the system are the roots of the numerator polynomial. Hence, if we have a transfer function

$$G(s) \quad = \quad \frac{(s+1)(s-1)}{(s+2)^2(s+3)(s+4)}$$

the poles of the system are at -2 (repeated), -3, and -4 while the zeros of the system are at -1, +1 (and ∞, but we don't usually talk about this). The poles of the transfer function are the same as the eigenvalues of the system. We care about the poles of the system since they indicate how fast the system will respond and the bandwidth of the system. The commands **pole(G)** and **zero(G)** will return the poles and zeros of transfer function G.

A.7 Roots and Polynomials

If we want the roots of a polynomial Q assigned to a variable r , we would use the Matlab command **roots**

```
r = roots(Q);
```

For example, if $Q(s) = s^3 + s + 1$ and we wanted the roots of $Q(s)$, we would type

```
Q = [1 0 1 1];
r = roots(Q);
```

and we would get an array

```
r =
    0.3412 + 1.1615i
    0.3412 - 1.1615i
   -0.6823
```

If we wanted to determine the polynomial with roots at $0.3412 \pm 1.1615j$, -0.6823 we would use the **poly** command

```
Q = poly([0.3412+1.1615*i 0.3412-1.1615*i -0.6823]);
```

or, in our case

```
Q = poly(r);
```

or

```
Q = poly([ r(1) r(2) r(3) ]);
```

If we want to polynomial with roots at $0.3412 \pm 1.1615j, -0.6823$ we can just type

```
Q = poly([ r(1) r(2) ]);
```

A.8 Root Locus Plots

To plot the root locus of a system with open loop transfer function $H(s)$, we use the **rlocus** command,

```
rlocus(H);
```

You will be able to click on a line and determine both the value of the gain K at that point and the corresponding closed loop pole values. If we want to know the values of the closed loop poles at a particular value of K, say $K = 10$, we type

```
r = rlocus(H,10)
```

A.9 Bode Plots, Gain and Phase Margins

To determine the gain and phase margin of a system with open loop transfer function $H(s)$, we use the **margin** command

```
margin(H)
```

To create the bode plot of a system with open loop transfer function $H(s)$, we use the **bode** command

```
bode(H)
```

There are a number of useful variations on the bode command. For example, if we want to view to bode plot over a specified range of frequencies, we type

```
w = logspace(0,2,100);     % create 100 logarithmically spaced points
                           %  between 1 (10^0) and 100 (10^2)
bode(H,w);
```

Sometimes we want the magnitude and phase of the transfer function $H(s)$. We can use the command

```
[Mag,Phase,w] = bode(H);
Mag = Mag(:);
Phase = Phase(:);
```

In this command, Matlab returns the magnitude (not in dB), phase,and frequencies the function was evaluated at, but the magnitude and phase are stored in a weird way. The command $Mag = Mag(:)$ forces Matlab to put them in a column. We can also specify which frequencies we want to evaluate the function at

```
[Mag,Phase] = bode(H,w);
```

If we then want to just plot the magnitude of the transfer function we can use

```
Mag = Mag(:);
Mag_dB = 20*log10(Mag);
semilogx(w,Mag_dB); grid;
```