ECE-320 Linear Control Systems
Homework 8

Due: Tuesday November 2, 2004

If matrix P is given as

$$P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then

$$P^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

and the *determinant* of P is given by $ad - bc$.

The general for for writing a continuous time state variable system is

$$\begin{aligned} \dot{\underline{x}}_(t) &= A\underline{x}(t) + B\underline{u}(t) \\ \underline{y}(t) &= C\underline{x}(t) + D\underline{u}(t) \end{aligned}$$

Now assume we are using state variable feedback, so that $\underline{u}(t) = k_{pf}\underline{r}(t) - \underline{kx}(t)$. Here $\underline{r}(t)$ is our new reference input, k_{pf} is a scaling factor, and $\underline{k} = [k_1\ k_2]$ is the feedback gain matrix. With this state variable feedback, we have the system

$$\dot{\underline{x}}(t) = A\underline{x}(t) + B(k_{pf}\underline{r}(t) - \underline{kx}(t))$$

or

$$\dot{\underline{x}}(t) = \tilde{A}\underline{x}(t) + \tilde{B}\underline{r}(t)$$

where $\underline{r}(t)$ is the new input. For $D = 0$, the transfer matrix is given by

$$G(s) = C\left[(sI - \tilde{A})^{-1}\right]\tilde{B}$$

For each of the systems below,

- determine the transfer function when there is state variable feedback
- determine if k_1 and k_2 exist to allow us to place the poles arbitrarily. That is, can we make the denominator look like $s^2 + a_1 s + a_2$ for any a_1 and any a_2.

You can use Maple to check your answers, but I expect you to be able to do this without Maple.

$\boxed{1}$ Let

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = [0\ 1]\,, D = 0$$

$Ans.\ G(s) = \frac{(s-1)k_{pf}}{(s-1)(s-1+k_2)}$

$\boxed{2}$ Let

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = [0\ 1]\,, D = 0$$

$Ans.\ G(s) = \frac{sk_{pf}}{s^2+(k_2-1)s+k_1}$

$\boxed{3}$ Let

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = [1\ 0]\,, D = 0$$

$Ans.\ G(s) = \frac{k_{pf}}{s^2+(k_2-1)s+(k_1-1)}$

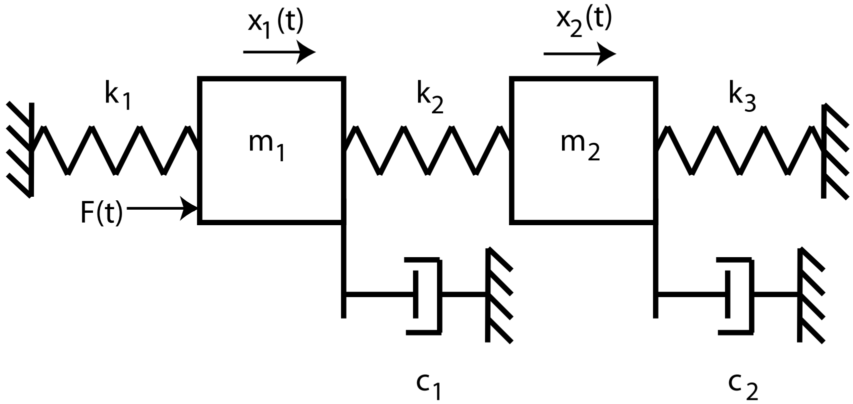
$\boxed{4}$ Let

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, C = [0\ 1]\,, D = 0$$

$Ans.\ G(s) = \frac{(s+1)k_{pf}}{(s+k_1)(s+k_2)-(k_1-1)(k_2-1)}$

5 Preparation for Lab:

Consider the following model of the two DOF system we will be using.



a) Draw freebody diagrams of the forces acting on the two masses.

b) The equations of motion for the two masses can be written

$$m_1\ddot{x}_1 + c_1\dot{x}_1 + (k_1+k_2)x_1 = F + k_2 x_2 \quad (1)$$

$$m_2\ddot{x}_2 + c_2\dot{x}_2 + (k_2+k_3)x_2 = k_2 x_1 \quad (2)$$

If we define $q_1 = x_1$, $q_2 = \dot{x}_1$, $q_3 = x_2$, and $q_4 = \dot{x}_2$, show that we get the following state equations

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\left(\frac{k_1+k_2}{m_1}\right) & -\left(\frac{c_1}{m_1}\right) & \left(\frac{k_2}{m_1}\right) & 0 \\ 0 & 0 & 0 & 1 \\ \left(\frac{k_2}{m_2}\right) & 0 & -\left(\frac{k_2+k_3}{m_2}\right) & -\left(\frac{c_2}{m_2}\right) \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m_1} \\ 0 \\ 0 \end{bmatrix} F$$

In order to get the A and B matrices, we need to determine all of the quantities in the above matrices.

c) Now we want to rewrite equations 1 and 2 in terms of ζ_1, ω_1, ζ_2, and ω_2 as

$$\ddot{x}_1 + 2\zeta_1\omega_1\dot{x}_1 + \omega_1^2 x_1 = \frac{k_2}{m_1}x_2 + \frac{1}{m_1}F \quad (3)$$

$$\ddot{x}_2 + 2\zeta_2\omega_2\dot{x}_2 + \omega_2^2 x_2 = \frac{k_2}{m_2}x_1 \quad (4)$$

We will get our initial estimates of the parameters ζ_1, ω_1, ζ_2 and ω_2 using the log-decrement method. Assuming we measure these parameters, show how $A_{2,1}$, $A_{2,2}$, $A_{4,3}$ and $A_{4,4}$ can be determined.

d) By taking the Laplace transforms of equations 3 and 4, show that we get the following transfer function

$$\frac{X_2(s)}{F(s)} = \frac{\left(\frac{k_2}{m_1 m_2}\right)}{(s^2 + 2\zeta_1\omega_1 s + \omega_1^2)(s^2 + 2\zeta_2\omega_2 s + \omega_2^2) - \frac{k_2^2}{m_1 m_2}} \tag{5}$$

e) It is more convenient to write this transfer function as

$$\frac{X_2(s)}{F(s)} = \frac{\left(\frac{k_2}{m_1 m_2}\right)}{(s^2 + 2\zeta_a\omega_a s + \omega_a^2)(s^2 + 2\zeta_b\omega_b s + \omega_b^2)} \tag{6}$$

By equating coefficients of powers of s in the denominators in these two transfer functions (equations 5 and 6), you should be able to write down four equations. The equations corresponding to the coefficients of s^3, s^2, and s do not seem to give us any new information, but they will be used later to get consistent estimates of ζ_1 and ω_1. The equation for the coefficients of s^0 will give us a new relationship for $\frac{k_2^2}{m_1 m_2}$ in terms of parameters we will be measuring.

f) We will actually be fitting the frequency response data to the following form of the transfer function:

$$\frac{X_2(s)}{F(s)} = \frac{K_2}{(\frac{1}{\omega_a^2}s^2 + \frac{2\zeta_a}{\omega_a}s + 1)(\frac{1}{\omega_b^2}s^2 + \frac{2\zeta_b}{\omega_b}s + 1)} \tag{7}$$

What is K_2 in terms of the parameters given in equation 6?

g) Using equation 6 and the Laplace transform of equation 4, show that we can write

$$\frac{X_1(s)}{F(s)} = \frac{\frac{1}{m_1}(s^2 + 2\zeta_2\omega_2 s + \omega_2^2)}{(s^2 + 2\zeta_a\omega_a s + \omega_a^2)(s^2 + 2\zeta_b\omega_b s + \omega_b^2)} \tag{8}$$

h) This equation is more convenient to write in the form

$$\frac{X_1(s)}{F(s)} = \frac{K_1(\frac{1}{\omega_2^2}s^2 + \frac{2\zeta_2}{\omega_2}s + 1)}{(\frac{1}{\omega_a^2}s^2 + \frac{2\zeta_a}{\omega_a}s + 1)(\frac{1}{\omega_b^2}s^2 + \frac{2\zeta_b}{\omega_b}s + 1)} \tag{9}$$

What is K_1 in terms of the quantities given in equation 8?

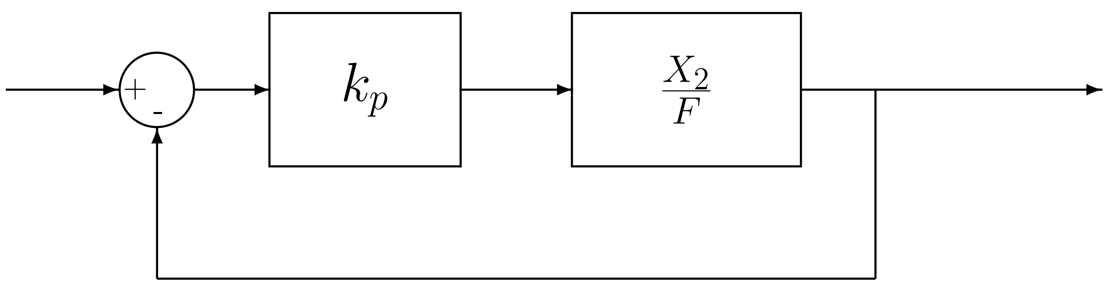
i) Show that

$$A_{4,1} = \frac{k_2}{m_2} = \frac{K_2}{K_1}\omega_2^2 \tag{10}$$

j) Show that

$$A_{2,3} = \frac{k_2}{m_1} = \frac{\omega_1^2\omega_2^2 - \omega_a^2\omega_b^2}{A_{4,1}} \tag{11}$$

k) All that's left is to find $\frac{1}{m_1}$, which is b_2. It's important to understand that this parameter also includes "scaling" on $F(s)$. Now assume we look at the closed loop response to a simple proportional type controller. Hence we have the system shown below:



For a step response of amplitude Amp, show that the steady state value of $x_2(t)$, $x_{2,ss}$ is

$$x_{2,ss} = \frac{K_2 k_p Amp}{1 + K_2 k_p} \tag{12}$$

(*Hint: It's easiest to use equation 7 for X_2/F*)

and that by rearranging this equation we get

$$b_2 = \frac{1}{m_1} = \frac{x_{2,ss}}{k_p(Amp - x_{2,ss})}\frac{\omega_a^2\omega_b^2}{A_{4,1}} \tag{13}$$

Summary

1) Fit frequency response data to

$$\frac{X_2(s)}{F(s)} = \frac{K_2}{(\frac{1}{\omega_a^2}s^2 + \frac{2\zeta_a}{\omega_a}s + 1)(\frac{1}{\omega_b^2}s^2 + \frac{2\zeta_b}{\omega_b}s + 1)} \tag{14}$$

this will give us estimates for K_2, ζ_a, ω_a, ζ_b, and ω_b.

2) Using the above parameters, fit frequency response data to

$$\frac{X_1(s)}{F(s)} = \frac{K_1(\frac{1}{\omega_2^2}s^2 + \frac{2\zeta_2}{\omega_2}s + 1)}{(\frac{1}{\omega_a^2}s^2 + \frac{2\zeta_a}{\omega_a}s + 1)(\frac{1}{\omega_b^2}s^2 + \frac{2\zeta_b}{\omega_b}s + 1)} \tag{15}$$

This will give us estimates for K_1, ζ_2, ω_2.

3) Using the relationships derived in part 8, find the values of ζ_1 and ω_1 consistent with the rest of the parameters. Note that if the estimate of ζ_1 is less than one fourth the estimate determined by the log-decrement method (initial estimate), the value of ζ_1 is set to one fourth the initial estimate.

4) Estimate all of the parameters in A.

5) Look at the step response to estimate b_2.