ECE-320 Linear Control Systems
Homework 7

Due: Tuesday October 26, 2004

1 Preparation for Lab

a) Assume we have the plant transfer function

$$G(s) = \frac{k_{clg}}{\frac{1}{\omega_n^2}s^2 + \frac{2\zeta}{\omega_n}s + 1}$$

If the input is $u(t)$ and the output is $x(t)$ we can represent this system with the differential equation

$$\frac{1}{\omega_n^2}\ddot{x}(t) + \frac{2\zeta}{\omega_n}\dot{x}(t) + x(t) = k_{clg}u(t)$$

Assume $q_1(t) = x(t)$ and $q_2(t) = \dot{x}(t)$. Show that in terms of these variables we can write a state variable description of the system as

$$\begin{aligned}
\frac{d}{dt}\begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix}\begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ k_{clg}\omega_n^2 \end{bmatrix} u(t) \\
y(t) &= [1\ 0]\underline{q}(t)
\end{aligned}$$

b) Assume we are using state variable feedback of the form $u(t) = k_{pf}r(t) - \underline{k}\underline{q}(t)$, so the new form of the system equations is

$$\begin{aligned}
\dot{\underline{q}}(t) &= (A - B\underline{k})\underline{q} + Bk_{pf}r(t) \\
y(t) &= C\underline{q}(t)
\end{aligned}$$

Assume the input $r(t)$ is a unit step with amplitude Amp. Show that in steady state for $y(t)$ to equal Amp we must choose the prefilter k_{pf} as (the ECP used k_{pf} for the prefilter)

$$k_{pf} = -1/[C(A - B\underline{k})^{-1}B]$$

You must recompute k_{pf} every time you change the state feedback gain vector $\underline{k}$!!!

c) Consider the system with transfer function

$$G_p(s) = \frac{9.29}{0.00087s^2 + 0.00118s + 1}$$

Determine the state variable model for this system (Find the A, B, C and D matrices.)

Now we need to choose the state feedback gain matrix $\underline{k}$. *We need to keep a few things in mind:*

- *The system must remain stable.*
- k_1 *is multiplying the position of the cart and* k_2 *is multiplying the derivative of the position, which is not measured directly but is calculated, and tends to be noisy. Hence, we want* $k_1 > k_2$.
- *To avoid positive feedback, we generally want* k_1 *and* k_2 *to be positive values. (Negative values will work fine in a simulation, but they tend to not work very well on our systems in the lab unless they have a small magnitude.)*
- *We need to try and keep the gains small to keep our motor from not frying.*

To determine if the system is stable, we could compute the transfer function for the close loop system and look at the poles, but you'll get to do that enough on next weeks homework, so , we'll be clever and assume we actually remember something from MA 221. Whenever we compute the transfer function, the denominator is the determinant of $sI - A$, *which means the poles of the closed loop system are actually equal to the eigenvalues of the* A *matrix. Hence, to determine the closed loop poles when we have state variable feedback, we need to compute the eigenvalues of* $A - B\underline{k}$.

We will use the program **state_variables_1cart.m** on the web site to simulate the state variable system both on this homework and in lab. The program **state_variables_1cart.m** has the following input arguments

- the amplitude of the step input (in cm)
- the system A matrix
- the system B matrix
- the system C matrix (D is assumed to be 0)
- the state feedback gain matrix $\underline{k} = [k_1 \ k_2]$.
- the length of time to run the simulation for.

- the filename containing the ECP data. If the ECP data is not yet available, enter '' (two single quotes).

d) By trial and error, find two different values of $\underline{k}$ so that

- Your system is stable (closed loop poles in the LHP)
- You system has a position error of less than 0.15
- Your system reaches steady state within 1 seconds (the faster the better)
- Your system has as little overshoot as you can manage
- k_2 is less than 0.05 and k_1 less than 1.0.

Simulate your system and turn in the plots.

An alternative method for determining the state feedback gains is to use the Linear Quadratic Regulator (LQR) method. The Linear Quadratic Regulator finds the gain $\underline{k}$ to minimize

$$J = \int_0^\infty \left[\underline{x}^T(t)Q\underline{x}(t) + u(t)Ru(t)\right] dt$$

where

$$\begin{aligned} \dot{\underline{x}}(t) &= A\underline{x}(t) + Bu(t) \\ u(t) &= -\underline{k}\underline{x}(t) \end{aligned}$$

Here Q is a (usually diagonal) matrix which penalizes deviations of states from their final values, and R is a scalar that penalizes gains that produce large control signals. For our second order systems, we are actually finding the state feedback gains to minimize

$$J = \int_0^\infty \left[q_{11}x_1(t)^2 + q_{22}x_2(t)^2 + Ru(t)^2\right] dt$$

This is very similar to the quadratic optimal control we already discussed in class for a single input single output system.

The Matlab routine **lqr** is used to estimate the feedback gains k_1 and k_2. The arguments to this routine are

- the A matrix of the system
- the B matrix of the system
- a penalty matrix Q
- a penalty matrix R

(Note there is one more possible argument, but we won't use it. Type help lqr for more information). In our case Q is a two by two positive definite matrix, and R is a scalar. Since Q is most likely a diagonal matrix, it's easiest to iterate using the following command in Matlab

```
> K = lqr(A,B,diag([q11 q22]),R)
```

where $q11$ and $q22$ are the desired diagonal elements of Q and R is a scalar. In general, as R gets larger (it may have to get very large), the size of the gains goes down. Also if the cart seems to move too much (i.e. has a large overshoot) we make $q11$ much larger than $q22$.

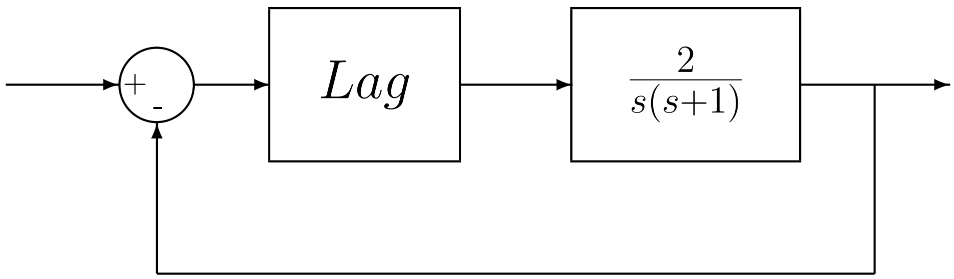
e) Using the LQR method, find two different values of $\underline{k}$ so that

- Your system is stable (closed loop poles in the LHP)
- You system has a position error of less than 0.15
- Your system reaches steady state within 1 seconds (the faster the better)
- Your system has as little overshoot as you can manage
- k_2 is less than 0.05 and k_1 less than 1.0.

Simulate your systems and turn in your plot.

As the order of the system gets larger (we will eventually be making all three carts move), it becomes much more difficult to find values of the feedback gain $\underline{k}$ *by trial and error, and the* LQR *method is much better.*

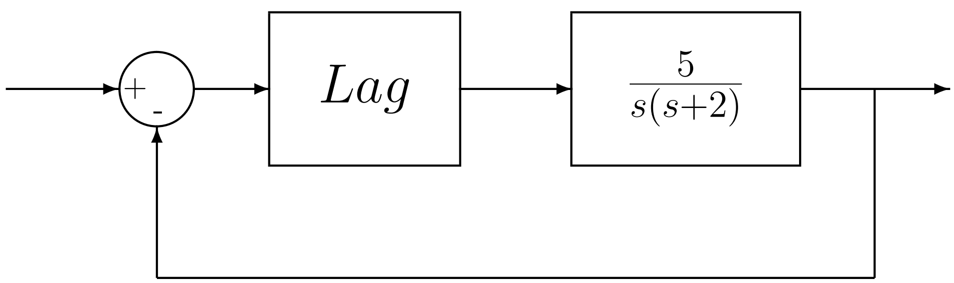
2 For the system shown below:



a) show that without the lag compensator, $K_v = 2$, and $e_v = 1/2$

b) Include the lag compensator, with $z = 0.01$, so the steady state velocity error will be 0.01, $e_v = 0.01$. (*Ans:* $p = 0.0002$)

3 For the system shown below:



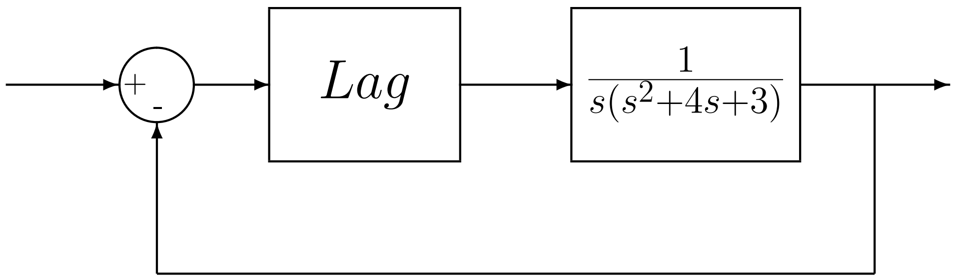
a) show that without the lag compensator, $e_p = 0$, $K_v = 5/2$, and $e_v = 2/5$

b) Include the lag compensator, with $z = 0.01$, so the steady state velocity error will be 0.01, $e_v = 0.01$. (*Ans:* $p = 0.00025$)

In the following problem, you need to simulate the closed loop system, given an open loop transfer function. Assume we have chosen values for z and p. Then to do the simulations in Matlab type something like:

```
G = tf(1,[1 4 3 0]);
To = feedback(G,1);    % original system with unity feedback
Gc = tf([1 z],[1 p]);
Tc = feedback(G*Gc,1); % compensated system with unity feedback
t = [0:0.01:35];
ustep = ones(1,length(t));
uramp = t;
%
% do the step response
%
yo = lsim(To,ustep,t);
yc = lsim(Tc,ustep,t);
%
orient tall % makes the figure take up the page, you might try orient landscape
%
subplot(2,1,1);
plot(t,yo,':',t,yc,'-',t,ustep,'.-');
grid; legend('Original','With Lag','Step');
title('Step Response'); xlabel('Time (s)');
%
%  do the ramp response
%
yo = lsim(To,uramp,t);
yc = lsim(Tc,uramp,t);
%
subplot(2,1,2);
plot(t,yo,':',t,yc,'-',t,uramp,'.-');
grid; legend('Original','With Lag',Ramp');
title('Ramp Response'); xlabel('Time (s)');
```

4 For the system shown below:



a) Show $e_p = 0$, $K_v = 1/3$, and $e_v = 3$.

b) Add a lag compensator in series with the plant in the feedforward loop. Show that $K_v = z/3p$ when this lag compensator is included in the system.

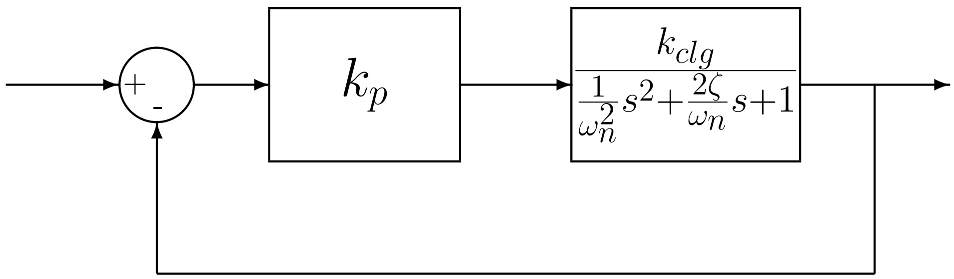
c) If we want $e_v = 0.1$ and $z = 0.1$, find the correct value for p. Plot, using Matlab, the step response for the original system and the system with the lag compensator *on the same plot*. Be sure to also plot the input signal. Then plot, using Matlab, the ramp response for the original system and the system using the lag compensator *on the same plot*. Be sure to also plot the input signal. For both plots, examine the response out to 35 seconds.

d) If we want $e_v = 0.1$ and $z = 0.01$, find the correct value for p. Plot, using Matlab, the step response for the original system and the system with the lag compensator *on the same plot*. Be sure to also plot the input signal. Then plot, using Matlab, the ramp response for the original system and the system using the lag compensator *on the same plot*. Be sure to also plot the input signal. For both plots, examine the response out to 35 seconds.

e) If we want $e_v = 0.1$ and $z = 0.001$, find the correct value for p. Plot, using Matlab, the step response for the original system and the system with the lag compensator *on the same plot*. Be sure to also plot the input signal. Then plot, using Matlab, the ramp response for the original system and the system using the lag compensator *on the same plot*. Be sure to also plot the input signal. For both plots, examine the response out to 35 seconds.

Note: You should notice that, for large z, the steady state velocity error is reduced much more quickly than with the smaller z, but the step response is also much worse. If the z is small, it takes much longer for the velocity error to be reduced, but the step response is not very different from that of the original system. You should have 6 graphs to turn in.

5 Consider the system below:



a) Show that the sensitivity of the closed loop system to k_{clg} is

$$S_{k_{clg}}^T(s) = \frac{s^2 + 2\zeta\omega_n s + \omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2(1 + k_p k_{clg})}$$

b) Show that the sensitivity of the closed loop system to ζ is

$$S_{\zeta}^T(s) = \frac{-2\zeta\omega_n s}{s^2 + 2\zeta\omega_n s + \omega_n^2(1 + k_p k_{clg})}$$

c) Show that the sensitivity of the closed loop system to ω_n is

$$S_{\omega_n}^T(s) = \frac{2s^2 + 2\zeta\omega_n s}{s^2 + 2\zeta\omega_n s + \omega_n^2(1 + k_p k_{clg})}$$

d) Plot the sensitivity of the closed loop system to these parameters as a function of frequency for $\omega = 1$ to 100 rad/sec for nominal values $k_p = 0.02$, $k_{clg} = 9$, $\omega_n = 27$, and $\zeta = 0.1$ All curves should be on one graph with different line types and a legend. To which parameter is the system most sensitive at low frequencies? At high frequencies?

If $T = \frac{2s}{s^2+2s+10}$, *we can plot the magnitude of the frequency response with the following*

```
T = tf([2 0],[1 2 10]);
w = logspace(0,2,1000);
[M,P] = bode(T,w);
Mdb = 20*log10(M(:));
semilogx(w,Mdb); grid;
```