Root Locus Rules

$\boxed{\text{1. Loci Branches}}$

$$\textbf{poles}\ (k = 0) \rightarrow \textbf{zeros}\ (k = \infty)$$

Continuous curves, which comprise the locus, start at each of the n poles of $G(s)H(s)$ for which $k = 0$. As k approaches ∞, the branches of the locus approach the m zeros of $G(s)H(s)$. Locus branches for excess poles extend to infinity.
The root locus is symmetric about the real axis.

$\boxed{\text{2. Real Axis Segments}}$

The root locus includes all points along the real axis to the left of an odd number of poles plus zeros of $G(s)H(s)$.

$\boxed{\text{3. Asymptotic Angles}}$

As $k \rightarrow \infty$, the branches of the locus become asymptotic to straight lines with angles

$$\theta = \frac{180^o + i360^o}{n - m}, \quad i = 0, \pm 1, \pm 2, ...$$

until all $(n - m)$ angles not differing by multiples of 360^o are obtained. n is the number of poles of $G(s)H(s)$ and m is the number of zeros of $G(s)H(s)$.

$\boxed{\text{4. Centroid of the Asymptotes}}$

The starting point on the real axis from the the asymptotic lines radiate is given by

$$\sigma = \frac{\sum_i p_i - \sum_j z_j}{n - m}$$

where p_i is the i^{th} pole of $G(s)H(s)$, z_j is the j^{th} zero of $G(s)H(s)$, n is the number of poles of $G(s)H(s)$ and m is the number of zeros of $G(s)H(s)$. This point is terms the centroid of the asymptotes.

$\boxed{\text{5. Leaving/Entering the Real Axis}}$

When two branches of the root locus leave or enter the real axis, they usually do so at angles of ± 90 degrees.

Table of Laplace Transforms

$f(t)$	$F(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
$tu(t)$	$\frac{1}{s^2}$
$\frac{t^{n-1}}{(n-1)!}u(t)\ (n = 1, 2, 3...)$	$\frac{1}{s^n}$
$t^n u(t)\ (n = 1, 2, 3, ...)$	$\frac{n!}{s^{n+1}}$
$e^{-at}u(t)$	$\frac{1}{s+a}$
$te^{-at}u(t)$	$\frac{1}{(s+a)^2}$
$\frac{1}{(n-1)!}t^{n-1}e^{-at}u(t)\ (n = 1, 2, 3, ...)$	$\frac{1}{(s+a)^n}$
$t^n e^{-at}u(t)\ (n = 1, 2, 3, ...)$	$\frac{n!}{(s+a)^{n+1}}$
$\sin(bt)u(t)$	$\frac{b}{s^2+b^2}$
$\cos(bt)u(t)$	$\frac{s}{s^2+b^2}$
$e^{-at}\sin(bt)u(t)$	$\frac{b}{(s+a)^2+b^2}$
$e^{-at}\cos(bt)u(t)$	$\frac{(s+a)}{(s+a)^2+b^2}$

Theorem *Strictly Proper Plant* Assume we have a strictly proper n^{th} order plant transfer function, $G_p(s) = N(s)/D(s)$. Since $G_p(s)$ is strictly proper we have the degree of $N(s) <$ the degree of $D(s)$. Since $G_p(s)$ is n^{th} order the degree of $D(s) = n$. Assume also that $N(s)$ and $D(s)$ have no common factors. Then for any polynomial $D_0(s)$ of degree $n+m$ a strictly proper controller $G_c(s) = B(s)/A(s)$ of degree m exists so that the characteristic equation of the resulting closed loop system is equal to $D_0(s)$. If $m = n-1$, the controller is unique. If $m \geq n$, the controller is not unique and some of the coefficients can be used to achieve other design objectives.

Theorem *Special case: degree $N(s)$ = degree $D(s)$.* Assume we have a proper n^{th} order plant transfer function, $G_p(s) = N(s)/D(s)$, where the degree of $D(s)$ = degree $N(s) = n$ Assume also that $N(s)$ and $D(s)$ have no common factors. Then for any polynomial $D_0(s)$ of degree $n+m$ a strictly proper controller $G_c(s) = B(s)/A(s)$ of degree m exists so that the characteristic equation of the resulting closed loop system is equal to $D_0(s)$. If $m = n$, the controller is unique. If $m \geq n+1$, the controller is not unique and some of the coefficients can be used to achieve other design objectives.

“Guidelines” for Phase Lead Compensator Design Using Bode Plots

The primary function of the lead compensator is to reshape the frequency response curve by adding phase to the system. The phase lead compensator also adds gain to the system.

$\boxed{1}$ Assume the compensator has the form

$$G_c(s) = K_c \frac{s+\frac{1}{T}}{s+\frac{1}{\alpha T}} = K_c \alpha \frac{Ts+1}{\alpha Ts+1} = K\frac{Ts+1}{\alpha Ts+1}$$

Determine K to satisfy the static error constant requirements (for e_p and e_v, etc.)

$\boxed{2}$ Using this value of K, draw the Bode diagram of $KG(s)H(s)$. Determine the phase margin.

$\boxed{3}$ Determine the necessary phase-lead angle to be added to the system. Add an additional 5^o to 12^o to the phase lead required, because the phase lead compensator shifts the phase crossover frequency to the right and decreases the phase margin. ϕ_m is then the total phase our compensator needs to add to the system.

$\boxed{4}$ Determine α using

$$\alpha = \frac{1-\sin(\phi_m)}{1+\sin(\phi_m)}$$

Determine the magnitude where $KG(j\omega)H(j\omega)$ is equal to $-20\log_{10}(\frac{1}{\sqrt{\alpha}}) = 10\log_{10}(\alpha)$. This is the new gain crossover frequency $\omega_m = \frac{1}{T\sqrt{\alpha}}$, or $T = \frac{1}{\omega_m\sqrt{\alpha}}$.

Note: If $\alpha < 0.05$, you will probably need two compensators. Choose a phase angle ϕ_m that produces an acceptable α. Finish the design, then treat $KG_c(s)G(s)H(s)$ as the system and go back to step $\boxed{2}$.

$\boxed{5}$ Determine the corner frequencies of the compensator as $z = \frac{1}{T}$ and $p = \frac{1}{\alpha T}$.

$\boxed{6}$ Determine $K_c = \frac{K}{\alpha}$.

$\boxed{7}$ Check the gain and phase margins to be sure they are satisfactory.