Name Solutions CM

ECE 300
Signals and Systems

Exam 3
17 February, 2009

This exam is closed-book in nature. Credit will not be given for work not shown. ***You may not use calculators!***

Problem 1	_______	/ 30
Problem 2	_______	/ 30
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Problem 4	_______	/ 10
Exam 3 Total Score:	_______	/ 100

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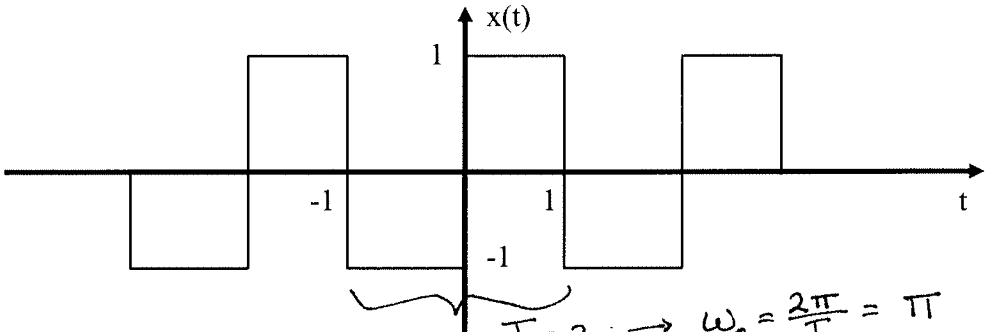
1. Fourier Series (30 points) The plot of x(t) can be used to answer parts (a) and (b) below.



$T_0 = 2 \longrightarrow \omega_0 = \frac{2\pi}{T_0} = \pi$

(a) Determine the Fourier Series coefficients for x(t). Do NOT express your answer as a sinc function, but do simplify your answer as much as possible.

$$c_k = \frac{1}{2}\left[\int_{-1}^{0} -e^{-jk\omega_0 t}\,dt + \int_{0}^{1} e^{-jk\omega_0 t}\,dt\right] = \frac{-1}{2jk\omega_0}\left[-e^{-jk\omega_0 t}\Big|_{-1}^{0} + e^{-jk\omega_0 t}\Big|_{0}^{1}\right]$$

$$= \frac{-1}{2jk\omega_0}\left[-(1-e^{jk\omega_0}) + (e^{-jk\omega_0}-1)\right] = \frac{-1}{2jk\omega_0}\left[e^{jk\omega_0}+e^{-jk\omega_0}-2\right]$$

$$= \frac{1}{jk\pi}\left[-\cos(k\pi)+1\right] = \frac{1}{jk\pi}\left[1-(-1)^k\right] \longrightarrow c_k = \begin{cases} 0 & k \text{ even} \\ \frac{2}{jk\pi} & k \text{ odd} \end{cases}$$

(b) You found the Fourier Coefficients of a waveform that is different from the one above to be as shown below. Express these coefficients as a sinc function with a possible gain phase term.

$$c_k = \frac{-3}{j2\pi k}\left[e^{-jk\frac{4\pi}{3}} - e^{-jk\frac{2\pi}{3}}\right] = \frac{-3}{j2\pi k} e^{-jk\frac{\frac{6\pi}{3}}{2}}\left[e^{-jk\frac{\frac{2\pi}{3}}{2}} - e^{jk\frac{\frac{2\pi}{3}}{2}}\right]$$

$$= \frac{3}{j2\pi k} e^{-jk\pi}\left[e^{jk\pi/3} - e^{-jk\pi/3}\right] = \frac{3}{\pi k} e^{-jk\pi}\sin\left(k\frac{\pi}{3}\right)$$

$$= e^{-jk\pi}\,\mathrm{sinc}\left(\frac{k}{3}\right)$$

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2. (**30 points**) An LTI system has the following transfer function

$$H(\omega)=\begin{cases}6e^{-j2\omega} & 4<|\omega|<6\\ 0 & \text{otherwise}\end{cases}$$

The input to the system is given by

$$x(t)=\operatorname{sinc}\left(\frac{5t}{\pi}\right)$$

a) What is $X(\omega)$?

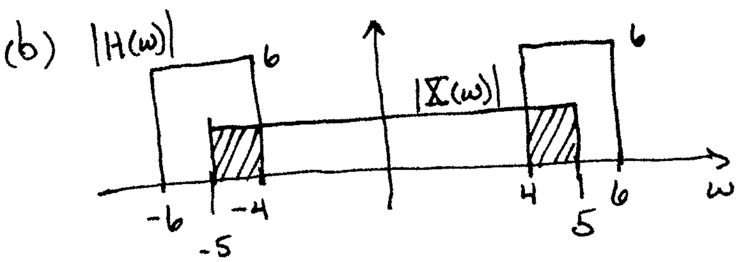
b) What is $y(t)$?

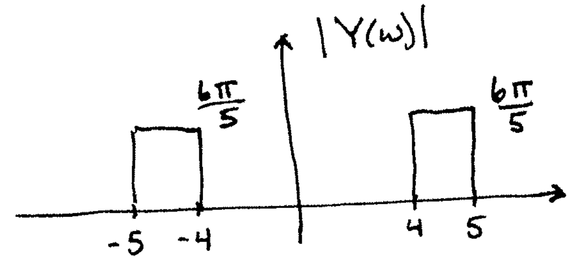
c) What is the energy in $x(t)$?

a) $x(t) = \operatorname{sinc}\left(\frac{5t}{\pi}\right) \longleftrightarrow X(\omega) = \frac{1}{W}\operatorname{rect}\left(\frac{\omega}{2\pi W}\right) = \frac{\pi}{5}\operatorname{rect}\left(\frac{\omega}{10}\right)$

$W = \frac{5}{\pi}$

(b)





$$Y(\omega) = |H(\omega)||X(\omega)|\,e^{j(\theta_x+\theta_H)}$$

$$= \frac{6\pi}{5}\left[\operatorname{rect}(\omega+4.5) + \operatorname{rect}(\omega-4.5)\right]e^{-j2\omega}$$

$$Y_1(\omega) = \operatorname{rect}(\omega) \longleftrightarrow y_1(t) = W\operatorname{sinc}(Wt) = \frac{1}{2\pi}\operatorname{sinc}\left(\frac{t}{2\pi}\right)$$

$2\pi W = 1$

$$Y_2(\omega) = \frac{12\pi}{5}\left[\frac{1}{2}Y_1(\omega+4.5) + \frac{1}{2}Y_1(\omega-4.5)\right] \longleftrightarrow y_2(t) = \frac{12\pi}{5}\,y_1(t)\cos(4.5t)$$

$$Y_3(\omega) = Y_2(\omega)\,e^{-j2\omega} \longleftrightarrow y_3(t) = y_2(t-2)$$

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(b) cont'd

$$y(t) = y_3(t) = \frac{12\pi}{5}\left(\frac{1}{2\pi}\right)\text{sinc}\left(\frac{t-2}{2\pi}\right)\cos\left(4.5\,(t-2)\right)$$

$$\boxed{= \frac{6}{5}\,\text{sinc}\left(\frac{t-2}{2\pi}\right)\cos\left(4.5t - 9\right)}$$

(c) $$E = \int_{-\infty}^{\infty} |x(t)|^2\,dt = \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2\,d\omega}_{\text{use this one}}$$

$$= \frac{1}{2\pi}\int_{-5}^{5}\left(\frac{\pi}{5}\right)^2 d\omega = \frac{1}{2\pi}\left(\frac{\pi^2}{25}\right)(10) = \boxed{\frac{\pi}{5}}$$

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3. (30 points) Determine $X(\omega)$ or $x(t)$. You must show your work. You may leave solutions in terms of integrals, derivatives, or convolutions (i.e. you do not have to explicitly solve them), but you must go further than the basic Fourier/Inverse Fourier Transform Definitions.

a) For $x(t) = e^{-4t^2}$ determine the corresponding Fourier transform $X(\omega)$

b) For $X(\omega) = \dfrac{2}{(1+j\omega)^2}$, determine the corresponding inverse Fourier transform $x(t)$

c) For $x(t) = \dfrac{2}{2+\left[\frac{2}{3}t-2\right]^2}$, determine the corresponding Fourier transform $X(\omega)$

(a) $\frac{1}{2\sigma^2} = 4 \quad \sigma^2 = \frac{1}{8} \quad X(\omega) = \frac{1}{\sqrt{8}}\sqrt{2\pi}\, e^{-\omega^2/16} = \boxed{\frac{\sqrt{\pi}}{2} e^{-\omega^2/16} = X(\omega)}$

(b) $X_1(\omega) = \frac{1}{1+j\omega} \Longleftrightarrow x_1(t) = e^{-t}u(t)$

$X(\omega) = 2X_1(\omega)X_1(\omega) \Longleftrightarrow x(t) = 2x_1(t) * x_1(t) = \boxed{2e^{-t}u(t) * e^{-t}u(t) = x(t)}$

(c) $x_1(t) = \frac{1}{\sqrt{2}} e^{-\sqrt{2}|t|} \Longleftrightarrow X_1(\omega) = \frac{2}{2+\omega^2}$

$x_2(t) = \frac{2}{2+t^2} = X_1(t) \Longleftrightarrow X_2(\omega) = 2\pi x_1(-\omega) = \sqrt{2}\,\pi e^{-\sqrt{2}|\omega|}$

$x_3(t) = x_2(t-2) = \frac{2}{2+(t-2)^2} \Longleftrightarrow X_3(\omega) = e^{-2j\omega}X_2(\omega) = \sqrt{2}\,\pi e^{-\sqrt{2}|\omega|} e^{-j2\omega}$

$x_4(t) = x_3\left(\frac{2}{3}t\right) = \frac{2}{2+\left(\frac{2}{3}t-2\right)^2} \Longleftrightarrow X_4(\omega) = \frac{3}{2}X_3\left(\frac{3\omega}{2}\right)$

$= \boxed{\frac{3\pi}{\sqrt{2}} e^{-\frac{3}{\sqrt{2}}|\omega|} e^{-j3\omega} = X(\omega)}$

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4. (10 points) Starting from the Fourier transform (or inverse transform) integrals,

$$X(\omega)=\int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega$$

If $x(t)\leftrightarrow X(\omega)$ then derive the corresponding Fourier Transform of $x(-(t-a))$

$$X(\omega) = \int_{-\infty}^{\infty} x(-t+a)\,e^{-j\omega t}\,dt$$

let $\lambda = -t+a$ $\quad d\lambda = -dt$ $\quad t = a-\lambda$

$$X(\omega) = \int_{+\infty}^{-\infty} x(\lambda)\,e^{-j\omega(a-\lambda)}(-d\lambda) = -\int_{+\infty}^{\infty} x(\lambda)e^{-j\omega a}e^{j\omega\lambda}\,d\lambda$$

$$= e^{-j\omega a}\int_{-\infty}^{\infty} x(\lambda)\,e^{-j(-\omega)\lambda}\,d\lambda = \boxed{e^{-j\omega a}X(-\omega)}$$