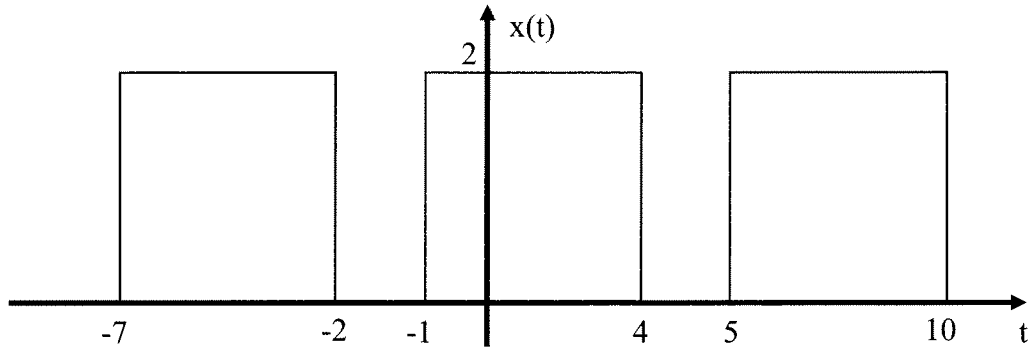
Name: ECE 300 Exam 2 Solns CM: W08-09

1. (30 points) The next few questions are based on the plot of x(t) below.



(a) What is the fundamental frequency of x(t) in r/s?

$$T_0 = 4-(-2) = 6$$

$$\omega_0 = \frac{2\pi}{T_0} = \frac{\pi}{3}$$

(b) Determine the Fourier Series coefficients for x(t). Express your answer in terms of a sinc function with a gain and phase term if necessary.

$$c_k = \frac{1}{T_0}\int_{-1.5}^{4.5} x(t)e^{-jk\omega_0 t}\,dt = \frac{2}{6}\int_{-1}^{4} e^{-jk\omega_0 t}\,dt = \frac{-1}{3jk\omega_0}\left[e^{-jk\omega_0 4} - e^{+jk\omega_0}\right]$$

$$= \frac{2}{3k\omega_0} e^{-j\frac{3}{2}k\omega_0}\left[\frac{e^{j5/2\,k\omega_0} - e^{-j5/2\,k\omega_0}}{2j}\right] = \frac{2}{3k\frac{2\pi}{6}} e^{-j\frac{3}{2}k\frac{2\pi}{6}} \sin\left(\frac{5}{2}k\frac{2\pi}{6}\right)$$

$$= \frac{2}{k\pi} e^{-j\frac{k\pi}{2}} \sin\left(\frac{10\pi k}{12}\right) = \frac{\frac{10}{12}}{\frac{10}{12}}\frac{2}{k\pi} e^{-jk\pi/2}\sin\left(\pi\frac{10}{12}k\right)$$

$$= \frac{20}{12} e^{-jk\pi/2}\,\mathrm{sinc}\left(\frac{10}{12}k\right)$$

(c) If a periodic signal x(t) is inserted into a system given by $y(t) = x(t+1.5) - 1$, express the Fourier coefficients of the output y(t), c_k^y, in terms of the input Fourier coefficients of x(t), c_k^x. This problem is independent of part (b).

$$x(t) = \sum_k c_k^x e^{jk\omega_0 t}$$

$$x(t+1.5) = \sum_k c_k^x e^{jk\omega_0(t+1.5)} = \sum_k c_k^x e^{jk\omega_0 1.5} e^{jk\omega_0 t}$$

$c_0^y = c_0^x - 1$

$c_k^y = c_k^x e^{jk\omega_0 1.5}$

$$y(t) = c_0^y + \sum_{k\neq 0} c_k^y e^{jk\omega_0 t} = x(t+1.5) - 1 = -1 + c_0^x + \sum_{k\neq 0} c_k^x e^{jk\omega_0 1.5} e^{jk\omega_0 t}$$

Name ______________________________ CM__________

2. (25 points) A periodic signal $x(t)$ is the input to an LTI system with output $y(t)$. The signal $x(t)$ has period 2 seconds, and is given over one period as

$$x(t) = e^{-t} \quad 0 < t < 2$$

$x(t)$ has the Fourier series representation

$$x(t) = \sum_k \frac{0.4323}{1 + jk\pi} e^{jk\pi t}$$

The system is an ideal lowpass filter with unit amplitude that eliminates all signals with frequency content higher than 1.25 Hz.

a) Find the average power in $x(t)$.

b) Determine an expression for the output, $y(t)$. Your expression for $y(t)$ must be real, and written in terms of sines and/or cosines.

c) Determine the average power in $y(t)$. *Be sure to show what you are calculating, not just the final answer!*

(a) $P_{ave}^{x} = \frac{1}{T_0}\int_{T_0} |x(t)|^2 dt = \frac{1}{2}\int_0^2 e^{-2t} dt = \frac{1}{4} e^{-2t}\Big|_0^2 = \frac{1-e^{-4}}{4} = \boxed{0.2454 = P_{ave}^{x}}$

(b) $f_0 = \frac{1}{2} \quad c_0^x = 0.4323 \quad c_1^x = 0.1311 \angle -72.34^\circ \quad c_2^x = 0.0679 \angle -80.96^\circ$

$y(t) = 0.4323 + 2(0.1311)\cos(\pi t - 72.34^\circ) + 2(0.0679)\cos(2\pi t - 80.96^\circ)$

$\boxed{y(t) = 0.4323 + 0.2622\cos(\pi t - 72.34^\circ) + 0.1358\cos(2\pi t - 80.96^\circ)}$

(c) $P_{ave}^{y} = (c_0^y)^2 + 2|c_1^y|^2 + 2|c_2^y|^2$

$= (0.4323)^2 + 2(0.1311)^2 + 2(0.0679)^2 = \boxed{0.2305 = P_{ave}^{y}}$

Name ______________________________ CM __________

3. (20 points) A periodic signal $x(t)$ with period $T_0 = 3$ seconds is the input to an LTI system with transfer function $H(j\omega)$. The relevant non-zero spectra are shown below. Determine an expression for the output of the system $y(t)$. All angles are multiples of 45 degrees. Your final answer must be in terms of sines and/or cosines. *You do not need to simplify you answer.*

$C_0^x = -2$

$C_1^x = 1\angle 90^\circ$

$C_2^x = 0.5\angle 45^\circ$

$C_3^x = 1\angle 135^\circ$

$H(0) = 1$

$H(\omega_0) = 2\angle 0^\circ$

$H(2\omega_0) = 3\angle -90^\circ$

$H(3\omega_0) = 4\angle 45^\circ$

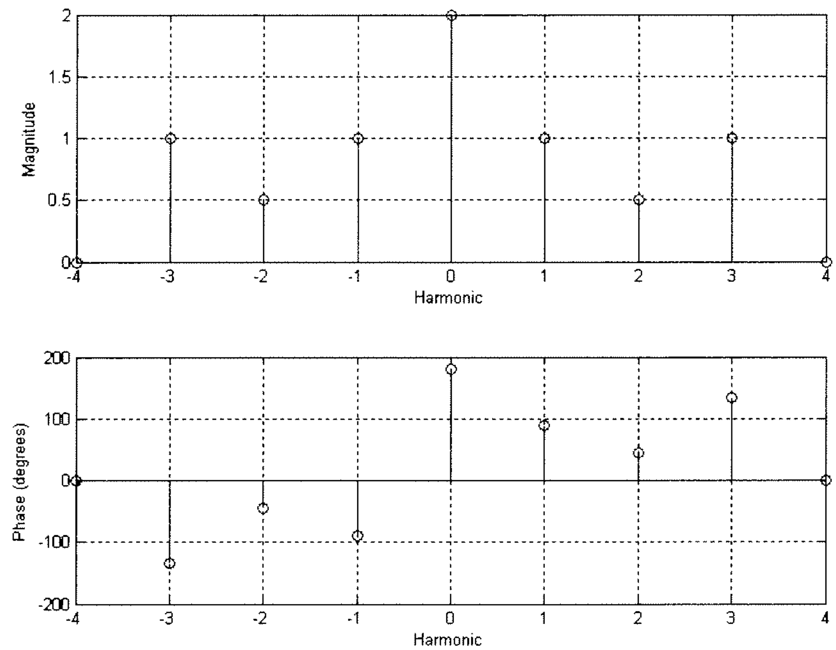


Figure 1. Spectrum of the input signal $x(t)$.

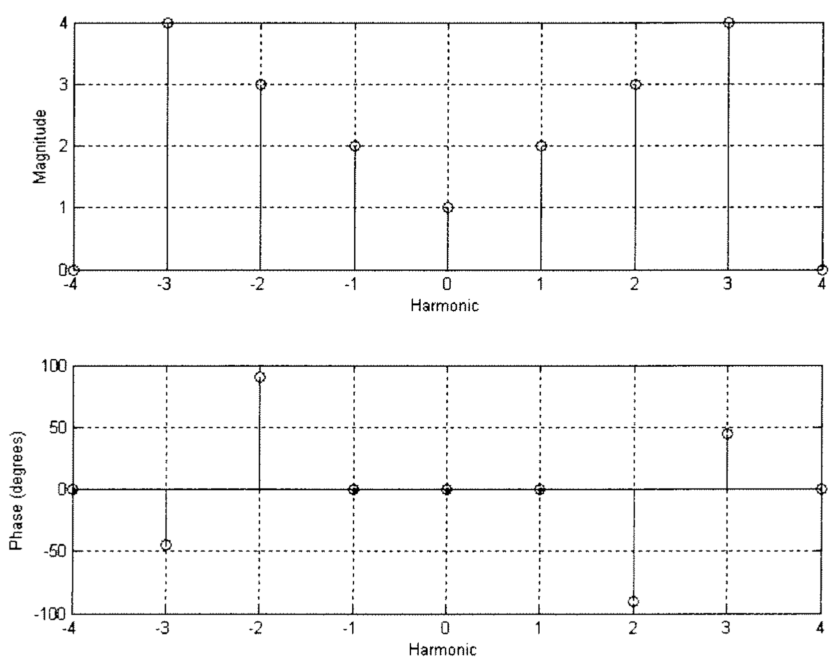


Figure 2. Samples of the transfer function $H(jk\omega_0)$

$$\boxed{\begin{aligned} y(t) = -2 &+ 2(1)(1)\cos\left(\tfrac{2\pi}{3}t + 90^\circ\right) + 2(0.5)(3)\cos\left(\tfrac{4\pi}{3}t + 45^\circ - 90^\circ\right) \\ &+ 2(1)(4)\cos\left(\tfrac{6\pi}{3}t + 45^\circ + 135^\circ\right) \end{aligned}}$$

Name ____ CM___________

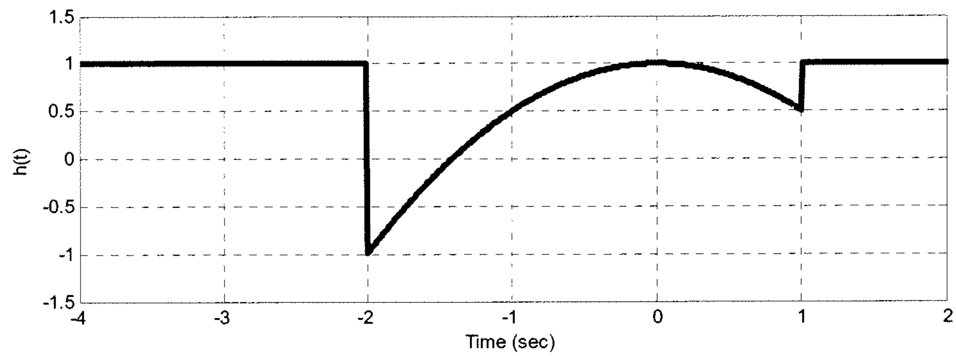
4. (25 points) Consider a causal linear time invariant system with impulse response

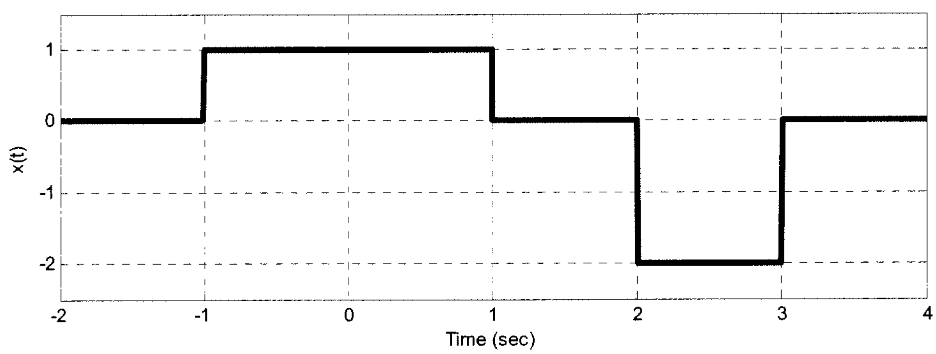
$$h(t) = \left[1 - 0.5t^2\right]\left[u(t+2) - u(t-1)\right]$$

The input to the system is

$$x(t) = u(t+1) - u(t-1) - 2u(t-2) + 2u(t-3)$$

These two functions are plotted below:





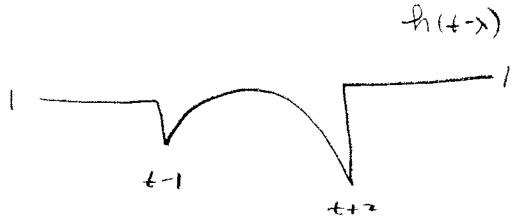
Using ***graphical convolution***, set up the integrals to determine the output $y(t)$ **for** $t \leq 1$. **Note that we are only interested in a limited range!!**

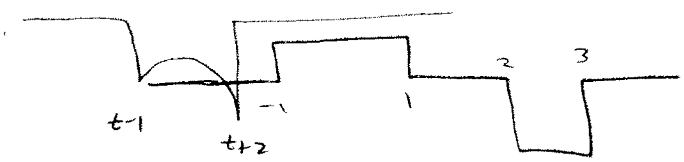
Specifically, you must

- Flip and slide $h(t)$
- Show graphs displaying $h(t-\lambda)$ relative to $x(\lambda)$ for each region of interest.
- Determine the ranges of t for which each part of your solution is valid.
- Set up any necessary integrals to compute $y(t)$. Your integrals must be complete and simplified as much as possible (no unit step functions)
- ***Do Not Evaluate the Integrals!!***

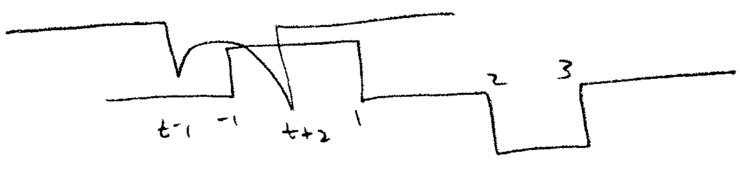
Name ______________________________ CM ____________

Solution (as drawn)

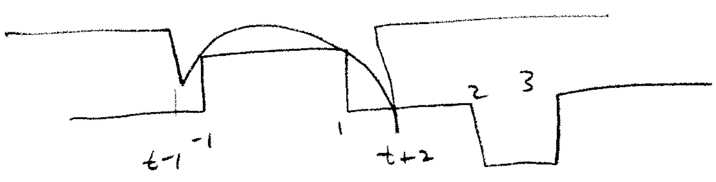




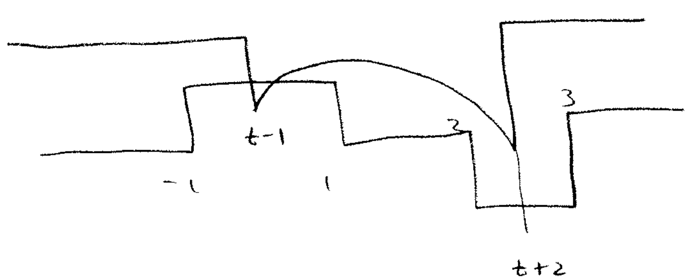
$t \le -3 \quad y(t) = \int_{-1}^{1} (1)\, d\lambda + \int_{2}^{3} (-2)\, d\lambda$



$-3 \le t \le -1 \quad y(t) = \int_{-1}^{t+2} [1 + 0.5(t-\lambda)^2]\, d\lambda + \int_{t+2}^{1} (1)\, d\lambda + \int_{2}^{3} (-2)\, d\lambda$



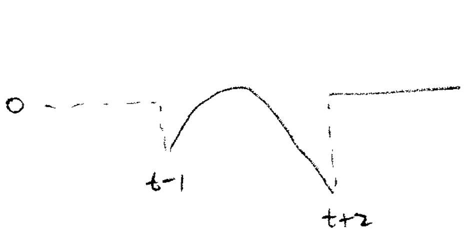
$-1 \le t \le 0 \quad y(t) = \int_{-1}^{1} [1 + 0.5(t-\lambda)^2]\, d\lambda + \int_{2}^{3} (-2)\, d\lambda$

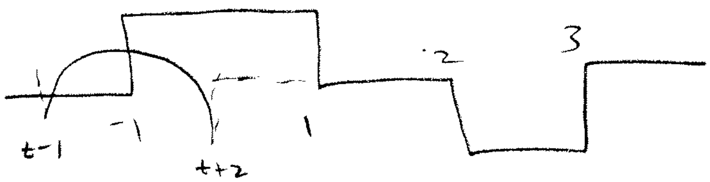


$0 \le t \le 1 \quad y(t) = \int_{-1}^{t-1} (1)\, d\lambda + \int_{t-1}^{1} [1 + 0.5(t-\lambda)^2]\, d\lambda + \int_{2}^{t+2} (-2)[1 + 0.5(t-\lambda)^2]\, d\lambda + \int_{t+2}^{3} (-2)\, d\lambda$

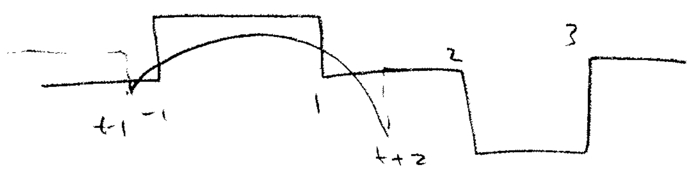
Name ____ CM____________

Solution (as written)

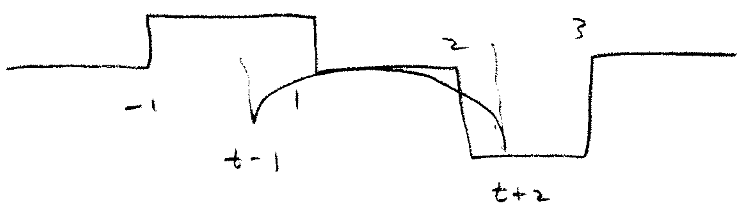




$-3 \le t \le -1 \quad y(t) = \int_{-1}^{t+2} \left[1 - 0.5(t-\lambda)^2\right] d\lambda$



$-1 \le t \le 0 \quad y(t) = \int_{-1}^{1} \left[1 - 0.5(t-\lambda)^2\right] d\lambda$



$0 \le t \le 1 \quad y(t) = \int_{t-1}^{1} \left[1 - 0.5(t-\lambda)^2\right] d\lambda$

$+ \int_{2}^{t+2} \left[1 - 0.5(t-\lambda)^2\right](-2)\, d\lambda$