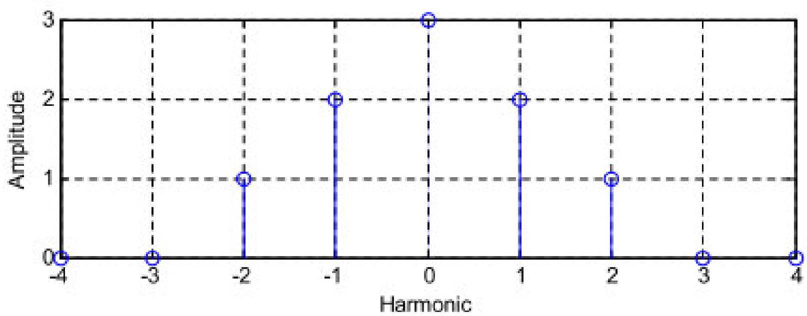
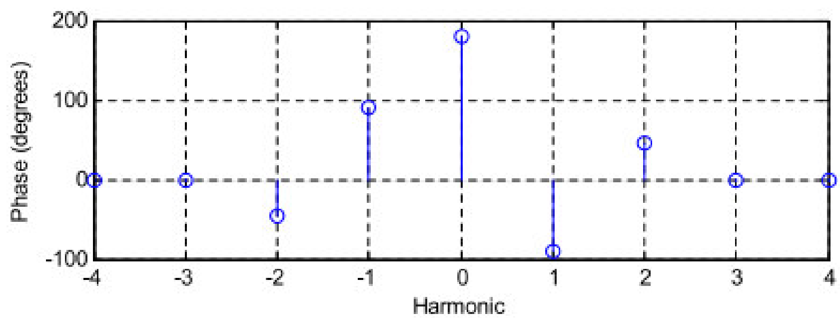
ECE300 Hw 6 Solutions W0809

Tuesday, January 27, 2009
9:22 AM

1. Assume $x(t)$, which has a fundamental period of 2 seconds, has the following spectrum (all phases are multiples of 45 degrees)





a) What is $x(t)$? Your expression must be real.

b) What is the average value of $x(t)$?

c) What is the average power in $x(t)$?

a) To be "real" means no $e^{jk\omega_o t}$ in expression for $x(t)$. So use

$$x(t) = C_o + \sum_{k=1}^{\infty} 2|c_k| \cos(k\omega_o t + \theta_k)$$

from the spectrum:

$$C_o = 3e^{j\pi} = -3$$

$$C_1 = 2e^{-j90^\circ}$$

$$C_2 = 1e^{j45^\circ}$$

$$C_3 = C_4 = 0$$

Also given $T_o = 2$ s $\rightarrow \omega_o = \frac{2\pi}{T_o} = \pi$ r/s

$$x(t) = -3 + 2|C_1|\cos(\pi t + \theta_1) + 2|c_2|\cos(2\pi t + \theta_2)$$

$$= -3 + 4\cos(\pi t - 90^\circ) + 2\cos(2\pi t + 45^\circ)$$

(b) The average value of $x(t)$ is given by C_0

$$C_0 = -3$$

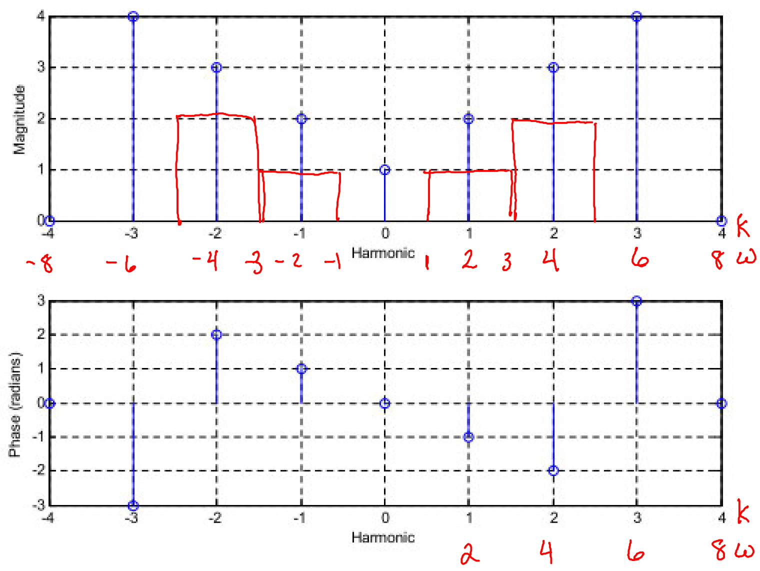
(c) The average Power in $x(t)$ is given by

$$P_{ave} = \frac{1}{T_0}\int_0^{T_0} |x(t)|^2 dt = \sum_k |C_k|^2 = |C_0|^2 + \sum_{k=1}^{\infty} 2|C_k|^2$$

Because the bandwidth of $x(t)$ is limited (for $k>2$ all $C_k=0$) this is easiest to find w/ C_k values

$$P_{ave} = 3^2 + 2(2^2) + 2(1)^2 = 9 + 8 + 2 = \boxed{19\ W}$$

2. Assume $x(t)$ has the spectrum shown below (the phase is shown in radians) and a fundamental frequency $\boxed{\omega_o = 2 \text{ rad/sec}:}$



$|H(\omega)|$ is in Red

Assume $x(t)$ is the input to a system with the transfer function

$$H(\omega) = \begin{cases} e^{-j\omega} & 1 \le |\omega| < 3 \\ 2e^{-j2\omega} & 3 < |\omega| < 5 \\ 0 & else \end{cases} \longrightarrow \text{only } k = 1, 2 \text{ terms pass thru filter}$$

Determine an expression for the steady state output $y(t)$. Be as specific as possible, simplifying all values and using actual numbers wherever possible.

$$y(t) = 2|C_1^x||H(\omega_o)| \cos(\omega_o t + \theta_{C_1^x} + \theta_{H(\omega_o)}) + 2|C_2^x||H(2\omega_o)| \cos(2\omega_o t + \theta_{C_2^x} + \theta_{H(2\omega_o)})$$

$$= 2(2)(1)\cos(2t - 1 - 2) + 2(3)(2)\cos(4t - 2 - 8)$$

$$= 4\cos(2t - 3) + 12\cos(4t - 10)$$

3. A periodic signal $x(t)$ is the input to an LTI system with output $y(t)$. The signal $x(t)$ has period 2 seconds, and is given over one period as

$$x(t) = e^{-t} \quad 0 < t < 2$$

$x(t)$ has the Fourier series representation

$$x(t) = \sum_k \frac{0.4323}{1 + jk\pi} e^{jk\pi t}$$

The system is an ideal highpass filter that eliminates all signals with frequency content less than 0.75 Hz.

a) Find the average power in $x(t)$.

b) Determine an expression for the output, $y(t)$. Your expression for $y(t)$ must be real.

(*Answer:* $y(t) = e^{-t} - 0.4323 - 0.2622\cos(\pi t - 1.2626)$)

c) Determine the average power in $y(t)$.

d) What fraction of the average power in $x(t)$ is contained in the DC and fundamental frequency components?

a) To find P_{ave} in $x(t)$

$$P_{ave} = \frac{1}{T_0}\int_0^{T_0} |x(t)|^2\,dt = \sum_k |C_k|^2$$

Note that because $C_k \neq 0$ for all k, this route would require an infinite sum. So use time domain instead.

$$P_{ave} = \frac{1}{2}\int_0^2 e^{-2t}\,dt = \frac{1}{4}\int_{-4}^{0} e^{u}\,du = \frac{1}{4}\left[e^0 - e^{-4}\right] = 1 - e^{-4} = 0.245\text{ W}$$

(b) An Ideal highpass filter has zero phase shift and a rectangular magnitude response with amplitude of 1. So it simply selects certain harmonics to pass through. In this case the cutoff freq is 0.75 Hz and $f_0 = 0.5$ Hz so the filter eliminates $k=0$ & $k=1$ terms from $x(t)$.

$$C_0 = 0.4323 \qquad C_1 = \frac{0.4323}{1+j\pi} = 0.1311\,e^{-j72^\circ}$$

$$y(t) = x(t) - C_0^x - 2|C_1^x|\cos(\omega_0 t + \theta_1^x) = \boxed{e^{-t} - 0.4323 - 0.2622\cos(\pi t - 72^\circ)}$$

(c) To find average power in $y(t)$ we already know the power for the e^{-t} term. We just need to subtract the power from the $k=0$ & $k=1$ terms of $x(t)$.

$$P_{ave} = 0.245 - |C_0^x|^2 - 2|C_1^x|^2 = \boxed{0.0241 \text{ W}}$$

(d) What fraction of average power in $x(t)$ is contained in DC and fundamental?

$$\frac{|C_0^x|^2 + 2|C_1^x|^2}{0.245} = 90\%$$

4. Assume $x(t) = t^2 \quad -\pi \le t \le \pi$ with Fourier Series representation

$$x(t) = \sum_k X_k e^{jkt}$$

where

$$X_k = \begin{cases} \dfrac{\pi^2}{3} & k = 0 \\ \dfrac{2(-1)^k}{k^2} & k \ne 0 \end{cases}$$

a) Assume $x(t)$ is the input to a system that eliminates all signals with frequencies outside the range 0.5 to 0.7 Hz. What is the output of the system $y(t)$ and what fraction of the average power in $x(t)$ is in $y(t)$? (Note: your answers must be real, no $e^{j\alpha}$ terms.)

b) Assume $x(t)$ is the input to a system that eliminates all signals with frequencies in the range 0.5 to 0.7 Hz. What is the output of the system $y(t)$ and what fraction of the average power in $x(t)$ is in $y(t)$?

(a) $T_0 = 2\pi \quad f_0 = \frac{1}{2\pi} = 0.159$ Hz $\longrightarrow$ So the filter will only pass the $k=4$ term at $4f_0 = 0.636$ Hz.

$3f_0 < 0.5$ Hz

$5f_0 > 0.7$ Hz.

$$C_4^x = \frac{2(-1)^4}{16} = \frac{2}{16} = \frac{1}{8} e^{j0}$$

$$y(t) = 2|C_4^x| \cos(4\omega_0 t + \theta_4^x) = 2\left(\frac{1}{8}\right)\cos(4t) = \frac{1}{4}\cos(4t)$$

$$P_{ave}^y = 2|C_4^x|^2 = 2\left(\frac{1}{64}\right) = \boxed{\frac{1}{32}}$$

$$P_{ave}^x = \frac{1}{2\pi}\int_{-\pi}^{\pi} t^4\,dt = \frac{1}{10\pi}\left[\pi^5 - (-\pi)^5\right] = \frac{2\pi^5}{10\pi} = \frac{\pi^4}{5} = 19.5\text{ W}$$

$$\frac{P_{ave}^y}{P_{ave}^x} = 0.16\ \%$$

(b) Now the filter eliminates only the $k=4$ term of $x(t)$

$$y(t) = x(t) - \frac{1}{4}\cos(4t)$$

The average Power in $y(t)$ can be found by

$$P_{ave}^{y} = 19.5\,W - 1/32 = 19.47\,W$$

$$P_{ave}^{x} = 19.5\,W$$

$$\frac{P_{ave}^{y}}{P_{ave}^{x}} = 99.8\%$$

5. Assume two periodic signals have the Fourier series representations

$$x(t) = \sum X_k e^{jk\omega_o t} \quad y(t) = \sum Y_k e^{jk\omega_o t}$$

For the following system (input/output) relationships:

a) $y(t) = bx(t-a)$
b) $y(t) = b\dot{x}(t-a)$
c) $y(t) = bx(t)\cos(\omega_o t)$ (Answer: $Y_n = \frac{b}{2}(X_{n-1} + X_{n+1})$)
d) $\ddot{y}(t) + 2\zeta\omega_n \dot{y}(t) + \omega_n^2 y(t) = K\omega_n^2 x(t)$

i) write Y_k in terms of the X_k
ii) If possible, determine the system transfer function $H(j\omega)$
iii) A system must be both linear and time-invariant to have a transfer function. If you cannot determine the transfer function, indicate which system property is not satisfied (**L** or **TI**).

(a) $y(t) = b\,x(t-a) = b \sum X_k e^{jk\omega_o(t-a)} = \sum b X_k e^{-jk\omega_o a} e^{jk\omega_o t}$

$= \sum Y_k e^{jk\omega_o t}$

$Y_k = b X_k e^{-jk\omega_o a}$

$H(j\omega) = \frac{Y_k}{X_k} = b e^{-jk\omega_o a}$

(b) $y(t) = b\,\dot{x}(t-a)$

$b\,x(t-a) = \sum b X_k e^{-jk\omega_o a} e^{jk\omega_o t}$

$b\,\dot{x}(t-a) = \sum jbk\omega_o X_k e^{-jk\omega_o a} e^{jk\omega_o t} = \sum Y_k e^{jk\omega_o t}$

$Y_k = jbk\omega_o X_k e^{-jk\omega_o a}$

$\frac{Y_k}{X_k} = j\,bk\omega_o\, e^{-jk\omega_o a}$

(c) $y(t) = b\,x(t)\cos(\omega_o t)$

$$= \frac{b}{2}\left(e^{j\omega_o t} + e^{-j\omega_o t}\right) \sum X_k e^{jk\omega_o t}$$

$$= \sum \frac{b}{2} X_k \left[e^{j(k+1)\omega_o t} + e^{j(k-1)\omega_o t}\right]$$

$$= \sum \frac{b}{2} X_{k-1} e^{jk\omega_o t} + \sum \frac{b}{2} X_{k+1} e^{jk\omega_o t}$$

$$= \sum \frac{b}{2}\left[X_{k-1} + X_{k+1}\right] e^{jk\omega_o t}$$

$$Y_k = \frac{b}{2}\left[X_{k-1} + X_{k+1}\right]$$

This is a linear system b/c we can write $\dfrac{Y_k}{X_{k-1}+X_{k+1}}$ but it is time varying, so we cannot write Y_k/X_k

(d) $\ddot{y} + 2\zeta\omega_n \dot{y} + \omega_n^2 y = x K \omega_n^2$

$\dot{y} = \sum jk\omega_o Y_k e^{jk\omega_o t}$ $\qquad \ddot{y} = \sum -(k\omega_o)^2 Y_k e^{jk\omega_o t}$

$$\sum\left[-(k\omega_o)^2 + 2\zeta\omega_n(jk\omega_o) + \omega_n^2\right] Y_k e^{jk\omega_o t} = \sum K\omega_n^2 X_k e^{jk\omega_o t}$$

$$\left[-(k\omega_o)^2 + 2\zeta\omega_n(jk\omega_o) + \omega_n^2\right] Y_k = k\omega_n^2 X_k$$

$$\frac{Y_k}{X_k} = \frac{k\omega_n^2}{(\omega_n^2 - (k\omega_o)^2) + j2\zeta\omega_n k\omega_o}$$

6. A periodic signal $x(t)$ with period T_0 has the constant component $c_0 = 2$. The signal $x(t)$ is applied to an LTI system with transfer function

$$H(j\omega) = \begin{cases} 10e^{-j5\omega} & |\omega| > \frac{\pi}{T_0} \\ 0 & otherwise \end{cases} \longrightarrow \text{for } |\omega| \le \frac{\omega_0}{2}$$

$\omega_0 = \frac{2\pi}{T_0}$

The filter eliminates DC but passes every thing else w/ given TF

The output of the system $y(t)$ can be written

$$y(t) = ax(t-b) + c$$

Determine the constants $a, b,$ and c.

$$y(t) = \sum_{k=-\infty}^{\infty} |c_k^x|\,|H(jk\omega_0)|\, e^{-j5k\omega_0} e^{jk\omega_0 t} = 10 \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} |c_k^x|\, e^{jk\omega_0(t-5)}$$

$$= 10\left[x(t-5) - c_0^x\right] = 10\,x(t-5) - 20$$

$a = 10$

$b = 5$

$c = -20$