Name ______________________________ CM__________

ECE 300
Signals and Systems

Exam 3
12 February, 2008

NAME Answers

This exam is closed-book in nature. You may use the provided table of Fourier Transform relationships, but no calculator is allowed.

Problem 1	________	/ 15
Problem 2	________	/ 40
Problem 3	________	/ 25
Problem 4	________	/ 20

Exam 3 Total Score: _______ / 100

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1. Fourier Transforms (15 points)

If $x(t)$ is given by the following function

$$x(t) = 2e^{\frac{-(t-1)^2}{4}}$$

Find the Fourier Transform $X(\omega) = \mathcal{F}\{x(t)\}$.

For $x_1(t) = e^{-t^2/2\sigma^2} \Longleftrightarrow X_1(\omega) = \sigma\sqrt{2\pi}\, e^{-\sigma^2\omega^2/2}$

$\sigma^2 = 2 \quad \sigma = \sqrt{2}$

$x_1(t) = e^{-t^2/4} \Longleftrightarrow X_1(\omega) = \sqrt{2}\sqrt{2\pi}\, e^{-2\omega^2/2} = 2\sqrt{\pi}\, e^{-\omega^2}$

$x_2(t) = 2x_1(t) = 2e^{-t^2/4} \Longleftrightarrow X_2(\omega) = 2X_1(\omega) = 4\sqrt{\pi}\, e^{-\omega^2}$

$x(t) = x_2(t-1) = 2e^{-(t-1)^2/4} \Longleftrightarrow X(\omega) = X_2(\omega)\, e^{-j\omega}$

$= \boxed{4\sqrt{\pi}\, e^{-\omega^2} e^{-j\omega} = X(\omega)}$

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2. Fourier Analysis of LTI Systems (40 points)

Assume $x(t) = 2\,\text{sinc}\left[\frac{3}{\pi}(t-3)\right]\cos(2(t-3))$ is the input to an LTI system with transfer function $H(\omega) = \begin{cases} 3e^{-j\omega 2} & |\omega| < 2 \\ 0 & else \end{cases}$

a) Determine the Fourier transform $X(\omega)$ of $x(t)$

b) Accurately sketch the magnitude and phase of $X(\omega)$

c) Determine the energy in $x(t)$

d) Accurately sketch the magnitude and phase of the system output in the frequency domain

e) Determine the system output $y(t)$

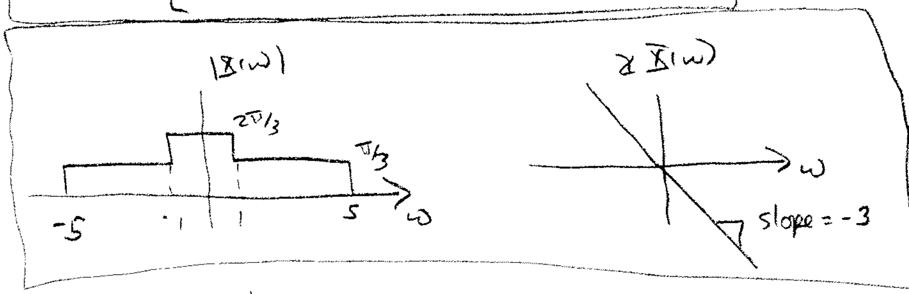
(a) $x_1(t) = 2\,\text{sinc}\left(\frac{3}{\pi}t\right) \longleftrightarrow X_1(\omega) = \frac{2\pi}{3}\,\text{rect}\left(\frac{\omega}{6}\right)$

$x_2(t) = x_1(t)\cos(2t) \longleftrightarrow X_2(\omega) = \frac{1}{2}X_1(\omega+2) + \frac{1}{2}X_1(\omega-2)$

$= \frac{\pi}{3}\,\text{rect}\left(\frac{\omega+2}{6}\right) + \frac{\pi}{3}\,\text{rect}\left(\frac{\omega-2}{6}\right)$

$$\boxed{X(\omega) = \left[\frac{\pi}{3}\,\text{rect}\left(\frac{\omega+2}{6}\right) + \frac{\pi}{3}\,\text{rect}\left(\frac{\omega-2}{6}\right)\right]e^{-j3\omega}}$$

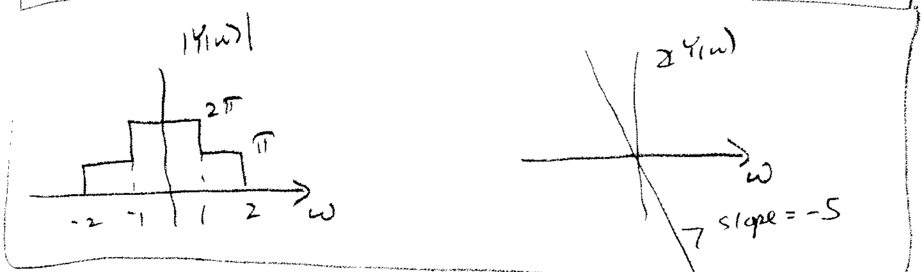
(b)



(c) $E_x = \frac{1}{2\pi}\left[2\int_0^1 \left(\frac{2\pi}{3}\right)^2 d\omega + 2\int_1^5 \left(\frac{\pi}{3}\right)^2 d\omega\right]$

$= \frac{1}{\pi}\left[\frac{4\pi^2}{9} + \frac{4\pi^2}{9}\right] = \boxed{\frac{8}{9}\pi = E_x}$

(d)



(e) $Y(\omega) = \left[\pi\,\text{rect}\left(\frac{\omega}{4}\right) + \pi\,\text{rect}\left(\frac{\omega}{2}\right)\right]e^{-j5\omega}$

$\text{rect}\left(\frac{\omega}{4}\right) \longleftrightarrow \frac{2}{\pi}\,\text{sinc}\left(\frac{2}{\pi}t\right) \quad 2\pi W = 4,\ W = \frac{2}{\pi}$

$\text{rect}\left(\frac{\omega}{2}\right) \longleftrightarrow \frac{1}{\pi}\,\text{sinc}\left(\frac{1}{\pi}t\right) \quad 2\pi W = 2 \quad W = 1/\pi$

$$\boxed{y(t) = 2\,\text{sinc}\left(\frac{2}{\pi}(t-5)\right) + \text{sinc}\left(\frac{1}{\pi}(t-5)\right)}$$

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3. Deriving Fourier Transform Pairs (25 points)
Starting from the definition of the Fourier Transform and Inverse Fourier Transform

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \text{ and}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

Derive the following Fourier Transform pairs. NOTE: YOU WILL ONLY GET CREDIT IF YOU START FROM THE INTEGRALS ABOVE. NO CREDIT WILL BE GIVEN FOR USING THE TRANSFORM TABLES.:

(a) If $x(t) \Leftrightarrow X(\omega)$ Show that $x(a(t-t_0)) \Leftrightarrow \frac{1}{a} e^{-jt_0\omega} X\left(\frac{\omega}{a}\right)$ for $a > 0$

$$\int_{-\infty}^{\infty} x(a(t-t_0)) e^{-j\omega t} dt$$

let $\lambda = a(t-t_0)$

$d\lambda = a\,dt$

$\frac{\lambda}{a} + t_0 = t$

$$\int_{-\infty}^{\infty} x(\lambda) e^{-j\omega\left(\frac{\lambda}{a}+t_0\right)} \frac{d\lambda}{a}$$

$$= \frac{1}{a} e^{-j\omega t_0} \int_{-\infty}^{\infty} x(\lambda) e^{-j\frac{\omega}{a}\lambda} d\lambda$$

$$= \boxed{\frac{1}{a} e^{-j\omega t_0} X\left(\frac{\omega}{a}\right)}$$

(b) If $x(t) = \text{rect}\left(\frac{t-1}{2}\right)$ show that $X(\omega) = 2e^{-j\omega}\text{sinc}\left(\frac{\omega}{\pi}\right)$

$$x(t) = \text{rect}\left(\frac{t-1}{2}\right) = u(t) - u(t-2)$$

$$\int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \int_0^2 e^{-j\omega t} dt = \frac{e^{-j\omega t}}{-j\omega}\Big|_0^2 = \frac{e^{-j\omega 2} - 1}{-j\omega} = \frac{1 - e^{-j\omega 2}}{j\omega}$$

$$= \frac{e^{-j\omega}}{\omega}\left[\frac{e^{+j\omega} - e^{-j\omega}}{2j}\right]2 = 2e^{-j\omega}\frac{\sin(\omega)}{\omega} = 2e^{-j\omega}\frac{\sin\left(\pi\frac{\omega}{\pi}\right)}{\left(\pi\frac{\omega}{\pi}\right)}$$

$$= \boxed{2e^{-j\omega}\,\text{sinc}\left(\frac{\omega}{\pi}\right) = X(\omega)}$$

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4. Fourier Series (20 points)
Assume $x(t)$ is a periodic signal with Fourier series representation

$$x(t) = \sum_{k=-\infty}^{k=\infty} \frac{1}{1+jk} e^{jk1.5t}$$

$x(t)$ is the input to an LTI system with transfer function

$$H(j\omega) = \frac{1}{1+j\omega\alpha}$$

Determine the value of α so that the ratio of the average power in the DC component of the output is equal to 10 times the average power in the second harmonic of the output signal.

$$P_{ave}^{0} = (C_0^Y)^2 \qquad P_{ave}^{2} = 2|C_2^Y|^2 \qquad P_{ave}^{0} = 10\, P_{ave}^{2}$$

$$(C_0^Y)^2 = 10 \cdot \left(2|C_2^Y|^2\right)$$

$$C_0^Y = C_0^x H(0) = (1)(1) = 1$$

$$C_2^Y = C_2^x H(j2\omega_0)$$

$$C_2^x = \frac{1}{1+j2} \qquad |C_2^x| = \frac{1}{\sqrt{1^2+2^2}} = \frac{1}{\sqrt{5}}$$

$$H(j2\omega_0) = H(j3) = \frac{1}{1+j3\alpha}$$

$$|H(j2\omega_0)| = \frac{1}{\sqrt{1^2+(3\alpha)^2}} = \frac{1}{\sqrt{1+9\alpha^2}}$$

So

$$1 = 20\, |C_2^x|^2 |H(j2\omega_0)|^2 = 20 \frac{1}{5} \frac{1}{1+9\alpha^2} = \frac{4}{1+9\alpha^2}$$

$$1+9\alpha^2 = 4$$

$$9\alpha^2 = 3$$

$$\alpha^2 = 1/3$$

$$\boxed{\alpha = \frac{1}{\sqrt{3}}}$$

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Some Potentially Useful Relationships

$$E_\infty = \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2 dt = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

$$e^{jx} = \cos(x) + j\sin(x) \qquad j = \sqrt{-1}$$

$$\cos(x) = \frac{1}{2}\left[e^{jx} + e^{-jx}\right] \qquad \sin(x) = \frac{1}{2j}\left[e^{jx} - e^{-jx}\right]$$

$$\cos^2(x) = \frac{1}{2} + \frac{1}{2}\cos(2x) \qquad \sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos(2x)$$

$$\mathrm{rect}\left(\frac{t-t_0}{T}\right) = u\left(t - t_0 + \frac{T}{2}\right) - u\left(t - t_0 - \frac{T}{2}\right)$$