Differential Equation Review

In this course you will be expected to be able to solve first order differential equations which can be solved using integrating factors without Maple, just as you are expected to be able to do basic calculus without Maple. Once you understand how to deal with these simple types of equations, and how they affect system properties, generalizing to other types of differential equations will not be difficult.

Differential Equations with Integrating Factors An integrating factor allows us to write one half of a first order differential equation as an exact derivative (something easy to integrate), and the other part as a function with no derivatives. In what follows we will assume the system starts at time t_o and ends at time *t.* In the first example we go over all of the details, but in the final two examples we just use results from **Example 1**.

Example 1: Consider the differential equation $\dot{y}(t)-y(t)=2$. We first rewrite the derivative as $\dot{y}(t)=\frac{dy}{dt}$, so we have $\frac{dy}{dt}-y(t)=2$. Next we want to write the left hand side of the equation as $\frac{d}{dt}\left[y(t)e^{a(t)}\right]$. Using basic properties from calculus we have

$$\frac{d}{dt}\left[y(t)e^{a(t)}\right]=e^{a(t)}\frac{dy(t)}{dt}+\frac{da(t)}{dt}e^{a(t)}y(t)=e^{a(t)}\left[\frac{dy(t)}{dt}+\frac{da(t)}{dt}y(t)\right]$$

We want the term in the brackets to look like our original equation, that is, we want

$$\left[\frac{dy(t)}{dt}+\frac{da(t)}{dt}y(t)\right]=\frac{dy(t)}{dt}-y(t)$$

Equating the two sides we get

$$\frac{da(t)}{dt}=-1$$

which gives us

$$a(t)=-t$$

At this point we have

$$\frac{d}{dt}[y(t)e^{-t}]=e^{-t}\left[\frac{dy(t)}{dt}-y(t)\right]$$

Now since we have (from our original differential equation)

$$\frac{dy}{dt} - y(t) = 2$$

we can multiple both sides of this equation by e^{-t} to get

$$e^{-t}\left[\frac{dy(t)}{dt} - y(t)\right] = e^{-t}[2]$$

or

$$e^{-t}\left[\frac{dy(t)}{dt} - y(t)\right] = \frac{d}{dt}\left[y(t)e^{-t}\right] = e^{-t}[2]$$

At this point we have the left hand side as an exact derivative

$$\frac{d}{dt}\left[y(t)e^{-t}\right] = 2e^{-t}$$

Now we want to integrate both sides of the equation

$$\int_{t_o}^{t} \frac{d}{dt}\left[y(t)e^{-t}\right]dt = \int_{t_o}^{t} 2e^{-\lambda}d\lambda$$

Integrating we have

$$y(t)e^{-t} - y(t_o)e^{-t_o} = -2e^{-t} + 2e^{-t_0}$$

Finally we get the solution

$$y(t) = y(t_o)e^{(t-t_o)} - 2 + 2e^{(t-t_o)}$$

<u>Example 2:</u> Consider the differential equation $\dot{y}(t) - 2ty(t) = x(t)$. From **Example 1**, we need $\frac{da(t)}{dt} = -2t$, or $a(t) = -t^2$. We then have $\frac{d}{dt}\left[y(t)e^{-t^2}\right] = e^{-t^2}x(t)$.
Integrating both sides we have

$$\int_{t_o}^{t} \frac{d}{dt}\left[y(t)e^{-t^2}\right]dt = \int_{t_o}^{t} x(\lambda)e^{-\lambda^2}d\lambda$$

or

$$y(t)e^{-t^2} - y(t_o)e^{-t_o^2} = \int_{t_o}^{t} x(\lambda)e^{-\lambda^2}\,d\lambda$$

Finally we have the solution

$$y(t) = y(t_o)e^{(t^2-t_o^2)} + \int_{t_o}^{t} x(\lambda)e^{t^2-\lambda^2}\,d\lambda$$

Note that we cannot go any further in the solution until we know $x(t)$.

__Example 3:__ Consider the differential equation $\dot{y}(t) + \frac{3}{2}\sqrt{t}\,y(t) = e^t x(t)$. From **Example 1**, we need $\frac{da(t)}{dt} = \frac{3}{2}\sqrt{t}$, or. $a(t) = t^{\frac{3}{2}}$ We then have $\frac{d}{dt}\left[y(t)e^{t^{\frac{3}{2}}} \right] = e^{t^{\frac{3}{2}}}e^t x(t)$. Integrating both sides we have

$$\int_{t_o}^{t} \frac{d}{dt}\left[y(t)e^{t^{\frac{3}{2}}} \right] dt = \int_{t_o}^{t} x(\lambda)e^{\lambda}e^{\lambda^{\frac{3}{2}}}\,d\lambda$$

or

$$y(t)e^{t^{\frac{3}{2}}} - y(t_o)e^{t_o^{\frac{3}{2}}} = \int_{t_o}^{t} x(\lambda)e^{\lambda}e^{\lambda^{\frac{3}{2}}}\,d\lambda$$

Finally we have the solution

$$y(t) = y(t_o)e^{-(t^{\frac{3}{2}}-t_o^{\frac{3}{2}})} + \int_{t_o}^{t} x(\lambda)e^{\lambda}e^{-(t^{\frac{3}{2}}-\lambda^{\frac{3}{2}})}d\lambda$$

Note that we cannot go any further in the solution until we know $x(t)$.