

ECE 300
Signals and Systems
Homework 8

Due Date: Tuesday May 7 at the beginning of class

Problems

1) Show that if $x(t)$ is real and even, then $X(\omega)$ is real and even, and if $x(t)$ is real and odd, then $X(\omega)$ is imaginary and odd.

2) By evaluating the Fourier transform integral directly, and using Euler's identity, show that the Fourier transform of $x(t) = \cos(t) \left[u(t + \frac{\pi}{2}) - u(t - \frac{\pi}{2}) \right]$ is

$$X(\omega) = \frac{2 \cos\left(\omega \frac{\pi}{2}\right)}{1 - \omega^2}$$

3) By evaluating the integral by hand, show that the Fourier transform of $x(t) = e^{-t}u(t)$ is given by

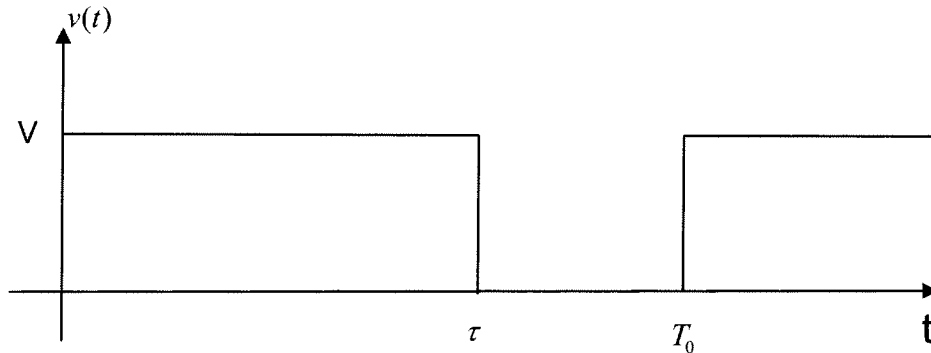
$$X(\omega) = \frac{1}{1 + j\omega} = \frac{1}{\sqrt{1 + \omega^2}} \angle -\tan^{-1}(\omega) \left[\frac{180}{\pi} \right] \text{degrees}$$

4) By evaluating the integral by hand, show that the Fourier transform of $x(t) = e^{-|t|}$ is

$$X(\omega) = \frac{2}{1 + \omega^2}$$

5) Most microcontrollers are capable of generating pulse width modulation (PWM) signals on one or more output pins. These signals are square waves where both the period and the duty cycle can be programmed in the microcontroller by the use of timers and different reference clocks. These PWM output signals can then be used in conjunction with lowpass filters to produce reasonable approximations to analog output signals. In this problem we will use what we have learned in the course to investigate how to do this. The signal $v(t)$ below is a PWM signal, shown for about one and a half periods. The signal has period T_0 , amplitude V (usually fixed at 5 or 3.3 volts), pulse duration τ , and

duty cycle $D = \frac{\tau}{T_0}$.



The Fourier series representation is $v(t) = \sum_{k=-\infty}^{k=\infty} e^{-\frac{jk\pi\tau}{T_0}} \left(\frac{V\tau}{T_0} \right) \text{sinc} \left(\frac{k\tau}{T_0} \right) e^{jk\omega_0 t}$

a) For the periodic signal $v(t)$, determine an expression for the **average power** in the periodic signal in terms of T_0 , τ , and V . Your answer must contain no sums or integrals.

b) Determine an expression for the **average value** of $v(t)$ in terms of T_0 , τ , and V .

c) It is the average value of $v(t)$ that we want to use as our analog output. Hence we need to design a lowpass filter that allows us to keep our DC term, and, ideally, remove all of the other harmonics. Let's assume we want to use a simple first order RC lowpass filter with transfer function

$$H(j\omega) = \frac{1}{j\omega RC + 1} = \frac{1}{j\omega \frac{\sqrt{\alpha}}{\omega_0} + 1}$$

where we have set $RC = \frac{\sqrt{\alpha}}{\omega_0}$ for convenience. Determine the value of α so that the average power in the first harmonic of the output signal is 20 dB lower than the average power in the DC component the output signal. Assume here that the fundamental frequency is $f_0 = 100 \text{ Hz}$, the duty cycle is $\frac{\tau}{T_0} = 0.8$, and $V = 5.0 \text{ volts}$.

d) For your value of α determined in part c and the parameter values given in part c, determine an expression for the first two terms (the DC and first harmonic) in the Fourier series representation of the output signal.

(Answer: $y(t) \approx 4 + 0.566 \cos(2\pi 100t - 3.77)$)

e) For your value of α determined in part c, and the parameter values given in part c, determine the **bandwidth** of the filter you designed. Be sure to include units!

6) (Matlab Problem) In this problem you will utilize your Matlab program **Complex_Fourier_Series.m** to demonstrate that as the period of a periodic function increases, the Fourier series approximates the Fourier transform. Use your answers for Problems 3 and 4 to plot the Fourier transform results.

- a) Use your program to determine the Fourier series for $x(t) = e^{-t}u(t)$ over the time interval $[-4,4]$. Hence the period $T=8$ in this case. You should be sure to look at the Fourier series representation to verify everything is correct.
- b) Construct a vector $W = [-N : N]\omega_0$ just as you did in lab. This will make plotting and evaluation functions much easier.
- c) Construct the vector $C = [\text{fliplr}(\text{conj}(c)) \ c_0 \ c]$. This will make plotting easier.
- d) Modify your code to plot the amplitude $T|c_k|$ versus $k\omega_0$ and the phase $\angle c_k$ (in degrees) versus $k\omega_0$. You should use the **subplot** command and plot both on one page. You should use the command **orient tall** before any plotting to use more of the page. Some Matlab commands you might find useful are **angle**, **length**, and **abs**. Instead of using the stem command, you should use the plot command and plot discrete points, like dots ('.').
- e) By using the **axis** command, limit the axes for magnitude plot to the range -8 to 8 radians/sec and from 0 to the maximum value of $T|c_k|$. The **max** command may prove useful here. Limit the axes for the phase plot to the range -8 to 8 radians/sec and from -90 to 90 degrees.
- f) Add plots of the magnitude and phase of $X(\omega)$ on the existing plots. It might be easiest if you define an anonymous function for this (just as you did for $(x(t))$). You may need the functions **sqrt** and **./** or **.*** Use a solid line type and be sure to add legends. If you have done everything correctly, and you use $N=100$ points in the Fourier series, you should get the plot shown in Figure 1. Be sure to modify the title and axes so they look like those in this figure (to get c_k type c_k , and to get ω_0 type ω_0)

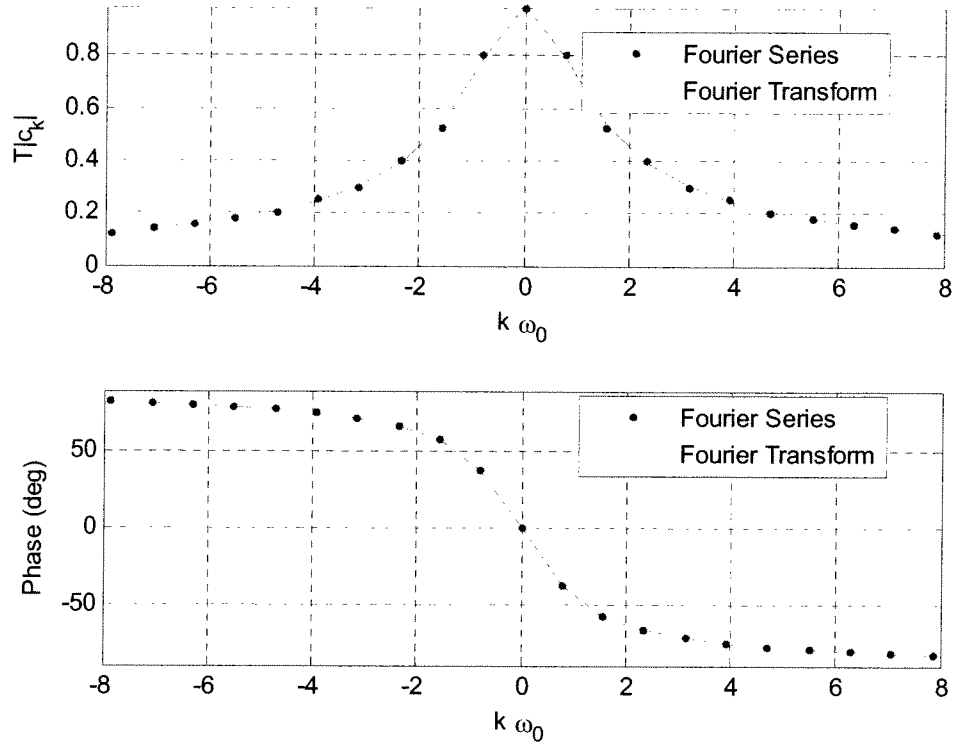


Figure 1: Example plots for part d.

- g) Change the duration of the periodic signal to $[-8,8]$ (so the period is $T = 16$) and $[-16,16]$ (so the period is $T = 32$) and rerun your code. Try and run a duration of $[-32,32]$ (so the period $T = 64$), though this may not work well. Turn in your plots. Keep the number of points at $N = 100$. Do not change $x(t)$. Here we are increasing the period of the function $x(t)$ to demonstrate that the Fourier transform is just a Fourier series in the limit as $T \rightarrow \infty$, $k\left(\frac{2\pi}{T}\right) = k\omega_0 \rightarrow \omega$, and $Tc_k \rightarrow X(\omega)$
- h) Redo the above for $x(t) = e^{-|t|}$. Turn in 3 plots for this part (4 plots if you can get $T=64$ to go...).

7) (Matlab Problem) In this problem we'll look at a real world situation when we have to truncate a signal. This actually happens more with digital signal processing, but we can get the basic idea using our continuous time abilities.

- a) Find an expression for the Fourier transform of $f(t) = \cos(4t) + \cos(5t)$.
- b) Now assume we look at $f(t)$ for a finite time, say T seconds. What we see is actually $y(t) = f(t)\text{rect}(t/T)$. Determine an expression for the Fourier transform of $y(t)$, and write your answers in terms of sinc functions.
- c) Plot, using **Matlab**, $Y(\omega)$ for ω between 0 and 10 when $T=1$, $T=6$, $T=10$, $T=20$, and $T=40$. Can you clearly tell there are two cosines present when you are looking at $Y(\omega)$ for all values of T ? What happens as T gets larger (you are looking at more and more data)? Think in terms of the width of the sinc function (the distance between the first nulls). Note: The **sinc** function exists in **Matlab**.

#1 ZTF Problem 4-4

a) assume $x(t)$ is real and even

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} x(t) \cos(\omega t) dt - j \int_{-\infty}^{\infty} x(t) \sin(\omega t) dt$$

↑ ↑
even * odd = odd
so integral is zero

$$= \int_{-\infty}^{\infty} x(t) \cos(\omega t) dt$$

$$= \int_{-\infty}^0 x(t) \cos(\omega t) dt + \int_0^{\infty} x(t) \cos(\omega t) dt$$

$$\omega t = -\tau$$
$$d\tau = -dt$$

$$= \int_{\infty}^0 x(-\tau) \cos(\omega(-\tau)) (-d\tau) + \int_0^{\infty} x(t) \cos(\omega t) dt$$

↑ ↑ ↑
even even switch
 limits on
 integral

$$= \int_0^{\infty} x(\tau) \cos(\omega \tau) d\tau + \int_0^{\infty} x(t) \cos(\omega t) dt$$

$$X(\omega) = 2 \int_0^{\infty} x(t) \cos(\omega t) dt$$

$$X(-\omega) = 2 \int_0^{\infty} x(t) \cos(-\omega t) dt = 2 \int_0^{\infty} x(t) \cos(\omega t) dt$$

so $X(\omega)$ is real and even

b) Assume $x(t)$ is real and odd

$$X(\omega) = \int_{-\infty}^{\infty} x(t) \cos(\omega t) dt - j \int_{-\infty}^{\infty} x(t) \sin(\omega t) dt$$

odd = odd, even
so integral is zero

$$X(\omega) = -j \int_{-\infty}^{\infty} x(t) \sin(\omega t) dt = -j \left[\int_{-\infty}^0 x(t) \sin(\omega t) dt + \int_0^{\infty} x(t) \sin(\omega t) dt \right]$$

$$\text{let } \sigma = -t \quad \int_{-\infty}^0 x(-\sigma) \sin(\omega(-\sigma)) (-d\sigma)$$

$$= \int_{-\infty}^0 [-x(\sigma)] [-\sin(\omega\sigma)] [-d\sigma]$$

$$= \int_0^{\infty} x(\sigma) \sin(\omega\sigma) d\sigma$$

$$\text{so } X(\omega) = -2j \int_0^{\infty} x(t) \sin(\omega t) dt$$

$$X(-\omega) = -2j \int_0^{\infty} x(t) \sin(-\omega t) dt = -X(\omega)$$

so $X(\omega)$ is imaginary and odd

#2

$$x(t) = \cos(t) \left[u\left(t + \frac{\pi}{2}\right) - u\left(t - \frac{\pi}{2}\right) \right]$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(t) e^{-j\omega t} dt$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{e^{jt} + e^{-jt}}{2} \right) e^{-j\omega t} dt = \frac{1}{2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[e^{j(1-\omega)t} + e^{-j(1+\omega)t} \right] dt$$

$$= \frac{1}{2} \left. \frac{e^{j(1-\omega)t}}{j(1-\omega)} \right|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \frac{1}{2} \left. \frac{e^{-j(1+\omega)t}}{-j(1+\omega)} \right|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{1}{2j} \left[\frac{e^{j(1-\omega)\frac{\pi}{2}} - e^{-j(1-\omega)\frac{\pi}{2}}}{(1-\omega)} \right] - \frac{1}{2j} \left[\frac{e^{-j(1+\omega)\frac{\pi}{2}} - e^{j(1+\omega)\frac{\pi}{2}}}{(1+\omega)} \right]$$

$$= \frac{1}{2j} \left[\frac{e^{j\frac{\pi}{2}} e^{-j\omega\frac{\pi}{2}} - e^{-j\frac{\pi}{2}} e^{j\omega\frac{\pi}{2}}}{(1-\omega)} \right] - \frac{1}{2j} \left[\frac{e^{-j\frac{\pi}{2}} e^{-j\omega\frac{\pi}{2}} - e^{j\frac{\pi}{2}} e^{j\omega\frac{\pi}{2}}}{(1+\omega)} \right]$$

$$= \frac{1}{2j} \left[\frac{j e^{-j\omega\frac{\pi}{2}} + j e^{j\omega\frac{\pi}{2}}}{(1-\omega)} \right] - \frac{1}{2j} \left[\frac{-j e^{-j\omega\frac{\pi}{2}} - j e^{j\omega\frac{\pi}{2}}}{1+\omega} \right]$$

$$= \frac{e^{j\omega\frac{\pi}{2}} + e^{-j\omega\frac{\pi}{2}}}{2} \left[\frac{1}{1-\omega} + \frac{1}{1+\omega} \right] = \cos(\omega\frac{\pi}{2}) \left[\frac{1+\omega+1-\omega}{1-\omega^2} \right]$$

$$= \frac{2\cos(\omega\frac{\pi}{2})}{1-\omega^2}$$

#3) $x(t) = e^{-t}u(t)$

$$\begin{aligned} X(\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt = \int_{-\infty}^{\infty} e^{-t}u(t)e^{-j\omega t} dt = \int_0^{\infty} e^{-t}e^{-j\omega t} dt \\ &= \int_0^{\infty} e^{-(1+j\omega)t} dt = \frac{e^{-(1+j\omega)t}}{-(1+j\omega)} \Big|_0^{\infty} = \boxed{\frac{1}{1+j\omega} = X(\omega)} \end{aligned}$$

#4) $x(t) = e^{-|t|}$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt = \int_{-\infty}^{\infty} e^{-|t|}e^{-j\omega t} dt$$

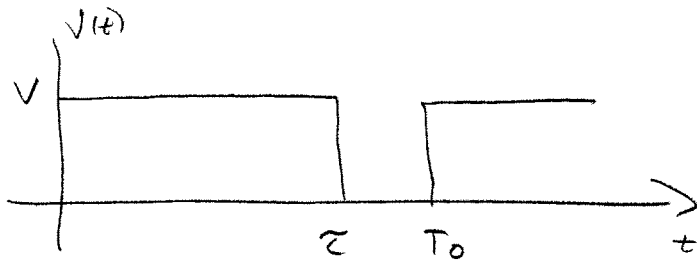
$$= \int_{-\infty}^0 e^te^{-j\omega t} dt + \int_0^{\infty} e^{-t}e^{-j\omega t} dt$$

$$= \int_{-\infty}^0 e^{t(1-j\omega)} dt + \int_0^{\infty} e^{-(1+j\omega)t} dt$$

$$= \frac{e^{t(1-j\omega)}}{1-j\omega} \Big|_{-\infty}^0 + \frac{e^{-(1+j\omega)t}}{-(1+j\omega)} \Big|_0^{\infty} = \frac{1}{1-j\omega} + \frac{1}{1+j\omega}$$

$$= \frac{1+j\omega + 1-j\omega}{(1+j\omega)(1-j\omega)} = \boxed{\frac{2}{1+\omega^2} = X(\omega)}$$

#5



$$v(t) = \sum_{k=-\infty}^{\infty} e^{-jk\pi\tau/T_0} \left(\frac{V\tau}{T_0} \right) \text{sinc} \left(\frac{k\tau}{T_0} \right) e^{jk\omega_0 t}$$

a) $P_{\text{ave}}^V = \frac{1}{T_0} \int_{-T_0}^{T_0} |v(t)|^2 dt = \frac{1}{T_0} \int_0^{2\tau} V^2 dt = \boxed{V^2 \frac{\tau}{T_0} = P_{\text{ave}}^V}$

b) $C_0^V = e^{-jk\pi\tau/T_0} \left(\frac{V\tau}{T_0} \right) \text{sinc} \left(\frac{k\tau}{T_0} \right) \Big|_{k=0} = \boxed{\frac{V\tau}{T_0} = C_0^V = V_{\text{ave}}}$

c) $v(t) \rightarrow \boxed{H(j\omega)} \rightarrow y(t)$

$$P_y^1 = 2|c_1^y|^2 \quad P_y^0 = |c_0^y|^2$$

$$20 \text{ dB} = 10 \log_{10} \left(\frac{|c_0^y|^2}{2|c_1^y|^2} \right)$$

$$2 = \log_{10} \left(\frac{|c_0^y|^2}{2|c_1^y|^2} \right)$$

$$\frac{|c_0^y|^2}{2|c_1^y|^2} = 10^2 = 100$$

$$|c_0^y|^2 = 200 |c_1^y|^2 \quad \text{or} \quad |c_1^y|^2 = \frac{1}{200} |c_0^y|^2$$

$$C_0^y = C_0^V H(0) = \left(\frac{V\tau}{T_0} \right) (1) = 5(0.8)(1) = 4.0$$

$$|c_1^y| = |c_1^V H(j\omega_0)| = |c_1^V| |H(j\omega_0)|$$

$$|H(j\omega_0)| = \left| \frac{1}{1+j\sqrt{\alpha}} \right| = \frac{1}{\sqrt{1+\alpha}}$$

$$|c_1^V| = \left| \frac{V\tau}{T_0} \right| \left| \text{sinc} \left(\frac{\tau}{T_0} \right) \right| = (4) (\text{sinc}(0.8)) = 4 \frac{\sin(\pi(0.8))}{\pi(0.8)} = 0.9355$$

$$|c_1^y| = \frac{0.9355}{\sqrt{1+\alpha}}$$

#5 (continued)

$$|c_1^y|^2 = \frac{1}{200} |c_0^y|^2$$

$$\left(\frac{0.9355}{1+\alpha}\right)^2 = \frac{1}{200} (4)^2 = \frac{16}{200} \quad \left(\frac{200}{10}\right) (0.9355)^2 = 1+\alpha$$

$$\alpha = 9.94$$

d) $c_0^y = 4$

$$c_1^y = c_1^v H(j\omega_0) = e^{-j\pi\tau/T_0} \left(\frac{\sqrt{\tau}}{T_0}\right) \text{sinc}\left(\frac{\tau}{T_0}\right) \frac{1}{1+j\sqrt{\alpha}}$$

$$= e^{-j0.8\pi} (0.9355) \frac{1}{1+j\sqrt{9.94}}$$

$$|c_1^y| = 0.2828$$

$$2|c_1^y| = 0.566$$

$$\angle c_1^y = -0.8\pi - 1.20365 = -3.577$$

$$y(t) \approx e^y + 2|c_1^y| \cos(\omega_0 t + \angle c_1^y)$$

$$y(t) \approx 4 + 0.566 \cos(2\pi \cdot 100t - 3.577)$$

e) $\alpha = 9.94 \quad \sqrt{\alpha} = 3.153$

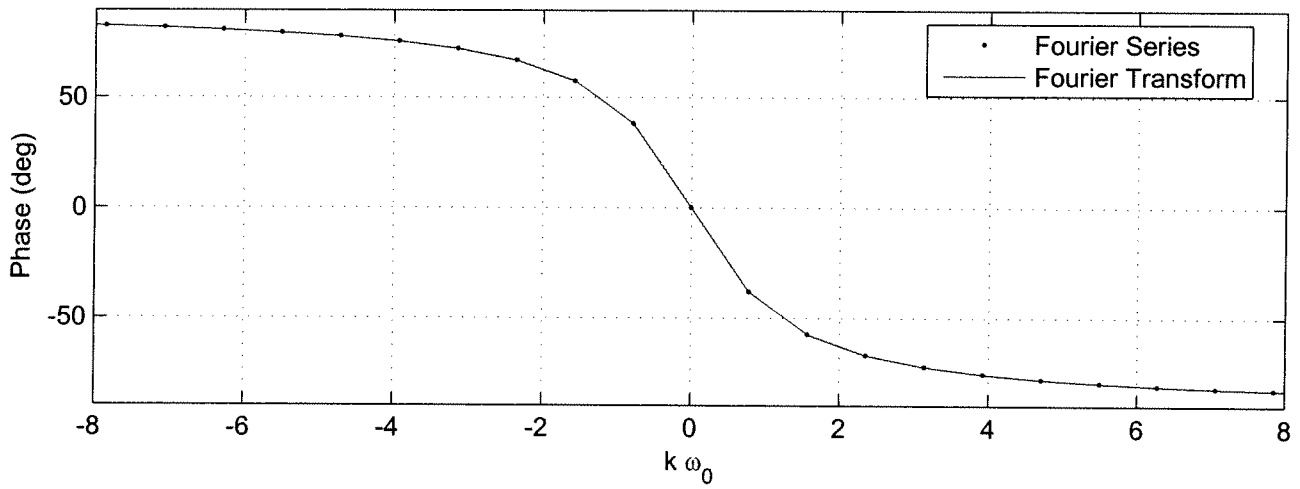
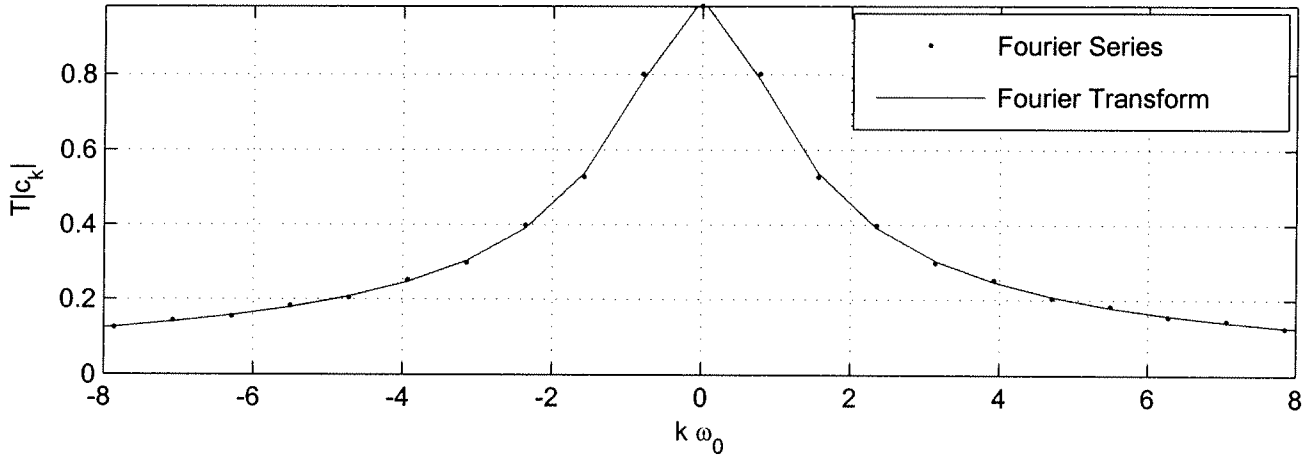
$$H(j\omega) = \frac{1}{j\omega \frac{\sqrt{\alpha}}{\omega_0} + 1} = \frac{1}{j\omega \frac{(3.153)}{2\pi \cdot 100} + 1} = \frac{1}{j\omega (0.00502) + 1}$$

$$\approx \frac{1}{j\omega \frac{1}{199.3} + 1}$$

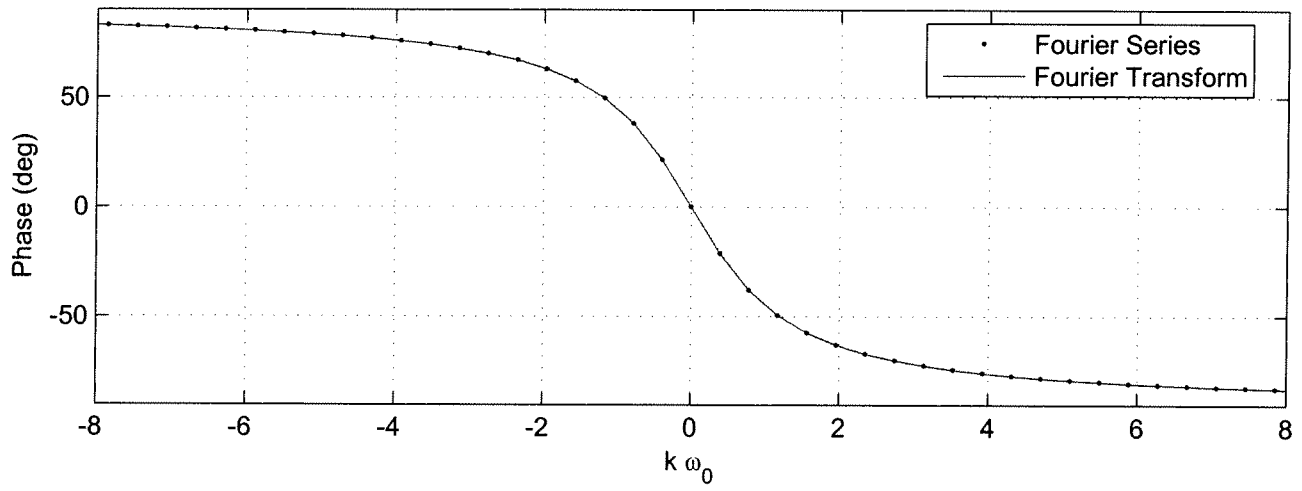
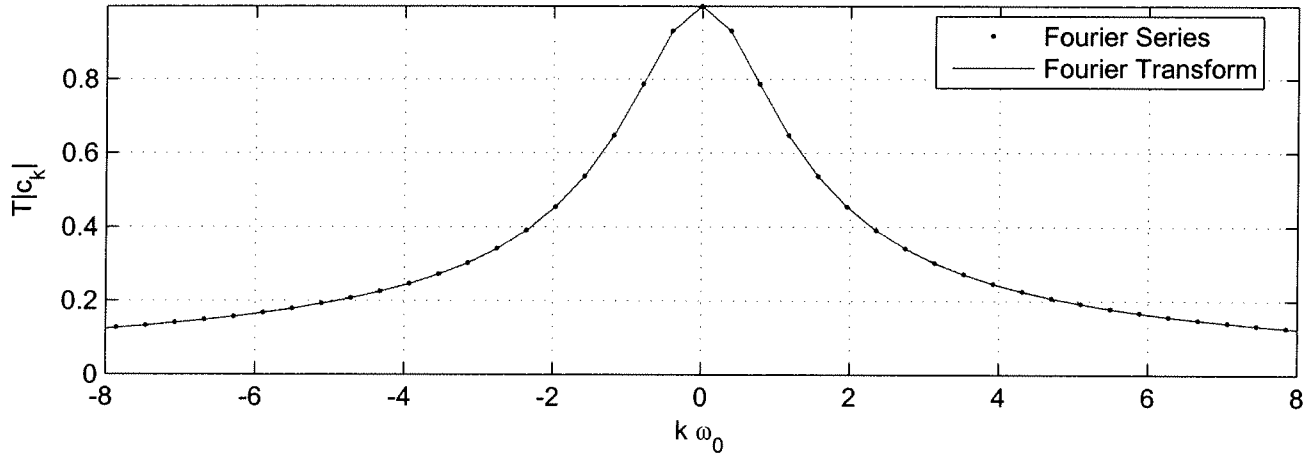
$$\text{Bandwidth} \approx 199.3 \text{ rad/sec}$$

$$\approx 31.7 \text{ Hz}$$

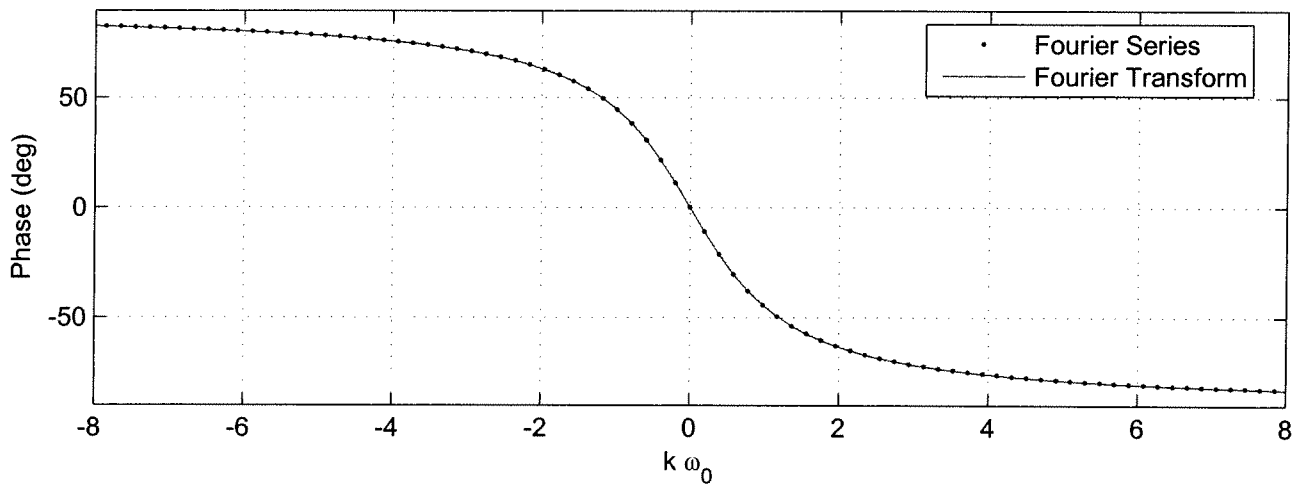
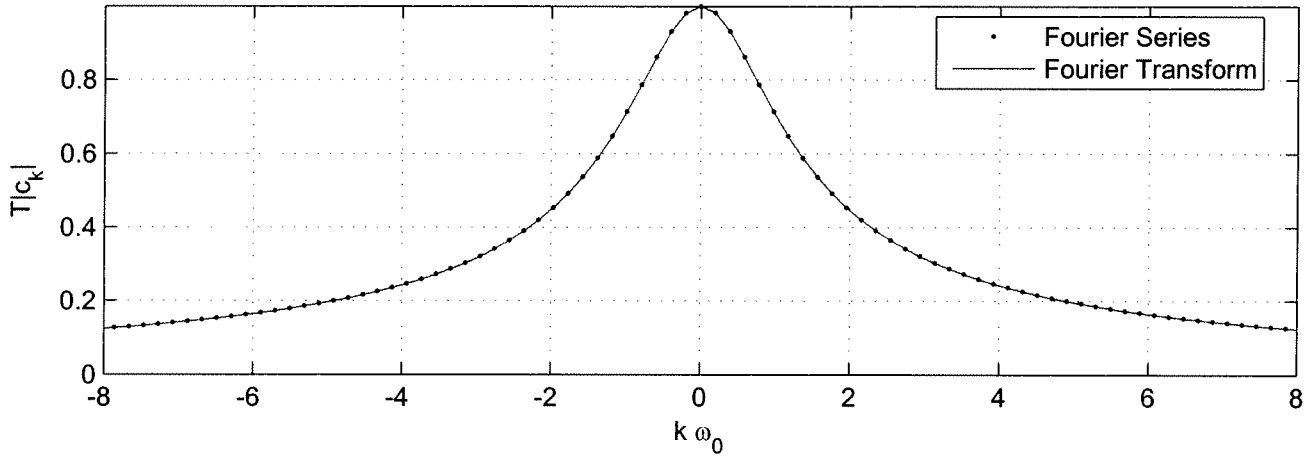
Fourier Series -> Fourier Transform, T = 8



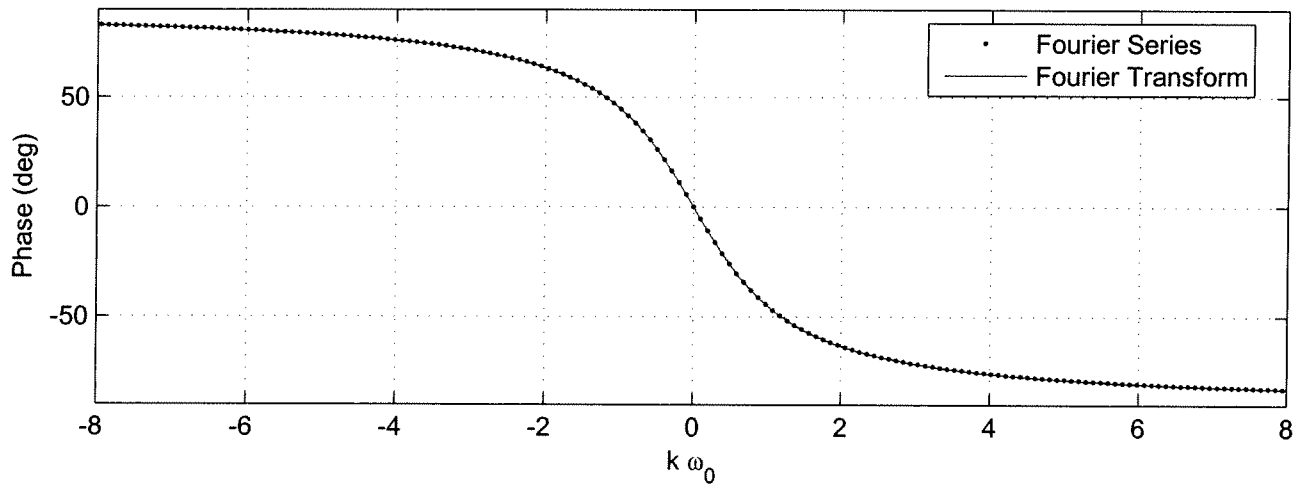
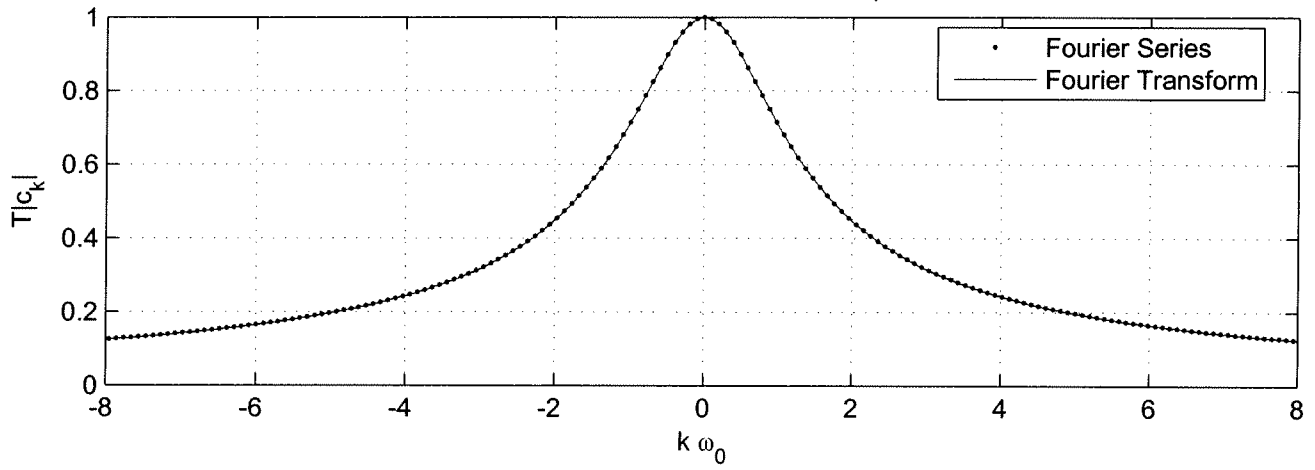
Fourier Series -> Fourier Transform, T = 16



Fourier Series -> Fourier Transform, T = 32



Fourier Series -> Fourier Transform, T = 64



```
%
% plot for problem 5 of homework 5
%
w = linspace(0,10,1000);

T = 1;
a = T/(2*pi);
Y1 = @(w) (T/2)*(sinc(a*(w+4))+sinc(a*(w-4))+sinc(a*(w+5))+sinc(a*(w-5)))

T = 6;
a = T/(2*pi);
Y6 = @(w) (T/2)*(sinc(a*(w+4))+sinc(a*(w-4))+sinc(a*(w+5))+sinc(a*(w-5)))

T = 10;
a = T/(2*pi);
Y10 = @(w) (T/2)*(sinc(a*(w+4))+sinc(a*(w-4))+sinc(a*(w+5))+sinc(a*(w-5)))

T = 20;
a = T/(2*pi);
Y20 = @(w) (T/2)*(sinc(a*(w+4))+sinc(a*(w-4))+sinc(a*(w+5))+sinc(a*(w-5)))

T = 40;
a = T/(2*pi);
Y40 = @(w) (T/2)*(sinc(a*(w+4))+sinc(a*(w-4))+sinc(a*(w+5))+sinc(a*(w-5)))

orient tall
subplot(5,1,1); plot(w,Y1(w)); grid; title('T = 1');
subplot(5,1,2); plot(w,Y6(w)); grid; title('T = 6');
subplot(5,1,3); plot(w,Y10(w)); grid; title('T = 10');
subplot(5,1,4); plot(w,Y20(w)); grid; title('T = 20');
subplot(5,1,5); plot(w,Y40(w)); grid; title('T = 40');
xlabel('\omega ');
```


#10

a) $f(t) = \cos(4t) + \cos(5t)$

$$F(\omega) = \pi [\delta(\omega+4) + \delta(\omega-4)] + \pi [\delta(\omega+5) + \delta(\omega-5)]$$

b) $y(t) = f(t) \text{rect}\left(\frac{t}{T}\right)$

$$Y(\omega) = \frac{1}{2\pi} F(\omega) * \mathcal{F}\left\{\text{rect}\left(\frac{t}{T}\right)\right\}$$

$$= \frac{1}{2\pi} F(\omega) * T \text{sinc}\left(\frac{T}{2\pi} \omega\right)$$

$$Y(\omega) = \frac{T}{2} \left\{ \text{sinc}\left(\frac{T}{2\pi}(\omega+4)\right) + \text{sinc}\left(\frac{T}{2\pi}(\omega-4)\right) + \text{sinc}\left(\frac{T}{2\pi}(\omega+5)\right) + \text{sinc}\left(\frac{T}{2\pi}(\omega-5)\right) \right\}$$

c) See Attached Plots

