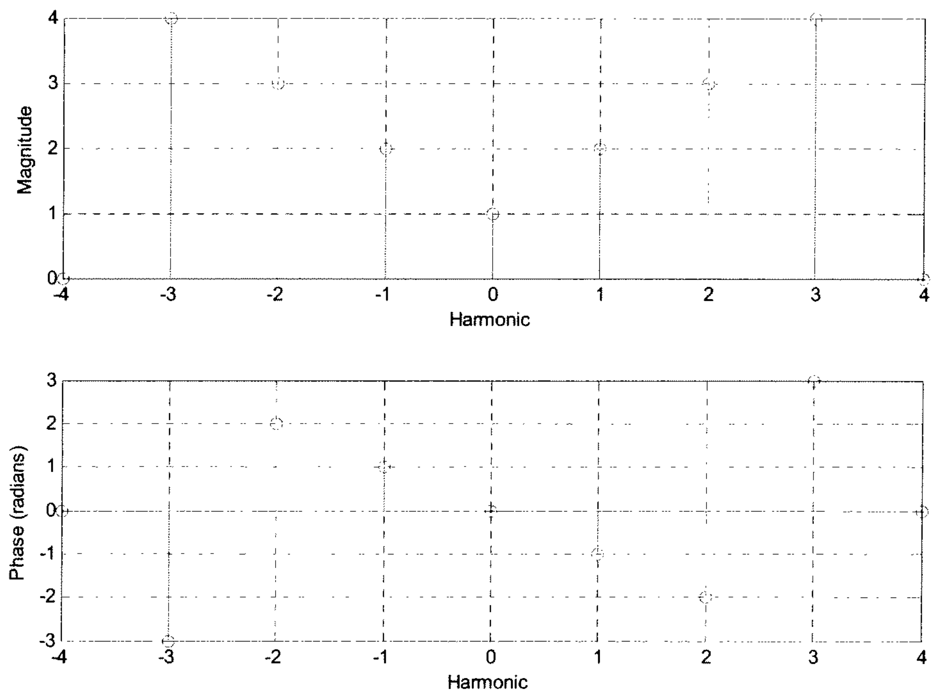
ECE 300
Signals and Systems
Homework 7

<u>Due Date:</u> Tuesday April 28, 2009 at the beginning of class

<u>Exam 2, Thursday April 30, 2009</u>

<u>Problems:</u>

1. Assume $x(t)$ has the spectrum shown below (the phase is shown in radians) and a fundamental frequency $\omega_o = 2\ \text{rad/sec}$:



Assume $x(t)$ is the input to a system with the transfer function

$$H(\omega) = \begin{cases} e^{-j\omega} & 1 \le |\omega| < 3 \\ 2e^{-j2\omega} & 3 < |\omega| < 5 \\ 0 & else \end{cases}$$

Determine an expression for the steady state output $y(t)$. Be as specific as possible, simplifying all values and using actual numbers wherever possible.

2. A periodic signal $x(t)$ is the input to an LTI system with output $y(t)$. The signal $x(t)$ has period 2 seconds, and is given over one period as

$$x(t) = e^{-t} \quad 0 < t < 2$$

$x(t)$ has the Fourier series representation

$$x(t) = \sum_k \frac{0.4323}{1 + jk\pi} e^{jk\pi t}$$

The system is an ideal highpass filter that eliminates all signals with frequency content less than 0.75 Hz.

a) Find the average power in $x(t)$.

b) Determine an expression for the output, $y(t)$. Your expression for $y(t)$ must be real.

(*Answer:* $y(t) = e^{-t} - 0.4323 - 0.2622\cos(\pi t - 1.2626)$)

c) Determine the average power in $y(t)$.

d) What fraction of the average power in $x(t)$ is contained in the DC and fundamental frequency components?

3. Assume $x(t) = t^2 \quad -\pi \le t \le \pi$ with Fourier Series representation

$$x(t) = \sum\nolimits_k c_k^x e^{jkt}$$

where

$$c_k^x = \begin{cases} \dfrac{\pi^2}{3} & k = 0 \\ \dfrac{2(-1)^k}{k^2} & k \ne 0 \end{cases}$$

a) Assume $x(t)$ is the input to a system that eliminates all signals with frequencies outside the range 0.5 to 0.7 Hz. What is the output of the system $y(t)$ and what fraction of the average power in $x(t)$ is in $y(t)$? (Note: your answers must be real, no e^{ja} terms.)

b) Assume $x(t)$ is the input to a system that eliminates all signals with frequencies in the range 0.5 to 0.7 Hz. What is the output of the system $y(t)$ and what fraction of the average power in $x(t)$ is in $y(t)$?

4. Assume two periodic signals have the Fourier series representations

$$x(t) = \sum X_k e^{jk\omega_o t} \quad y(t) = \sum Y_k e^{jk\omega_o t}$$

For the following system (input/output) relationships:

a) $y(t) = bx(t-a)$
b) $y(t) = b\dot{x}(t-a)$
c) $y(t) = bx(t)\cos(\omega_o t)$ (Answer: $Y_n = \frac{b}{2}(X_{n-1} + X_{n+1})$)
d) $\ddot{y}(t) + 2\zeta\omega_n \dot{y}(t) + \omega_n^2 y(t) = K\omega_n^2 x(t)$

i) write Y_k in terms of the X_k
ii) If possible, determine the system transfer function $H(j\omega)$
iii) A system must be both linear and time-invariant to have a transfer function. If you cannot determine the transfer function, indicate which system property is not satisfied (**L** or **TI**).

5. A periodic signal $x(t)$ with period T_0 has the constant component $c_0 = 2$. The signal $x(t)$ is applied to an LTI system with transfer function

$$H(j\omega) = \begin{cases} 10e^{-j5\omega} & |\omega| > \dfrac{\pi}{T_0} \\ 0 & otherwise \end{cases}$$

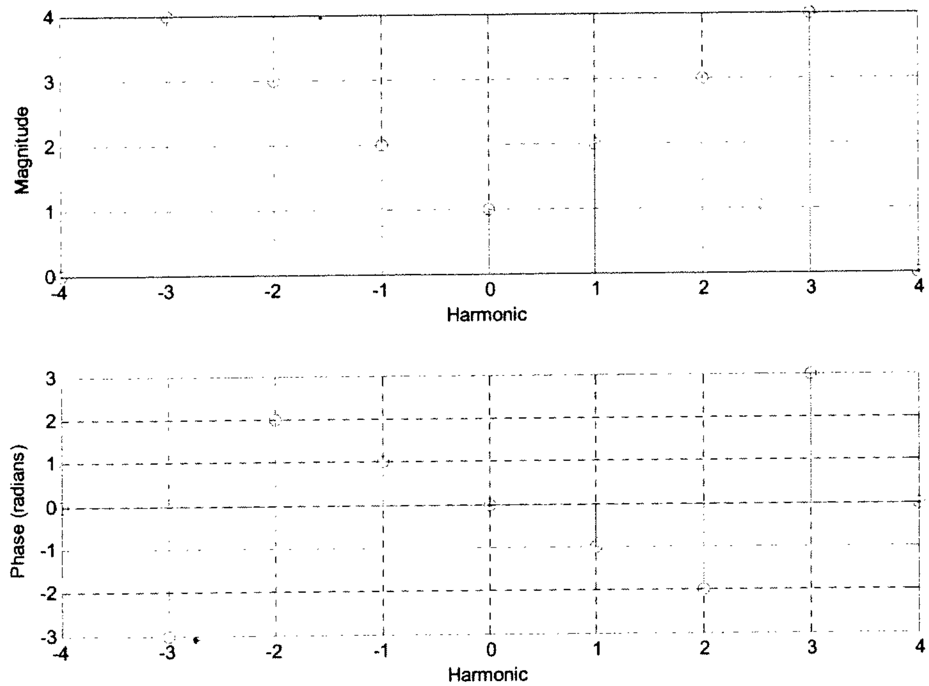
The output of the system $y(t)$ can be written

$$y(t) = ax(t-b) + c$$

Determine the constants $a, b,$ and c.

#1

$$H(\omega) = \begin{cases} e^{-j\omega} & 1 \le |\omega| < 3 \\ 2e^{-j2\omega} & 3 < |\omega| 5 \\ 0 & \text{else} \end{cases} \qquad \omega_0 = 2\,\text{rad/sec}$$



$$Y_0 = X_0 H(0) = 0$$

$$Y_1 = X_1 H(1\omega_0) = (2e^{-j1})(e^{-j2}) = 2e^{-j3} = 2 \measuredangle -3 \text{ rad}$$

$$Y_2 = X_2 H(2\omega_0) = (3e^{-j2})(2e^{-j8}) = 6e^{-j10} = 2 \measuredangle -10 \text{ rad}$$

$$Y_3 = X_3 H(3\omega_0) = 0$$

$$y(t) = Y_0 + 2|Y_1|\cos(\omega_0 t + \measuredangle Y_1) + 2|Y_2|\cos(2\omega_0 t + \measuredangle Y_2) + 0 + \cdots$$

$$\boxed{y(t) = 4\cos(2t-3) + 12\cos(4t-10)}$$

#2 $x(t) = e^{-t} \quad 0 \le t \le 2$ $\quad T_0 = 2 \quad f_0 = \frac{1}{2} = 0.5\,Hz$

$$x(t) = \sum_{k=-\infty}^{\infty} \frac{0.4323}{1+jk\pi} e^{jk\pi t}$$

a) $$P_{ave}^{x} = \frac{1}{T_0}\int_{T_0} |x(t)|^2 dt = \frac{1}{2}\int_0^2 e^{-2t} dt = \frac{1}{2} \left. \frac{e^{-2t}}{-2} \right|_0^2 = \frac{1}{4}\left(1 - e^{-4}\right) = 0.2454$$

$$\boxed{P_{ave}^{x} = 0.2454}$$

b) The high pass filter removes signals with frequency content below 0.75 Hz. Let's figure out what they are

It removes $k=0$, $k=\pm 1$

$$c_0^x = 0.4323$$

$$c_1^x = \frac{0.4323}{1+j\pi} = 0.13112 \measuredangle\, 1.2626 \text{ rad}$$

$$y(t) = e^{-t} - 0.4323 - 2(0.13112)\cos(\pi t - 1.2626)$$

$$\boxed{y(t) = e^{-t} - 0.4323 - 0.26225\cos(\pi t - 1.2626)}$$

c) $$P_{ave}^{y} = P_{ave}^{x} - |c_0^x|^2 - 2|c_1^x|^2 = 0.2454 - (0.4323)^2 - 2(0.13112)^2$$

$$\boxed{P_{ave}^{y} = 0.02413}$$

d) $$\frac{|c_0^x|^2 + 2|c_1^x|^2}{P_x^{ave}} = \frac{(0.4323)^2 + 2(0.13112)^2}{0.2454} = 0.90166 \approx 90\%$$

#3 $x(t) = t^2 \quad -\pi \le t \le \pi$

$x(t) = \sum X_K e^{jKt} \qquad X_K = \begin{cases} \frac{\pi^2}{3} & K = 0 \\ \frac{2(-1)^K}{K^2} & K \neq 0 \end{cases}$

a) Bandpass filter, removes everything outside range 0.5 to 0.7 Hz

$\omega_0 = 1 \text{ rad/sec} = 2\pi f_0 \qquad f_0 = 1/2\pi = 0.159 \text{ Hz}$

K	$f = Kf_0$
0	0
1	0.159
2	0.318
3	0.477
4	0.636 — only term
5	0.795

$y(t) = 2|c_4^x| \cos(4\omega_0 t + \measuredangle c_4^x)$

$c_4^x = \frac{2(-1)^4}{4^2} = \frac{2}{16} \measuredangle 0^\circ = \frac{1}{8} \measuredangle 0^\circ = 0.125 \measuredangle 0^\circ$

$\boxed{y(t) = 0.25 \cos(4t)}$

$P_{ave}^x = \frac{1}{2\pi} \int_{-\pi}^{\pi} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} t^4 dt = \frac{1}{2\pi} \frac{t^5}{5} \Big|_{-\pi}^{\pi} = \frac{2(\pi^5)}{5(2\pi)} = 19.48$

$\frac{2|c_4^x|^2}{P_{ave}^x} = \frac{2|0.125|^2}{19.48} \times 100\% = 0.16\%$ of total power

b) $y(t) = t^2 - 0.125\cos(4t)$

$\frac{P_{ave}^y}{P_{ave}^x} = 100\% - 0.16\% = \boxed{99.84\%}$

#4

$x(t) = \sum X_K e^{jKw_0t} \qquad y(t) = \sum Y_K e^{jKw_0t}$

a) $y(t) = b\,x(t-a)$

$\sum Y_K e^{jKw_0t} = b \sum X_K e^{jKw_0(t-a)} = \sum b X_K e^{-jKw_0a} e^{jKw_0t}$

$$\boxed{Y_K = X_K b e^{-jKw_0a} \qquad H(jw) = b e^{-jwa}}$$

b) $y(t) = b\,\dot{x}(t-a)$

$\sum Y_K e^{jKw_0t} = \frac{d}{dt} \sum b e^{-jKw_0a} X_K e^{jKw_0t}$

$= \sum b e^{-jKw_0a}\, jKw_0 X_K e^{jKw_0t}$

$$\boxed{Y_K = b e^{-jKw_0a} jKw_0 X_K \qquad H(jw) = b\,jw\,e^{-jwa}}$$

c) $y(t) = b\,x(t)\cos(w_0t)$

$Y_K = \frac{1}{T_0} \int_{T_0} y(t) e^{-jKw_0t}\,dt$

$= \frac{1}{T_0} \int_{T_0} b\,x(t) \cos(w_0t) e^{-jKw_0t}\,dt$

$= \frac{1}{T_0} \int_{T_0} b\,x(t) \left[\frac{e^{jw_0t}}{2} + \frac{e^{-jw_0t}}{2}\right] e^{-jKw_0t}\,dt$

$= \frac{b}{2}\left[\frac{1}{T_0} \int_{T_0} x(t) e^{-j(K-1)w_0t}\,dt + \frac{1}{T_0} \int_{T_0} x(t) e^{-j(K+1)w_0t}\,dt\right]$

$$\boxed{Y_K = \frac{b}{2}\left[X_{K-1} + X_{K+1}\right]}$$

not TI

d) $\ddot{y}(t) + \frac{2\zeta}{\omega_n}\dot{y}(t) + \frac{1}{\omega_n^2} y(t) = K x(t)$

$\sum Y_K (jKw_0)^2 e^{jKw_0t} + \sum Y_K \frac{2\zeta}{\omega_n}(jKw_0) e^{jKw_0t} + \sum Y_K \frac{1}{\omega_n^2} e^{jKw_0t} = \sum X_n K e^{jKw_0t}$

$$\boxed{Y_K = \frac{K}{(jKw_0)^2 + \frac{2\zeta}{\omega_n}(jKw_0) + \frac{1}{\omega_n^2}} X_K \qquad H(jw) = \frac{K}{(jw)^2 + \frac{2\zeta}{\omega_n}(jw) + \frac{1}{\omega_n^2}}}$$

(#5) $c_0 = 2$ $\quad H(j\omega) = \begin{cases} 10e^{-j5\omega} & |\omega| > \frac{\pi}{T_0} \\ 0 & \text{else} \end{cases}$

$x(t)$ is periodic with period T_0, $\omega_0 = \frac{2\pi}{T_0}$

$y(t)$ can be written $y(t) = a\,x(t-b) + c$

Since $\omega_0 = \frac{2\pi}{T_0}$ we can write $H(j\omega) = \begin{cases} 10e^{-j5\omega} & |\omega| > \frac{\omega_0}{2} \\ 0 & \text{else} \end{cases}$

so the filter removes the dc value of $x(t)$, scales by 10, and delays by 5

so $\boxed{y(t) = 10\,x(t-5) - 20 \qquad a = 10,\; b = 5,\; c = -20}$