Name Solution CM

ECE 300
Signals and Systems

Exam 3
19 MAY, 2009

This exam is closed-book in nature. You are not to use a calculator or computer during the exam. Do not write on the back of any page, use the extra pages at the end of the exam.

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Exam 3 Total Score: _______ / 100

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1. (45 points) Random Fourier transform problems.

a) **(10 points)** If $x(t) = \dfrac{2}{2 + j(t-2)}$, determine $X(\omega)$

$x_1(t) = 2e^{-2t}u(t) \longleftrightarrow X_1(\omega) = \dfrac{2}{2+j\omega}$

$x_2(t) = X_1(t) = \dfrac{2}{2+jt} \longleftrightarrow X_2(\omega) = 2\pi\, x_1(-\omega) = 4\pi e^{2\omega} u(-\omega)$

$x(t) = x_2(t-2) = \dfrac{2}{2+j(t-2)} \longleftrightarrow X(\omega) = e^{-j2\omega} X_2(\omega) = \boxed{4\pi e^{2\omega} e^{-j2\omega} u(-\omega)}$

$X(\omega) = \text{rect}\left(\frac{2}{3}(\omega-1)\right)$

b) **(10 points)** If $X(\omega) = rect\left(\dfrac{2\omega - 2}{3}\right)$, determine $x(t)$

$X_1(\omega) = \text{rect}\left(\frac{2}{3}\omega\right) \quad \frac{\omega}{2\pi W} = \frac{2}{3}W \quad W = \frac{3}{4\pi}$

$\longleftrightarrow x_1(t) = \frac{3}{4\pi}\,\text{sinc}\left(\frac{3}{4\pi}t\right)$

$X(\omega) = X_1(\omega - 1) = \text{rect}\left(\frac{2}{3}(\omega-1)\right) \longleftrightarrow x(t) = e^{jt} x_1(t)$

$= \boxed{\frac{3}{4\pi} e^{jt}\,\text{sinc}\left(\frac{3}{4\pi}t\right)}$

c) **(10 points)** If $x(t)$ and $y(t)$ are related through the relationship $\dot{y}(t) = x(t-b) \star e^{-t}u(t-c)$ determine the transfer function for the system.

$\mathcal{F}\{\dot{y}(t)\} = j\omega Y(\omega) \qquad \mathcal{F}\{x(t-b)\} = e^{-j\omega b} X(\omega)$

$\mathcal{F}\{e^{-t}u(t-c)\} = \mathcal{F}\{e^{-(t-c+c)}u(t-c)\} = e^{-c}\mathcal{F}\{e^{-(t-c)}u(t-c)\}$

$= \dfrac{e^{-c}e^{-j\omega c}}{1+j\omega}$

$j\omega Y(\omega) = \left[e^{-j\omega b}X(\omega)\right]\left[\dfrac{e^{-c}e^{-j\omega c}}{1+j\omega}\right]$

$\boxed{\dfrac{Y(\omega)}{X(\omega)} = H(\omega) = \dfrac{e^{-j\omega c}e^{-j\omega b}e^{-c}}{j\omega(1+j\omega)}}$

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d) **(10 points)** If we have the Fourier transform pair $x(t) \leftrightarrow X(\omega)$, use the definition of the Fourier transform or inverse Fourier transform to show $tx(t) \leftrightarrow j\frac{dX(\omega)}{d\omega}$ if the Fourier transform of $tx(t)$ exists. In this problem you are to prove this relationship from the Fourier transform (or inverse Fourier transform) definitions.

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$\frac{dX(\omega)}{d\omega} = \int_{-\infty}^{\infty} x(t) \frac{d}{d\omega} e^{-j\omega t} dt = \int_{-\infty}^{\infty} x(t)(-jt) e^{-j\omega t} dt$$

$$j\frac{dX(\omega)}{d\omega} = \int_{-\infty}^{\infty} [t\, x(t)] e^{-j\omega t} dt = \mathcal{F}\{t\, x(t)\}$$

e) **(5 points)** If $X(\omega) = \frac{1}{T}\text{sinc}\left(\frac{2\omega T}{\pi}\right)$ determine the location of the first nulls.

need $\frac{2\omega T}{\pi} = \pm 1$ $\qquad \boxed{\omega = \pm \frac{\pi}{2T}}$

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2. (25 points) The periodic function $x(t)$ is defined over one period ($T_0 = 4$ seconds) as

$$x(t) = \begin{cases} 2 & -2 \le t \le 0 \\ 0 & 0 \le t \le 2 \end{cases}$$

Determine the complex Fourier series coefficients, c_k.

Be sure to simplify your answer as much as possible and use a <u>*sinc*</u> *function if appropriate. Recall that* $1 = e^0$.

$$\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{4} = \frac{\pi}{2}$$

$$c_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}\,dt = \frac{1}{4}\int_{-2}^{0} 2e^{-jk\pi/2\, t}\,dt = \frac{1}{2}\int_{-2}^{0} e^{-jk\frac{\pi}{2}t}\,dt$$

$$= \frac{1}{2}\left.\frac{e^{-jk\pi/2\, t}}{-jk\pi/2}\right|_{-2}^{0} = \frac{1}{(\pi\, k/2)}\,\frac{1 - e^{jk\pi}}{(-2j)}$$

$$= \frac{1}{\pi(k/2)}\,\frac{e^{jk\pi} - 1}{2j} = \frac{e^{jk\pi/2}\left[e^{jk\frac{\pi}{2}} - e^{-jk\pi/2}\right]}{\pi(k/2)\quad 2j}$$

$$= e^{jk\pi/2}\,\frac{\sin\left(\pi \cdot \frac{k}{2}\right)}{\pi \cdot k/2} = \boxed{e^{jk\pi/2}\,\mathrm{sinc}\left(\frac{k}{2}\right) = c_k}$$

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3. (30 points) Assume $x(t) = 4\,\text{sinc}\left[\frac{1}{\pi}(t-2)\right]\cos(4(t-2))$ is the input to an LTI system

with transfer function $H(\omega) = \begin{cases} \frac{1}{\pi}e^{-j\omega 3} & |\omega| < 4 \\ 0 & else \end{cases}$

a) Determine the Fourier transform $X(\omega)$ of $x(t)$

b) Accurately sketch the magnitude and phase of $X(\omega)$

c) Determine the system output $y(t)$

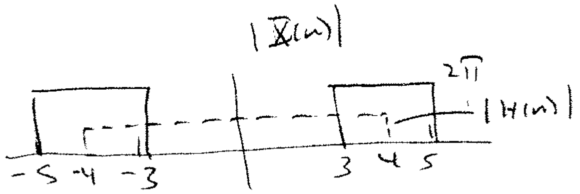
for $x_1(t) = 4\,\text{sinc}\left(\frac{t}{\pi}\right) \xleftrightarrow{W=\frac{1}{\pi}} X_1(\omega) = 4\pi\,\text{rect}\left(\frac{\omega}{2\pi\cdot\frac{1}{\pi}}\right) = 4\pi\,\text{rect}\left(\frac{\omega}{2}\right)$

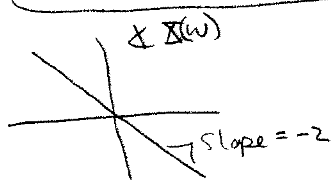
for $x_2(t) = x_1(t)\cos(4t) \longleftrightarrow X_2(\omega) = \frac{1}{2}X_1(\omega+4) + \frac{1}{2}X_1(\omega-4)$

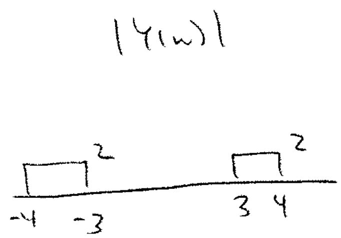
$= 2\pi\,\text{rect}\left(\frac{\omega+4}{2}\right) + 2\pi\,\text{rect}\left(\frac{\omega-4}{2}\right)$

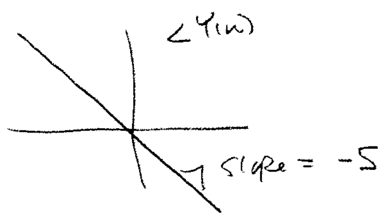
for $x(t) = x_2(t-2) \longleftrightarrow X(\omega) = e^{-j2\omega}X_2(\omega)$

$$\boxed{X(\omega) = 2\pi\left[\text{rect}\left(\frac{\omega+4}{2}\right) + \text{rect}\left(\frac{\omega-4}{2}\right)\right]e^{-j2\omega}}$$









$Y(\omega) = 2\left[\text{rect}(\omega+3.5) + \text{rect}(\omega-3.5)\right]e^{-j5\omega}$

for $Y_1(\omega) = \text{rect}(\omega) \xleftrightarrow{W=\frac{1}{2\pi}} y_1(t) = \frac{1}{2\pi}\text{sinc}\left(\frac{t}{2\pi}\right)$

for $Y_2(\omega) = 4\,\text{rect}(\omega) \longleftrightarrow y_2(t) = \frac{2}{\pi}\text{sinc}\left(\frac{t}{2\pi}\right)$

for $y_3(t) = y_2(t)\cos(3.5t) \longleftrightarrow Y_3(\omega) = 2\,\text{rect}(\omega+3.5) + 2\,\text{rect}(\omega-3.5)]$

with time shift $\boxed{y(t) = \frac{2}{\pi}\text{sinc}\left(\frac{t-5}{2\pi}\right)\cos(3.5(t-5))}$