

ECE 300
Signals and Systems
 Homework 9

Due Date: Thursday May 10 at the beginning of class

You are free to use the Fourier transform tables handed out in class or in the book for any of the following problems.

Problems

- At the end of this assignment are two sets of plots: one is a group of time-domain functions, $x(t)$, labeled **a-h**, and a group of frequency domain functions, $|X(\omega)|$, the magnitude of the Fourier transform. These are labeled **1-8**. You are to determine which time domain function goes with which frequency domain function. Specifically, you need to fill in the following table (and show some justification for your answers)

$x(t)$	a	b	c	d	e	f	g	h
$ X(\omega) $								

Hint: $e^{-at} \cos(bt)u(t) \leftrightarrow \frac{1}{(s+a)^2 + b^2}$

- K & H, Problem 3.21 (parts a, b, and c only).
- K & H, Problem 3.22. Note that $p_4(\omega) = \text{rect}\left(\frac{\omega}{4}\right)$.
- K & H, Problem 3.24
- K & H, Problem 3.27
- In this problem we'll look at a real world situation when we have to truncate a signal. This actually happens more with digital signal processing, but we can get the basic idea using our continuous time abilities.
 - Find an expression for the Fourier transform of $f(t) = \cos(4t) + \cos(5t)$.
 - Now assume we look at $f(t)$ for a finite time, say T seconds. What we see is actually $y(t) = f(t)\text{rect}(t/T)$. Determine an expression for the Fourier transform of $y(t)$, and write your answers in terms of sinc functions.

- c) Plot, using **Matlab**, $Y(\omega)$ for ω between 0 and 10 when $T=1$, $T=6$, $T=10$, $T=20$, and $T=40$. Can you clearly tell there are two cosines present when you are looking at $Y(\omega)$ for all values of T ? What happens as T gets larger (you are looking at more and more data)? Think in terms of the width of the sinc function (the distance between the first nulls). Note: The **sinc** function exists in **Matlab**.

7. (**Matlab/Prelab Problem**) A Butterworth filter has the property that it is maximally flat in the *passband*. An n th order Butterworth filter has the magnitude squared response

$$|H(\omega)|^2 = \frac{1}{1 + \left(\frac{\omega}{\omega_p}\right)^{2n}}$$

where ω_p is the *passband frequency*. At this frequency the power has been reduced by one half or 3 dB,

$$|H(\omega_p)|^2 = \frac{1}{1 + \left(\frac{\omega_p}{\omega_p}\right)^{2n}} = \frac{1}{2} \quad \text{or} \quad 10 \log_{10} |H(\omega_p)|^2 = 10 \log_{10} \left(\frac{1}{2}\right) = -3 \text{ dB}$$

To determine the required order of a filter we often look at the desired *stopband frequency*, ω_s . Usually we want to indicate the minimum required power difference between the passband and the stopband, Δ . Δ is the *rejection*. Hence we have

$$\Delta = 20 \log_{10} |H(0)| - 20 \log_{10} |H(\omega_s)|$$

or

$$\Delta = -10 \log_{10} \frac{1}{1 + \left(\frac{\omega_s}{\omega_p}\right)^{2n}}$$

The ratio $\frac{\omega_s}{\omega_p}$ is called the *transition ratio*.

a) Show that we can write

$$n = \frac{\ln \left(10^{\frac{\Delta}{20}} - 1 \right)}{2 \ln \left(\frac{\omega_s}{\omega_p} \right)}$$

Note that n must be an integer, so we always round up (to the next larger integer).

b) For $\omega_p = 10$ rad/sec, $\omega_s = 20$ rad/sec, and $\Delta = 18$ dB, determine the required Butterworth filter order for this filter. (Remember it must be an integer). Using the Table at the end of this problem, plot the Bode plot of your Butterworth filter and verify that all frequencies $\omega > \omega_s$ have magnitude (power) less than Δ_{\max} .

Matlab's **tf** command and **Bode** commands will be really useful here. Note that you can click on the curve on the Bode plot to read it more accurately. Turn in your plot.

c) Matlab's command **r = pole(H)**, where H is the transfer function of the Butterworth filter, returns the poles of the transfer function in the array **r**. Using Matlab's commands **abs** and **angle**, relate the magnitude of the poles to ω_p , then plot the pole locations in the complex plane on a circle with radius ω_p . (Note that **angle** returns angles in radians, and you probably want angle in degrees.) Note that the pole locations are all separated by an angle θ . What is this angle?

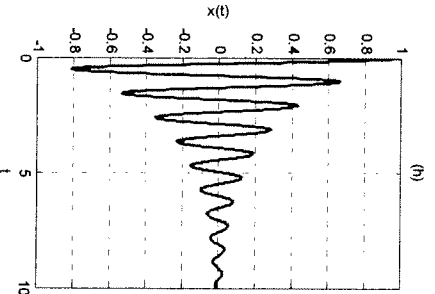
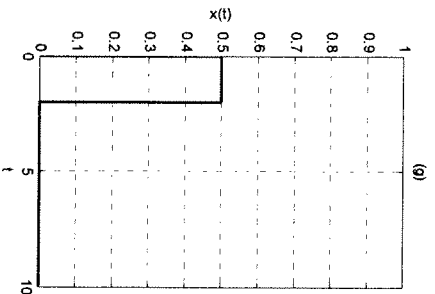
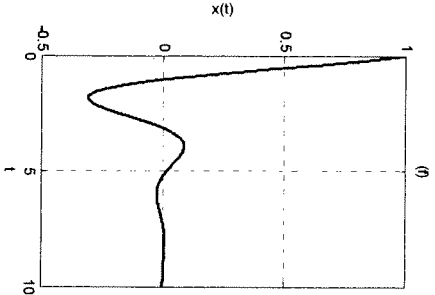
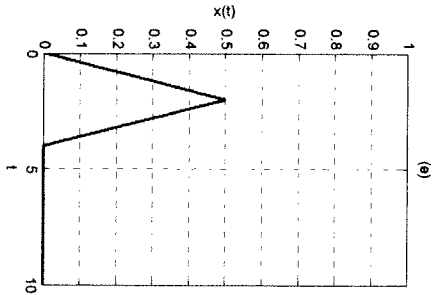
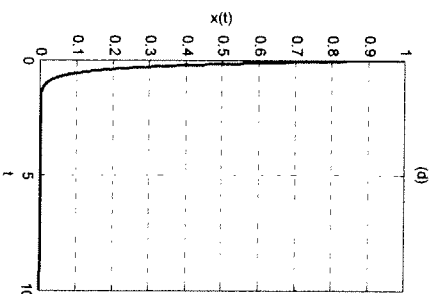
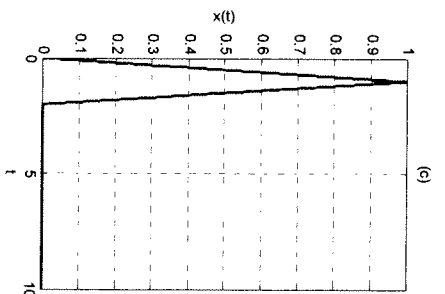
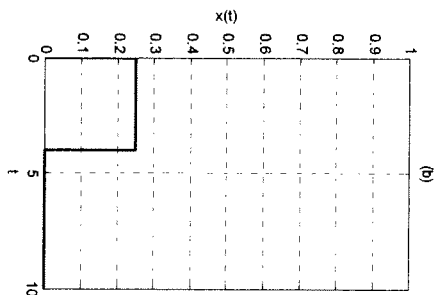
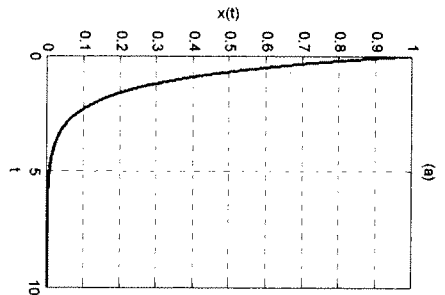
d) For $\omega_p = 15$ rad/sec, $\omega_s = 35$ rad/sec, and $\Delta = 28$ dB, determine the required Butterworth filter order for this filter. (Remember it must be an integer). Using the Table at the end of this problem, plot the Bode plot of your Butterworth filter and verify that all frequencies $\omega > \omega_s$ have magnitude (power) less than Δ_{\max} .

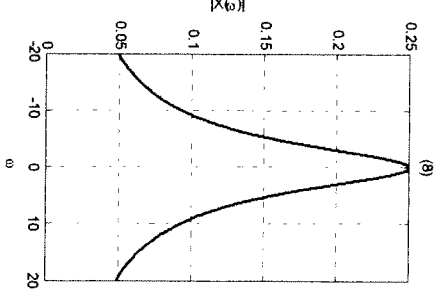
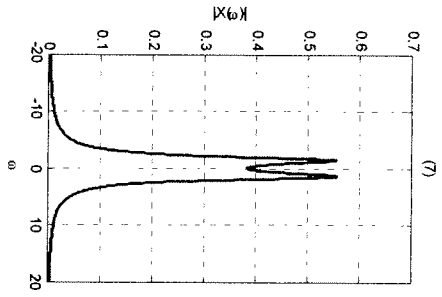
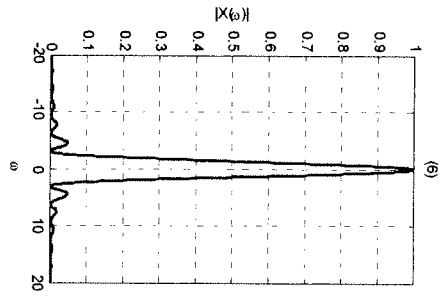
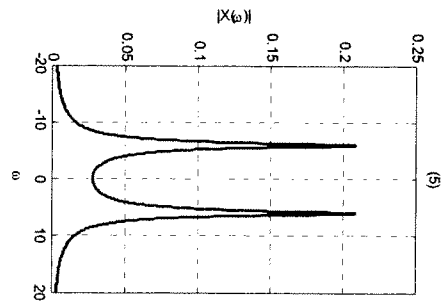
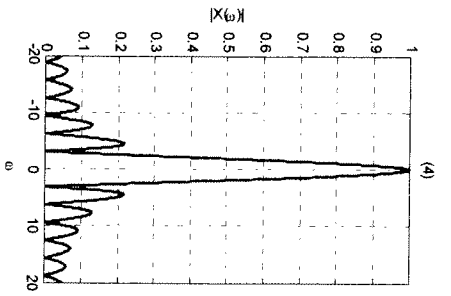
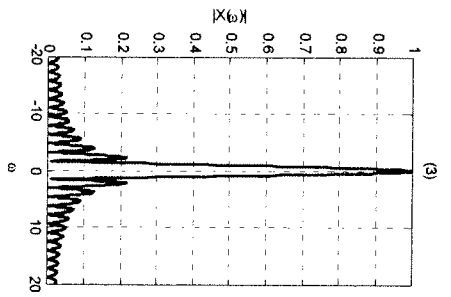
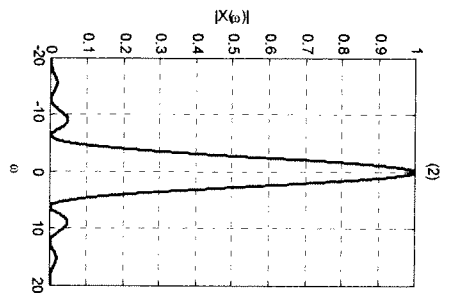
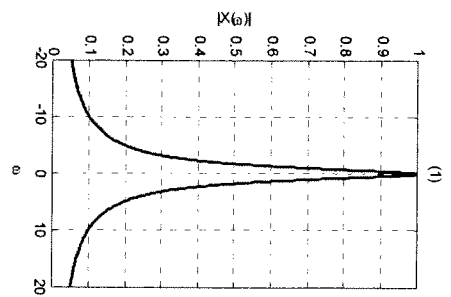
Matlab's **tf** command and **Bode** commands will be really useful here. Turn in your plot.

e) Matlab's command **r = pole(H)**, where H is the transfer function of the Butterworth filter, returns the poles of the transfer function in the array **r**. Using Matlab's commands **abs** and **angle**, relate the magnitude of the poles to ω_p , then plot the pole locations in the complex plane on a circle with radius ω_p . (Note that **angle** returns angles in radians, and you probably want angle in degrees.) Note that the pole locations are all separated by an angle θ . What is this angle?

n	denominator(s)
1	$\frac{s}{\omega_p} + 1$
2	$\left(\frac{s}{\omega_p}\right)^2 + 1.414\left(\frac{s}{\omega_p}\right) + 1$
3	$\left(\frac{s}{\omega_p}\right)^3 + 2\left(\frac{s}{\omega_p}\right)^2 + 2\left(\frac{s}{\omega_p}\right) + 1$
4	$\left(\frac{s}{\omega_p}\right)^4 + 2.6131\left(\frac{s}{\omega_p}\right)^3 + 3.4142\left(\frac{s}{\omega_p}\right)^2 + 2.6131\left(\frac{s}{\omega_p}\right) + 1$
5	$\left(\frac{s}{\omega_p}\right)^5 + 3.2361\left(\frac{s}{\omega_p}\right)^4 + 5.2361\left(\frac{s}{\omega_p}\right)^3 + 5.2361\left(\frac{s}{\omega_p}\right)^2 + 3.2361\left(\frac{s}{\omega_p}\right) + 1$

Table 1: Denominators of Buterworth filter for filter orders 1-5. The numerator for the Butterworth filter is 1.





#1

For the exponential signals, after 5 times constant the signal has basically reached its final value, so $e^{-5\tau} \approx 0$

For signals (a) and (d)

(a) looks like $5\tau \approx 5$ $\tau = 1$ $x(t) \approx e^{-t} u(t)$

(d) looks like $5\tau \approx 1$ $\tau \approx 1/5$ $x(t) \approx e^{-5t} u(t)$

For $x(t) \approx e^{-t} u(t)$ $X(\omega) \approx \frac{1}{1+j\omega}$ $|X(\omega)| \approx \frac{1}{\sqrt{1+\omega^2}}$ $|X(0)| \approx 1$

For $x(t) \approx e^{-5t} u(t)$ $X(\omega) \approx \frac{1}{5+j\omega}$ $|X(\omega)| \approx \frac{1}{\sqrt{25+\omega^2}}$ $|X(0)| \approx 0.2$

The only frequency domain signals with the correct shape are (c) and (e) clearly, by looking at $|X(0)|$ we get $a \leftrightarrow 1$, $d \leftrightarrow 8$

For signals (b), (g), (c), and (e)

(b) looks like $0.25 \text{rect}(\frac{t-2}{4})$ and (g) looks like $0.5 \text{rect}(\frac{t-1}{2})$

(c) looks like $\Lambda(\frac{t-1}{2})$ and (e) looks like $0.5 \Lambda(\frac{t-2}{4})$

For $x(t) \approx 0.25 \text{rect}(\frac{t-2}{4})$ $X(\omega) \approx (0.25)(4) \text{sinc}(\frac{4}{2\pi}\omega) e^{-j2\omega}$

or $|X(\omega)| \approx |\text{sinc}(\frac{2\omega}{\pi})|$

For $x(t) \approx 0.5 \text{rect}(\frac{t-1}{2})$ $X(\omega) \approx (0.5)(2) \text{sinc}(\frac{2}{2\pi}\omega) e^{-j\omega}$

or $|X(\omega)| \approx |\text{sinc}(\frac{\omega}{\pi})|$

For $x(t) \approx \Lambda(\frac{t-1}{2})$ $X(\omega) \approx \frac{2}{2} \text{sinc}^2(\frac{2}{4\pi}\omega) e^{-j\omega}$

$|X(\omega)| \approx \text{sinc}^2(\frac{\omega}{2\pi})$

For $x(t) \approx 0.5 \Lambda(\frac{t-2}{4})$ $X(\omega) \approx 0.5(\frac{4}{2}) \text{sinc}^2(\frac{4}{4\pi}\omega) e^{-j2\omega}$

$|X(\omega)| \approx \text{sinc}^2(\frac{\omega}{\pi})$

#1 (continued)

all of these functions have the same amplitude, so that won't help. Looking at the figures (3) and (4) must correspond to sinc functions and (2) and (6) to sinc^2 functions. The easiest way to figure this out is look at what happens to the "side lobes", all of these are less than 1, so when they are squared they become even smaller.

now let's look at the null locations

for (b) nulls at $\frac{2\omega}{\pi} = \pm 1, \pm 2, \dots$ $\omega = \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \dots$
 ≈ 12 nulls between 0 and 20
 must be (3)

for (g) nulls at $\frac{\omega}{\pi} = \pm 1, \pm 2, \dots$ $\omega = \pi, 2\pi, \dots$
 ≈ 6 nulls between 0 and 20
 must be (4)

for (c) nulls at $\frac{\omega}{2\pi} = \pm 1, \pm 2, \dots$ $\omega = 2\pi, 4\pi, 6\pi, \dots$
 ≈ 3 nulls between 0 and 20
 must be (2)

for (e) nulls at $\frac{\omega}{\pi} = \pm 1, \pm 2, \dots$ ≈ 6 nulls between 0 and 20
 must be (6)

so $b \leftrightarrow 3$ $g \leftrightarrow 4$ $c \leftrightarrow 2$ $e \leftrightarrow 6$

all that's left is (f) and (h)

#1 continued

For (A) and (B) we have

$$\mathcal{X}(s) = \frac{1}{(s+a)^2 + b^2} \Leftrightarrow x(t) = e^{-at} \cos(bt) u(t)$$

$$\mathcal{X}(j\omega) = \frac{1}{(j\omega+a)^2 + b^2} \Leftrightarrow x(t) = e^{-at} \cos(bt) u(t)$$

For (b) there are approximately 9.5 periods between 0 and 10.

each period is approximately $\frac{10}{9.5} = 1.05$ seconds $\approx T$

$$b = \frac{2\pi}{T} \approx \frac{2\pi}{1.05} \approx 6$$

$$5\tau \approx 10 \text{ so } \tau \approx 10/5 = 2 \quad a \approx 0.5$$

$$\text{so } x(t) \approx e^{-0.5t} \cos(6t) \Leftrightarrow \mathcal{X}(s) \approx \frac{1}{(s+0.5)^2 + 6^2}$$

$$|\mathcal{X}(j\omega)| \approx \frac{1}{6^2 + 0.25} \approx 0.03$$

For (c) there are approximately 3 periods between 0 and 10,

each period is approximately $\frac{10}{3} \approx 3.3$ seconds $\approx T$

$$b \approx \frac{2\pi}{T} \approx \frac{2\pi}{3.3} \approx 1.9$$

$$5\tau \approx 8 \quad \tau \approx 8/5 \quad a \approx 0.63$$

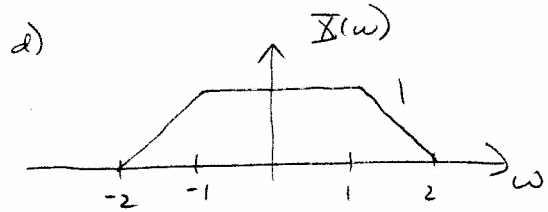
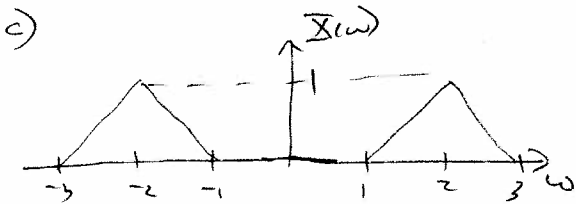
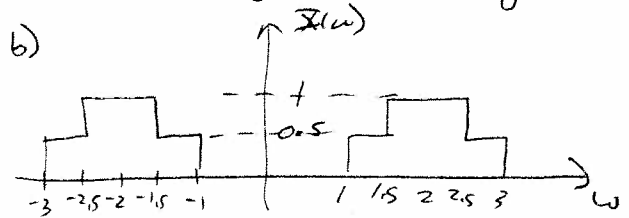
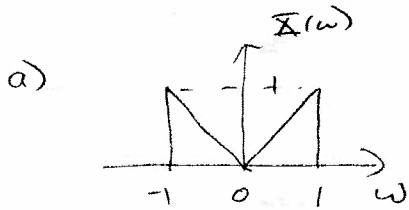
$$\text{so } x(t) \approx e^{-0.63t} \cos(1.9t) \quad \mathcal{X}(s) \approx \frac{1}{(s+0.63)^2 + 1.9^2}$$

$$|\mathcal{X}(j\omega)| \approx \frac{1}{1.9^2 + 0.63^2} \approx 0.25$$

Comparing, $h \leftrightarrow s$ and $f \leftrightarrow \tau$

Summary	a	b	c	d	e	f	g	h
	1	3	2	8	6	7	4	5

3.21 Compute the inverse Fourier transforms of the following



a) $X(\omega) = \text{rect}\left(\frac{\omega}{2}\right) - \mathcal{L}\left(\frac{\omega}{2}\right)$

$$\frac{1}{W} \text{rect}\left(\frac{\omega}{2\pi W}\right) \leftrightarrow \text{sinc}(Wt)$$

$$\text{rect}\left(\frac{\omega}{2\pi W}\right) = \text{rect}\left(\frac{\omega}{2}\right) \quad \pi W = 1 \quad W = \frac{1}{\pi}$$

$$\pi \text{rect}\left(\frac{\omega}{2}\right) \leftrightarrow \text{sinc}\left(\frac{t}{\pi}\right) \quad \text{rect}\left(\frac{\omega}{2}\right) \leftrightarrow \frac{1}{\pi} \text{sinc}\left(\frac{t}{\pi}\right)$$

$$\frac{1}{B} \mathcal{L}\left(\frac{\omega}{4\pi B}\right) \leftrightarrow \text{sinc}^2(Bt)$$

$$\mathcal{L}\left(\frac{\omega}{4\pi B}\right) = \mathcal{L}\left(\frac{\omega}{2}\right) \rightarrow B = \frac{1}{2}\pi$$

$$2\pi \mathcal{L}\left(\frac{\omega}{2}\right) \leftrightarrow \text{sinc}^2\left(\frac{t}{2\pi}\right) \quad \mathcal{L}\left(\frac{\omega}{2}\right) \leftrightarrow \frac{1}{2\pi} \text{sinc}^2\left(\frac{t}{2\pi}\right)$$

$$X(t) = \frac{1}{\pi} \text{sinc}\left(\frac{t}{\pi}\right) - \frac{1}{2\pi} \text{sinc}^2\left(\frac{t}{2\pi}\right)$$

b) $X(\omega) = 0.5 \text{rect}\left(\frac{\omega-2}{2}\right) + 0.5 \text{rect}\left(\frac{\omega-2}{1}\right) + 0.5 \text{rect}\left(\frac{\omega+2}{2}\right) + 0.5 \text{rect}\left(\frac{\omega+2}{1}\right)$

$$\frac{1}{W} \text{rect}\left(\frac{\omega}{2\pi W}\right) \leftrightarrow \text{sinc}(Wt)$$

$$\text{rect}\left(\frac{\omega}{2}\right) = \text{rect}\left(\frac{\omega}{2\pi W}\right) \quad W = \frac{1}{\pi}$$

$$\text{rect}\left(\frac{\omega}{1}\right) = \text{rect}\left(\frac{\omega}{2\pi W}\right) \quad W = \frac{1}{2\pi}$$

$$\text{rect}\left(\frac{\omega}{2}\right) \leftrightarrow \frac{1}{\pi} \text{sinc}\left(\frac{t}{\pi}\right)$$

$$\text{rect}\left(\frac{\omega}{1}\right) \leftrightarrow \frac{1}{2\pi} \text{sinc}\left(\frac{t}{2\pi}\right)$$

$$X(t) = \left[\frac{1}{\pi} \text{sinc}\left(\frac{t}{\pi}\right) + \frac{1}{2\pi} \text{sinc}\left(\frac{t}{2\pi}\right) \right] \cos(2t)$$

#2 (3.21) (continued)

$$c) X(\omega) = \mathcal{L}\left(\frac{\omega-2}{2}\right) + \mathcal{L}\left(\frac{\omega+2}{2}\right)$$

$$\frac{1}{B} \mathcal{L}\left(\frac{\omega}{4\pi B}\right) \leftrightarrow \text{sinc}^2(Bt)$$

$$4\pi B = 2 \quad B = \frac{1}{2\pi}$$

$$\mathcal{L}\left(\frac{\omega}{2}\right) \leftrightarrow \frac{1}{2\pi} \text{sinc}^2\left(\frac{t}{2\pi}\right)$$

$$X(t) = \frac{1}{\pi} \text{sinc}^2\left(\frac{t}{2\pi}\right) \cos(2t)$$

$$d) X(\omega) = 2 \mathcal{L}\left(\frac{\omega}{4}\right) - \mathcal{L}\left(\frac{\omega}{2}\right)$$

$$\frac{1}{B} \mathcal{L}\left(\frac{\omega}{4\pi B}\right) \leftrightarrow \text{sinc}^2(Bt)$$

$$\mathcal{L}\left(\frac{\omega}{2}\right) \leftrightarrow \frac{1}{2\pi} \text{sinc}^2\left(\frac{t}{2\pi}\right) \quad B = \frac{1}{2\pi}$$

$$\mathcal{L}\left(\frac{\omega}{4}\right) = \mathcal{L}\left(\frac{\omega}{4\pi B}\right) \quad B = \frac{1}{\pi}$$

$$\mathcal{L}\left(\frac{\omega}{4}\right) \leftrightarrow \frac{1}{\pi} \text{sinc}^2\left(\frac{t}{\pi}\right)$$

$$X(t) = \frac{2}{\pi} \text{sinc}^2\left(\frac{t}{\pi}\right) - \frac{1}{2\pi} \text{sinc}^2\left(\frac{t}{2\pi}\right)$$

3.22 Compute the inverse Fourier transform of the following signals

a) $X(\omega) = \cos(4\omega) \quad -\infty < \omega < \infty$

c) $X(\omega) = \text{rect}(\frac{\omega}{4}) \cos(\frac{\pi\omega}{2})$

b) $X(\omega) = \sin^2(3\omega) \quad -\infty < \omega < \infty$

d) $X(\omega) = \frac{\sin(\frac{\omega}{2})}{j\omega+2} e^{-j2\omega} \quad -\infty < \omega < \infty$

a) $X(\omega) = \cos(4\omega) = \frac{1}{2} e^{j\omega 4} + \frac{1}{2} e^{-j\omega 4}$

$$x(t) = \frac{1}{2} \delta(t+4) + \frac{1}{2} \delta(t-4)$$

b) $X(\omega) = \sin^2(3\omega) = \left[\frac{e^{j3\omega} - e^{-j3\omega}}{2j} \right] \left[\frac{e^{j3\omega} - e^{-j3\omega}}{2j} \right]$

$$= \frac{1}{4} \left[e^{j6\omega} - 2 + e^{-j6\omega} \right]$$

$$x(t) = -\frac{1}{4} \delta(t+6) + \frac{1}{2} \delta(t) - \frac{1}{4} \delta(t-6)$$

c) $X(\omega) = \text{rect}\left(\frac{\omega}{4}\right) \cos\left(\frac{\pi\omega}{2}\right) = \text{rect}\left(\frac{\omega}{4}\right) \left[\frac{e^{j\frac{\pi\omega}{2}} + e^{-j\frac{\pi\omega}{2}}}{2} \right]$

$$\text{rect}\left(\frac{\omega}{2\pi W}\right) = \text{rect}\left(\frac{\omega}{4}\right) \Rightarrow 2\pi W = 4 \quad W = \frac{4}{2\pi} = \frac{2}{\pi}$$

$$\frac{1}{W} \text{rect}\left(\frac{\omega}{2\pi W}\right) \leftrightarrow \text{sinc}(Wt)$$

$$\frac{\pi}{2} \text{rect}\left(\frac{\omega}{4}\right) \leftrightarrow \text{sinc}\left(\frac{2t}{\pi}\right)$$

$$\text{rect}\left(\frac{\omega}{4}\right) \leftrightarrow \frac{2}{\pi} \text{sinc}\left(\frac{2t}{\pi}\right)$$

$$\frac{\text{rect}\left(\frac{\omega}{4}\right) e^{j\frac{\pi\omega}{2}}}{2} + \frac{\text{rect}\left(\frac{\omega}{4}\right) e^{-j\frac{\pi\omega}{2}}}{2} \leftrightarrow \frac{1}{\pi} \text{sinc}\left(\frac{2(t+\frac{\pi}{2})}{\pi}\right) + \frac{1}{\pi} \text{sinc}\left(\frac{2(t-\frac{\pi}{2})}{\pi}\right)$$

$$x(t) = \frac{1}{\pi} \text{sinc}\left(\frac{2t+\pi}{\pi}\right) + \frac{1}{\pi} \text{sinc}\left(\frac{2t-\pi}{\pi}\right)$$

(#3) 3.22 (continued)

$$d) X(\omega) = \frac{\sin(\frac{\omega}{2}) e^{-j\omega 2}}{j\omega + 2} \quad -\infty < \omega < \infty$$

$$= \left[\frac{e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}}{2j} \right] e^{-j\omega 2} \frac{1}{j\omega + 2}$$

$$= \left[\frac{e^{-j\frac{3\omega}{2}} - e^{-j\frac{5\omega}{2}}}{2j} \right] \frac{1}{j\omega + 2}$$

$$= \frac{1}{2j} \frac{1}{j\omega + 2} e^{-j\frac{3\omega}{2}} - \frac{1}{2j} \frac{1}{j\omega + 2} e^{-j\frac{5\omega}{2}}$$

$$V(t) = e^{-2t} u(t) \leftrightarrow V(\omega) = \frac{1}{j\omega + 2}$$

$$\text{So } x(t) = \frac{1}{2j} e^{-2(t-\frac{3}{2})} u(t-\frac{3}{2}) - \frac{1}{2j} e^{-2(t-\frac{5}{2})} u(t-\frac{5}{2})$$

#4

8.24 Given $X(\omega) = \frac{1}{j\omega + b}$

Find the Fourier transform of the following

a) $v(t) = x(st-4)$

e) $v(t) = \frac{d^2}{dt^2} x(t)$

b) $v(t) = t^2 x(t)$

f) $v(t) = x(t) * x(t)$

c) $v(t) = x(t) e^{j2t}$

g) $v(t) = x(t)^2$

d) $v(t) = x(t) \cos(4t)$

h) $v(t) = \frac{1}{jt-b}$

Note $x(t) = e^{-bt} u(t)$

a) $v(t) = x(st-4) = x(s(t-\frac{4}{s}))$

if $x(t) \leftrightarrow X(\omega) = \frac{1}{j\omega + b}$

$$x(st) \leftrightarrow \frac{1}{s} X\left(\frac{\omega}{s}\right) = \frac{\frac{1}{s}}{j\left(\frac{\omega}{s}\right) + b} = \frac{1}{j\omega + sb}$$

$$x\left(s\left(t - \frac{4}{s}\right)\right) \leftrightarrow \frac{1}{s} X\left(\frac{\omega}{s}\right) e^{-j\omega \frac{4}{s}} = \boxed{\frac{e^{-j\omega \frac{4}{s}}}{j\omega + sb} = V(\omega)}$$

b) $v(t) = t^2 x(t)$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$\frac{d}{d\omega} X(\omega) = \int_{-\infty}^{\infty} -jt x(t) e^{-j\omega t} dt$$

$$\frac{d^2}{d\omega^2} X(\omega) = \int_{-\infty}^{\infty} (-jt)^2 x(t) e^{-j\omega t} dt$$

$$\mathcal{F}\{-t^2 x(t)\} = \frac{d^2}{d\omega^2} X(\omega)$$

$$\mathcal{F}\{t^2 x(t)\} = -\frac{d^2}{d\omega^2} X(\omega)$$

$$\frac{d}{d\omega} X(\omega) = \frac{d}{d\omega} (j\omega + b)^{-1} = \frac{-j}{(j\omega + b)^2}$$

$$\frac{d^2}{d\omega^2} X(\omega) = \frac{(-j)(-2j)}{(j\omega + b)^3} = \frac{-2}{(j\omega + b)^3}$$

$$-\frac{d^2}{d\omega^2} X(\omega) = \boxed{\frac{2}{(j\omega + b)^3} = V(\omega)}$$

#4

3.24 (continued)

c) $v(t) = x(t) e^{2t}$

$$V(\omega) = X(\omega - 2) = \frac{1}{j(\omega - 2) + b} = V(\omega)$$

d) $v(t) = x(t) \cos(4t)$

$$V(\omega) = \frac{1}{2} X(\omega + 4) + \frac{1}{2} X(\omega - 4)$$

$$= \frac{1}{2} \frac{1}{j(\omega + 4) + b} + \frac{1}{2} \frac{1}{j(\omega - 4) + b} = V(\omega)$$

e) $v(t) = \frac{d^2}{dt^2} x(t)$

$$\frac{d^2}{dt^2} x(t) \leftrightarrow (j\omega)^2 X(\omega) \Rightarrow V(\omega) = \frac{-\omega^2}{j\omega + b}$$

f) $v(t) = x(t) * x(t)$

$$V(\omega) = X(\omega) X(\omega) = \frac{1}{(j\omega + b)^2} = V(\omega)$$

g) $v(t) = x(t)^2$

$$V(\omega) = \frac{1}{2\pi} X(\omega) * X(\omega) \quad \text{This is easier in the time domain}$$

$$x(t) = e^{-bt} u(t) \quad x^2(t) = e^{-2bt} u(t)$$

$$V(\omega) = \frac{1}{j\omega + 2b}$$

h) $v(t) = \frac{1}{jt - b}$

$$x(t) = e^{-bt} u(t) \leftrightarrow X(\omega) = \frac{1}{j\omega + b}$$

$$\text{First use duality } X(t) \leftrightarrow 2\pi X(-\omega)$$

$$x_1(t) = \frac{1}{jt + b} \leftrightarrow 2\pi e^{b\omega} u(-\omega) = X_1(\omega)$$

$$x_1(-t) \leftrightarrow X_1(-\omega) \text{ and } -x_1(-t) = -X_1(-\omega)$$

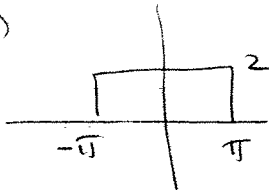
$$\text{so } \frac{1}{jt - b} \leftrightarrow -2\pi e^{-b\omega} u(\omega) = V(\omega) \quad \frac{1}{jt + b} \leftrightarrow -2\pi e^{-b\omega} u(\omega)$$

#5

$$3.27) \quad X(\omega) = \begin{cases} 2 & |\omega| < \pi \\ 0 & \text{else} \end{cases}$$

$$V(\omega) = X(\omega - \omega_0) + X(\omega + \omega_0)$$

- a) determine $x(t)$ in closed form
 b) determine $V(t)$ in closed form

a) $X(\omega)$ 

$$X(\omega) = 2 \operatorname{rect}\left(\frac{\omega}{2\pi}\right)$$

$$\operatorname{rect}\left(\frac{\omega}{2\pi W}\right) = \operatorname{rect}\left(\frac{\omega}{2\pi}\right) \Rightarrow W=1$$

$$\frac{1}{W} \operatorname{rect}\left(\frac{\omega}{2\pi W}\right) \leftrightarrow \operatorname{sinc}(Wt)$$

$$\text{so } \operatorname{rect}\left(\frac{\omega}{2\pi}\right) \leftrightarrow \operatorname{sinc}(t)$$

$$2 \operatorname{rect}\left(\frac{\omega}{2\pi}\right) \leftrightarrow \boxed{2 \operatorname{sinc}(t) = x(t)}$$

b) for $w(t) = x(t) \cos(\omega_0 t) \leftrightarrow W(\omega) = \frac{1}{2} X(\omega + \omega_0) + \frac{1}{2} X(\omega - \omega_0)$

for $V(t) = 2 x(t) \cos(\omega_0 t) \leftrightarrow V(\omega) = X(\omega + \omega_0) + X(\omega - \omega_0)$

$$V(t) = 2 x(t) \cos(\omega_0 t) = \boxed{4 \operatorname{sinc}(t) \cos(\omega_0 t) = V(t)}$$

#10

a) $f(t) = \cos(4t) + \cos(5t)$

$F(\omega) = \pi [\delta(\omega+4) + \delta(\omega-4)] + \pi [\delta(\omega+5) + \delta(\omega-5)]$

b) $y(t) = f(t) \text{rect}(\frac{t}{T})$

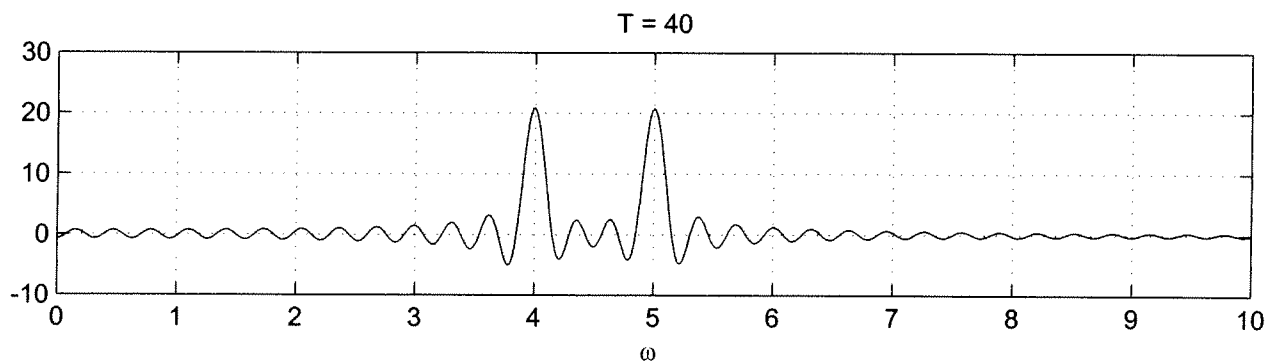
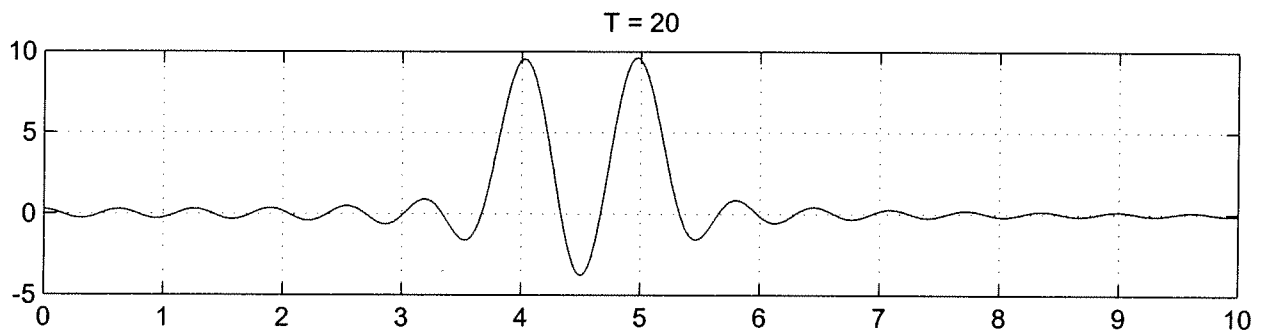
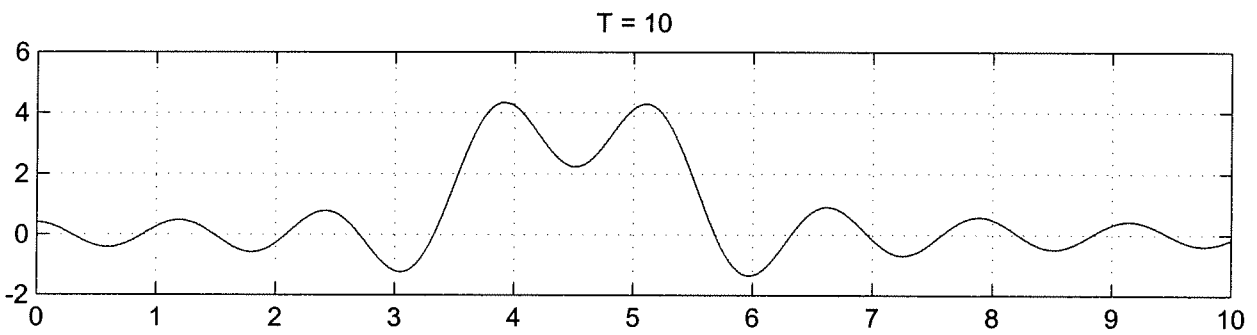
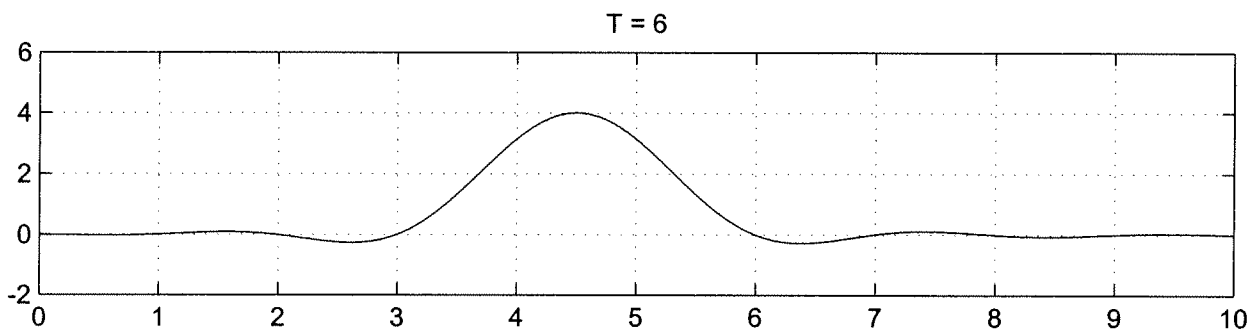
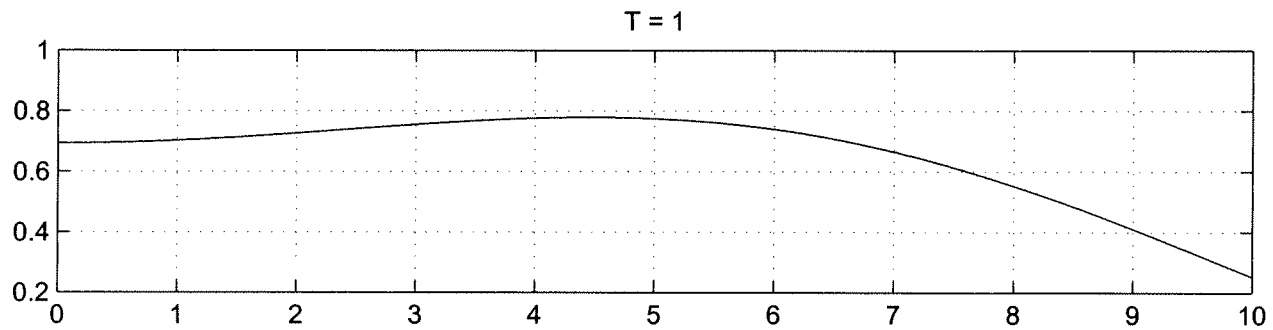
$Y(\omega) = \frac{1}{2\pi} F(\omega) * \mathcal{F}\{\text{rect}(\frac{t}{T})\}$

$= \frac{1}{2\pi} F(\omega) * T \text{sinc}(\frac{T}{2\pi} \omega)$

$Y(\omega) = \frac{T}{2} \left\{ \text{sinc}\left(\frac{T}{2\pi}(\omega+4)\right) + \text{sinc}\left(\frac{T}{2\pi}(\omega-4)\right) + \text{sinc}\left(\frac{T}{2\pi}(\omega+5)\right) + \text{sinc}\left(\frac{T}{2\pi}(\omega-5)\right) \right\}$

c) See Attached Plots

CAMPUS



```
%  
% plot for problem 5 of homework 5  
%
```

```
w = linspace(0,10,1000);
```

```
T = 1;
```

```
a = T/(2*pi);
```

```
Y1 = @(w) (T/2)*(sinc(a*(w+4))+sinc(a*(w-4))+sinc(a*(w+5))+sinc(a*(w-5)))
```

```
T = 6;
```

```
a = T/(2*pi);
```

```
Y6 = @(w) (T/2)*(sinc(a*(w+4))+sinc(a*(w-4))+sinc(a*(w+5))+sinc(a*(w-5)))
```

```
T = 10;
```

```
a = T/(2*pi);
```

```
Y10 = @(w) (T/2)*(sinc(a*(w+4))+sinc(a*(w-4))+sinc(a*(w+5))+sinc(a*(w-5)))
```

```
T = 20;
```

```
a = T/(2*pi);
```

```
Y20 = @(w) (T/2)*(sinc(a*(w+4))+sinc(a*(w-4))+sinc(a*(w+5))+sinc(a*(w-5)))
```

```
T = 40;
```

```
a = T/(2*pi);
```

```
Y40 = @(w) (T/2)*(sinc(a*(w+4))+sinc(a*(w-4))+sinc(a*(w+5))+sinc(a*(w-5)))
```

```
orient tall
```

```
subplot(5,1,1); plot(w,Y1(w)); grid; title('T = 1');
```

```
subplot(5,1,2); plot(w,Y6(w)); grid; title('T = 6');
```

```
subplot(5,1,3); plot(w,Y10(w)); grid; title('T = 10');
```

```
subplot(5,1,4); plot(w,Y20(w)); grid; title('T = 20');
```

```
subplot(5,1,5); plot(w,Y40(w)); grid; title('T = 40');
```

```
xlabel('\omega');
```

#7

(a) $\Delta = -10 \log_{10} \frac{1}{1 + (\frac{\omega_s}{\omega_p})^{2n}}$

$10^{-\Delta/10} = \frac{1}{1 + (\frac{\omega_s}{\omega_p})^{2n}}$

$1 + (\frac{\omega_s}{\omega_p})^{2n} = 10^{+\Delta/10}$

$(\frac{\omega_s}{\omega_p})^{2n} = 10^{+\Delta/10} - 1$

$2n \ln(\frac{\omega_s}{\omega_p}) = \ln[10^{+\Delta/10} - 1]$

$n = \frac{\ln[10^{+\Delta/10} - 1]}{2 \ln(\frac{\omega_s}{\omega_p})}$

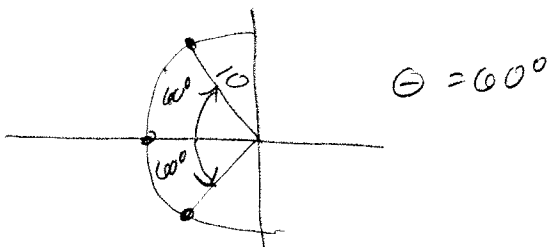
(b) $\omega_p = 10 \text{ rad/sec}$ $\omega_s = 20 \text{ rad/sec}$ $\Delta = +18 \text{ dB}$

$n = \frac{\ln[10^{+18} - 1]}{2 \ln(\frac{20}{10})} = 2.978$ need $n = 3$ See plot

(c) For this filter the poles are at $-10, -5 \pm j 8.660$

The magnitudes for each of these is $10 \text{ rad/sec} = \omega_p$

The phases are $-180, 120, -120$ poles separated by 60°



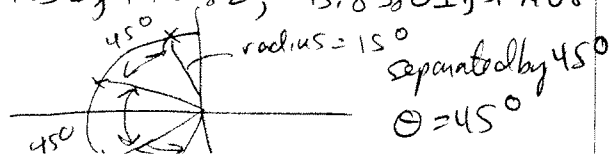
(d) $\omega_p = 15 \text{ rad/sec}$ $\omega_s = 35 \text{ rad/sec}$ $\Delta = +28 \text{ dB}$

$n = \frac{\ln[10^{28} - 1]}{2 \ln(\frac{35}{15})} = 3.8$ need $n = 4$ See plot

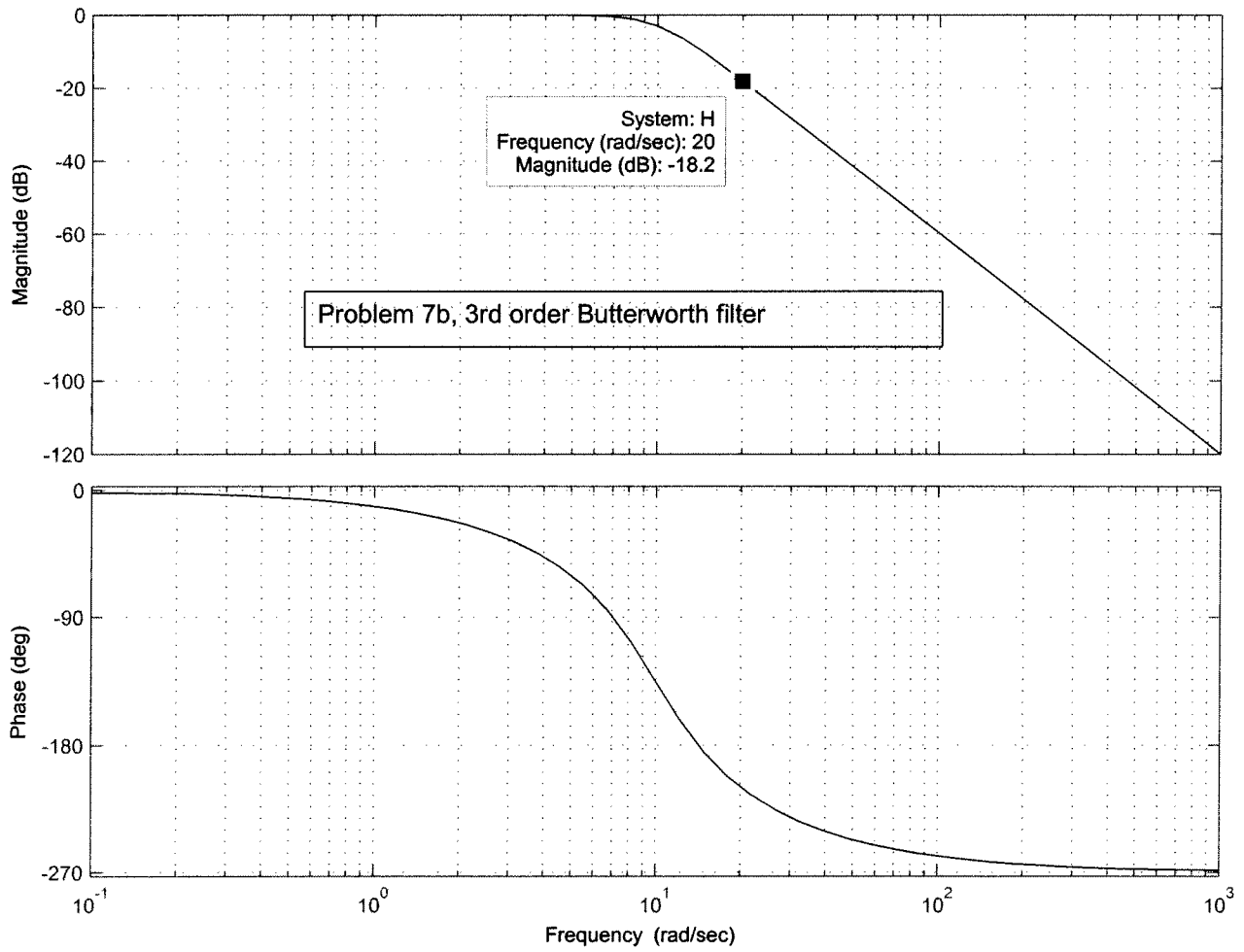
(e) For this filter the poles are at $-5.7403 \pm j 13.8582, -13.8580 \pm j 5.7408$

The magnitudes are all $15 \text{ rad/sec} = \omega_p$

The phases are $\pm 112.5^\circ, \pm 157.5^\circ$



Bode Diagram



Bode Diagram

