

ECE 300
Signals and Systems
Homework 3

Due Date: Thursday March 22, 2007 at the beginning of class

Problems

1. K & H Problem 1.15. *Hint: draw pictures and think about when the delta function is included in your interval. The answer to **d** is 1/2.*

2. For systems modeled by the following equations, determine the systems impulse response, $h(t)$. Assume the system is initially at rest (has zero initial energy), that is, $h(t_0) = 0$.

a) $\dot{y}(t) + 2y(t) = x(t)$ b) $y(t) = \int_{-\infty}^t e^{5(t-\lambda)} x(\lambda) d\lambda$ c) $y(t) = 5x(t)$

3. An LTI system responds to the following inputs with the corresponding outputs:

If (input) $x(t) = u(t)$ then (output) $y(t) = (1 - e^{-2t})u(t)$

If (input) $x(t) = \cos(2t)$ then (output) $y(t) = 0.3953 \cos(2t - 71.56^\circ)$

Find (the output) $y(t)$ for the following inputs:

a) $x(t) = 2u(t) - 2u(t-1)$ b) $x(t) = 4 \cos(2(t-2))$ c) $x(t) = 5u(t) + 10 \cos(2t - 6)$

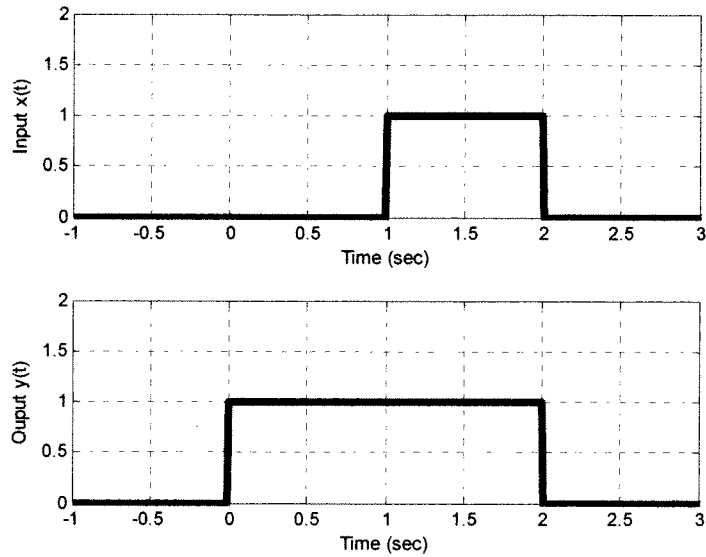
d) $x(t) = \delta(t)$ e) $x(t) = tu(t)$

*Hint: Use the results from problem k & H 1.23 for parts **d** and **e**.*

4. For the following system models, determine if the model represents a BIBO stable system. If the system is not BIBO stable, give an input $x(t)$ that demonstrates this.

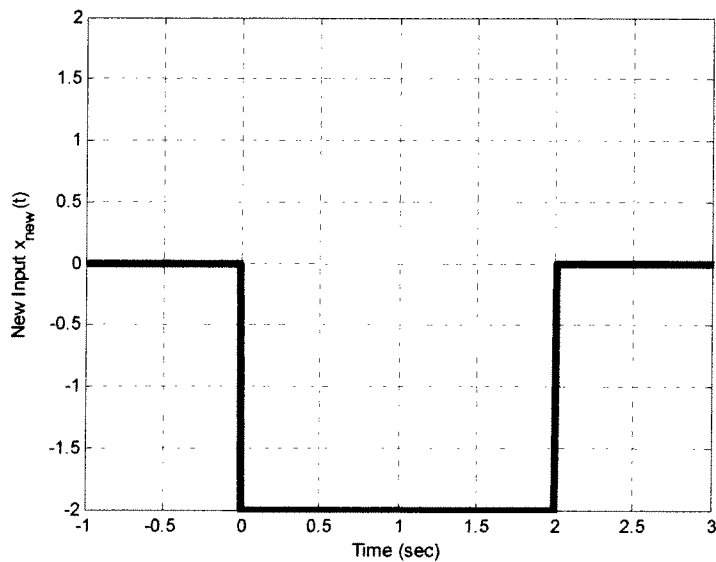
a) $y(t) = \dot{x}(t) + x(t)$ b) $y(t) = \sin\left(\frac{1}{x(t)}\right)$
c) $y(t) = x^2(t)$ d) $y(t) + y(t)x(t) = x(t)$

5. Assume we know we have an LTI system, and that for input $x(t)$ (shown below) we get output $y(t)$ (also shown below).



a) Is the system causal? Yes or No (Circle one, no work required)

b) Now assume the input to this LTI system is $x_{new}(t)$, shown below. Sketch and accurately label the output of the system.



6. Matlab/Pre-Lab: Experimentation with Sound, Small Signal Models

You need to build the circuit (in the lab handout) for the lab and bring it to lab. One circuit per group.

The MATLAB program `linear_systems_and_sounds.m` allows you to specify an input signal (lines 11-12) and system (line 24), and listen to how the sound changes as the signal passes through the system. We will use the sounds produced to try to identify the systems as linear or nonlinear. Recall from class that a system is linear if and only if an input signal at frequency f produces an output signal at frequency f . The phase angle and amplitude between the input and output may change, but the frequency cannot. Just think about how transfer functions affected periodic signals when you were using phasors to represent sinusoids- the transfer function could change the magnitude and phase but not the frequency of the input signal. *Note that we can only hear sounds when the speaker is vibrating.*

a) Using the code as is (i.e. with $y=x$), investigate how the frequency of the sinusoidal input signal affects the sound of the signal. Specifically, describe the sound you hear for input signal frequencies f_0 of 200, 1000, and 5000 Hz.. Note that you need to run the program, wait for the input sound (and graph) to appear, then hit enter and you will hear the output sound and see the output plotted in the lower panel.

b) Now change the system so the output is zero, i.e., $y = 0*x$ (this will make sure the y vector is the correct length). What do you hear for the system output, if anything?

c) Now change the system so the output is $y=0*x+10$; What do you hear? How fast (at what frequency) is the speaker vibrating?

d) For each of the mathematical models of systems in the following table, assume x is a periodic signal with frequency 200 Hz. For each of these models, determine if the mathematical model of the system is linear or not, and if you would classify the system models as linear or not based on both listening to and looking at the input and output. Discrete-time solutions for model (i) is included in the code, and is commented out. Uncomment the correct lines when you need to run these models.

e) For many system, such as transistors, we are really interested in **small signal models**. For these systems we only care about the behavior of the system about a set or bias point for an input with a small amplitude. For $x(t)$ small, we can use a Taylor series approximation of $y(x)$ around $x(t) = 0$ of the form

$$y(x) \approx y_0 + v x(t)$$

where y_0 and v are determined by the Taylor series expansion of $y(x)$ about the point $x = 0$. If we subtract the bias point (y_0) from both sides of the equation and rename variables we have

$$y(x) - y_0 \equiv \Delta y(x) = vx(t)$$

This is now a description of a linear system between input $x(t)$ and output $\Delta y(x)$. This relationship is only approximately linear if the input $x(t)$ is sufficiently small that the Taylor series approximation is valid.

For the systems that are mathematically nonlinear, but appeared/sounded linear, determine the small signal linear approximation, that is, determine y_0 and v .

Note that our input amplitude is 1, and hence something like $z = x/10$ would be considered small. The following formula's for Taylor series about the point $z = 0$ may be helpful:

Taylor series (general form) $f(z) \approx f(0) + z \left. \frac{df}{dz} \right|_{z=0}$

$$y(z) = [a + bz]^c \approx a^c + ca^{c-1}bz \quad y(z) = \sin(az) \approx az$$

$$y(z) = \log(a + bz) = \log(a) + \frac{bz}{a+b} \quad y(z) = e^{az} \approx 1 + az$$

(Note that log here means natural log, typing log in Matlab takes the natural log of the function.)

As a check, for those systems where the small signal model is valid for this particular input signal, if you look at the plots from *linear_systems_and_sounds.m*, the output signal should be centered at y_0 and should oscillate between $y_0 + v$ and $y_0 - v$. **Show your work in deriving the small signal model, do not just look at the system output!**

Don't forget that if you want to process element by element, you may need to use `.`, `./`, or `.^` in Matlab*

Fill-in and turn in the following table with this homework assignment

System	Mathematical Model	Mathematical Model Linear? (Y/N)	Sounds/Looks Linear?(Y/N)	Small Signal Model (if valid)
a	$y(t) = \frac{1}{2+x(t)}$	N	N	
b	$y(t) = \frac{1}{2 + \frac{x(t)}{10}}$	N	Y	$y - 0.5 = -0.025x$ Δy
c	$y(t) = \sin(2x(t))$	N	N	
d	$y(t) = \sin\left(\frac{x(t)}{10}\right)$	N	Y	$y = 0.1x$
e	$y(t) = x(t) $	N	N	
f	$y(t) = e^{x(t)}$	N	N	
g	$y(t) = \log(1.1 + x(t))$	N	N	
h	$y(t) = \left(10.1 + \frac{x(t)}{10}\right)^{0.2}$	N	Y	$y - 1.59 = 0.0031x$ Δy
i	$\dot{y}(t) + ay(t) = bx^2(t)$	N	N	

#1

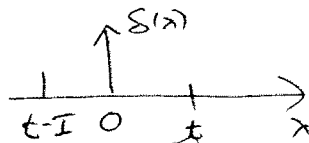
K+H 1.15

$$y(t) = \frac{1}{I} \int_{t-I}^t x(\lambda) d\lambda \quad I > 0$$

continuous-time moving average filter

(a) assume $x(t) = \delta(t)$ so $y(t) = \frac{1}{I} \int_{t-I}^t \delta(\lambda) d\lambda$

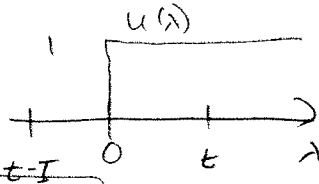
draw a picture!


 $y(t) = \frac{1}{I}$ if $\delta(\lambda)$ is in the region of integration

$$y(t) = \begin{cases} \frac{1}{I} & 0 < t < I \\ 0 & \text{else} \end{cases}$$

(b) assume $x(t) = u(t)$ so $y(t) = \frac{1}{I} \int_{t-I}^t u(\lambda) d\lambda$

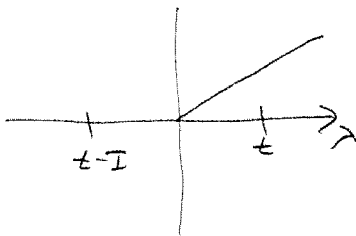
draw a picture!



$$y(t) = \begin{cases} \frac{t}{I} & 0 < t < I \\ 1 & t \geq I \\ 0 & \text{else} \end{cases}$$

(c) assume $x(t) = tu(t)$ so $y(t) = \frac{1}{I} \int_{t-I}^t \lambda u(\lambda) d\lambda$

draw a picture!



$$\text{for } 0 < t < I \quad y(t) = \frac{1}{I} \int_0^t \lambda^2 d\lambda = \frac{t^2}{2I}$$

$$\text{for } t \geq I \quad y(t) = \frac{1}{I} \int_{t-I}^t \lambda^2 d\lambda = \frac{t^2 - (t-I)^2}{2I}$$

$$y(t) = \begin{cases} \frac{t^2}{2I} & 0 < t \leq I \\ \frac{t^2 - (t-I)^2}{2I} & t \geq I \\ 0 & \text{else} \end{cases}$$

$$\textcircled{d} \text{ for } t > I \quad y(t) = \frac{t^2 - (t-I)^2}{2I} = \frac{t^2 - [t^2 - 2tI + I^2]}{2I}$$
$$= \frac{2tI - I^2}{2I} = \frac{2It - I^2}{2I} = t - \frac{I}{2}$$

input

$$x(t) = t$$

output

$$y(t) = t - I/2$$

delay is $I/2$

#2

(a) $y'(t) + 2y(t) = x(t)$

$$\frac{d}{dt} (e^{2t} y(t)) = (y'(t) + 2y(t)) e^{2t} = e^{2t} x(t)$$

$$\int_{t_0}^t \frac{d}{d\lambda} (e^{2\lambda} y(\lambda)) d\lambda = \int_{t_0}^t e^{2\lambda} x(\lambda) d\lambda$$

$$e^{2t} y(t) - e^{2t_0} y(t_0) = \int_{t_0}^t e^{2\lambda} x(\lambda) d\lambda$$

$$y(t) = e^{2(t-t_0)} y(t_0) + \int_{t_0}^t e^{2(\lambda-t)} x(\lambda) d\lambda$$

for impulse response $x(\lambda) = \delta(\lambda)$ $y(t) = h(t)$

$$h(t) = e^{2(t-t_0)} h(t_0) + \int_{t_0}^t e^{2(\lambda-t)} \delta(\lambda) d\lambda =$$

$$h(t_0) = 0$$

$$h(t) = \int_{t_0}^t e^{2(\lambda-t)} \delta(\lambda) d\lambda = e^{-2t} \text{ as long as } t_0 \leq 0^- \text{ and } t > 0$$

$$\boxed{h(t) = e^{-2t} u(t)}$$

(b) $y(t) = \int_{-\infty}^t e^{s(t-\lambda)} x(\lambda) d\lambda$

for impulse response $x(\lambda) = \delta(\lambda)$ $y(t) = h(t)$

$$h(t) = \int_{-\infty}^t e^{s(t-\lambda)} \delta(\lambda) d\lambda = \boxed{e^{st} u(t) = h(t)}$$

need $t > 0$

(c) $\boxed{h(t) = 5\delta(t)}$

#3: LTI system

$$x_1(t) = u(t) \longrightarrow y_1(t) = (1 - e^{-2t})u(t)$$

$$x_2(t) = \cos(2t) \longrightarrow y_2(t) = 0.3953 \cos(2t - 71.56^\circ)$$

$$a) x_{\text{new}}(t) = 2u(t) - 2u(t-1) \longrightarrow y_{\text{new}}(t) = 2y_1(t) - 2y_1(t-1)$$

$$y_{\text{new}}(t) = 2(1 - e^{-2t})u(t) - 2(1 - e^{-2(t-1)})u(t-1)$$

$$b) x_{\text{new}}(t) = 4\cos(2(t-2)) \longrightarrow 4y_2(t-2) = 1.5812 \cos(2(t-2) - 71.56^\circ) = y_{\text{new}}(t)$$

$$c) x_{\text{new}}(t) = 5x_1(t) + 10x_2(t-3)$$

$$y_{\text{new}}(t) = 5(1 - e^{-2t})u(t) + 3.953 \cos(2(t-3) - 71.56^\circ)$$

$$d) \text{ For } x_{\text{new}}(t) = \delta(t) = \frac{d}{dt} x_1(t) \quad y_{\text{new}}(t) = \frac{d}{dt} y_1(t)$$

$$\begin{aligned} \frac{d}{dt} y_1(t) &= 2e^{-2t}u(t) - (1 - e^{-2t})\delta(t) \\ &= 2e^{-2t}u(t) - (1 - e^{-2t}) \Big|_{t=0} \delta(t) = 2e^{-2t}u(t) = y_{\text{new}}(t) \end{aligned}$$

$$e) \text{ for } x_{\text{new}}(t) = tu(t) = r(t) = \int_{-\infty}^t x_1(\lambda) d\lambda$$

$$y_{\text{new}}(t) = \int_{-\infty}^t y_1(\lambda) d\lambda = \int_{-\infty}^t (1 - e^{-2\lambda})u(\lambda) d\lambda = \int_0^t (1 - e^{-2\lambda}) d\lambda$$

$$= \left(\lambda + \frac{1}{2}e^{-2\lambda} \right) \Big|_0^t = \left(t + \frac{1}{2}e^{-2t} - \frac{1}{2} \right) u(t) = y_{\text{new}}(t)$$

↑
only valid if $t > 0$

#4

$$a) y(t) = x(t) + x(t)$$

not BIBO stable for $x(t) = u(t)$

$$\text{at } t=0 \quad y(0) = i(0) + u(0) = 5(0) + u(0) = \infty$$

$$b) y(t) = \sin\left(\frac{t}{x(t)}\right)$$

no matter what $x(t)$ is, even $x(t) = 0$, $-1 \leq y(t) \leq 1$

BIBO stable

$$c) y(t) = x^2(t)$$

for $|x(t)| < m$ $|y(t)| \leq m^2$ so BIBO stable

$$d) y(t) + y(t)x(t) = x(t)$$

$$y(t)[1 + x(t)] = x(t)$$

$$y(t) = \frac{x(t)}{1 + x(t)}$$

$$\text{for } x(t) = -1 \quad y(t) = \infty$$

not BIBO stable

#5

$$\text{for } x(t) = u(t-1) - u(t-2)$$

$$y(t) = u(t) - u(t-2)$$

a) system is not causal, output occurs before the input

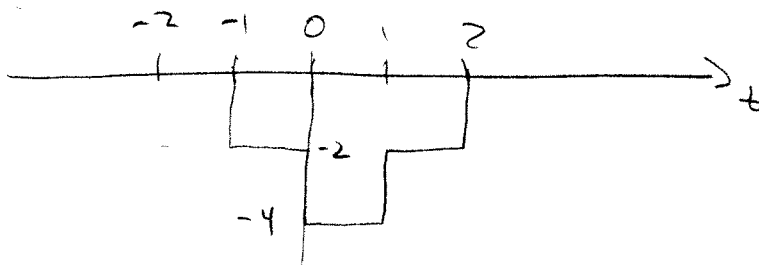
b) write new input in terms of known input

$$x_{\text{new}} = -2x(t+1) - 2x(t)$$

$$\text{so } y_{\text{new}} = -2y(t+1) - 2y(t)$$

$$= -2[u(t+1) - u(t-1)] - 2[u(t) - u(t-2)]$$

$$y_{\text{new}} = -2u(t+1) - 2u(t) + 2u(t-1) + 2u(t-2)$$



#6

$$(b) \quad y(x) = \frac{1}{2 + \frac{x}{10}} = (2 + \frac{x}{10})^{-1} \quad y(0) = \frac{1}{2}$$

$$\frac{dy}{dx} = -1(2 + \frac{x}{10})^{-2} \frac{1}{10}$$

$$\left. \frac{dy}{dx} \right|_{x=0} = -\frac{1}{4} \cdot \frac{1}{10} = -\frac{1}{40} = -0.0250$$

$$\Delta y = y(x) - 0.5 = -0.0250 x(x)$$

$y(x)$ should swing between 0.525 and 0.475 (approximately)

$$a=2, b=1, c=-1 \quad z = \frac{x}{10} \quad y(x) \approx 2^{-1} + (-1)2^{-2}z = \frac{1}{2} - \frac{1}{40}x$$

$$(d) \quad y(x) = \sin\left(\frac{x}{10}\right) \quad \Delta y = y(x) - 0 = \frac{1}{10} x(x)$$

$y(x)$ should swing between -0.1 and 0.1

$$(h) \quad y(x) = (10.1 + \frac{x}{10})^{0.2}$$

$$y(0) = 10.1^{0.2} = 1.59$$

$$\frac{dy}{dx} = 0.2(10.1 + \frac{x}{10})^{-0.8} \frac{1}{10}$$

$$\left. \frac{dy}{dx} \right|_{x=0} = \frac{0.2}{(10.1)^{0.8}} \frac{1}{10} = 0.0031$$

$$\Delta y = y(x) - 1.59 = 0.0031 x(x)$$

$y(x)$ should swing between 1.593 and 1.587