ECE 300
Signals and Systems
Homework 10

<u>Due Date:</u> ***<u>Wednesday</u>*** May 16 at 7 PM **(beginning of Q/A)**
Note: Exam 3 Thursday May 17 , Lab Practical Friday May 18

<u>Note</u>: Use the Fourier transform table given out in class.

<u>Problems</u>

1. Find the fraction of the total signal energy (as a percentage) contained between 100 and 300 Hz in the signal $x(t)$ given below:

$x(t) = 5\,\text{sinc}\left(\dfrac{t}{0.002}\right) + 5\,\text{sinc}\left(\dfrac{t}{0.001}\right)$ *Answer* 56%

2. Using the ***<u>duality property</u>***, find the corresponding Fourier transform for the following: **a)** $g(t) = \text{sinc}^2(Bt)$ **b)** $g(t) = \text{sinc}(Wt)$ **c)** $g(t) = \delta(t)$ **d)** $g(t) = \cos(\omega_0 t)$ **<u>Do not</u>** just look up the pairs from the table (though you can use any other pairs except the one you are trying to find).

3. K & H, Problem 5.16 (**a, b, c** only)

4. Consider a linear time invariant system with transfer function given by

$$H(\omega) = \begin{cases} 5e^{-j2\omega} & |\omega| \le 2 \\ 0 & else \end{cases}$$

with input $x(t) = \dfrac{8}{\pi}\text{sinc}^2\left(\dfrac{2(t-1)}{\pi}\right)$. The output of the system is $y(t)$.

a) Determine $X(\omega)$.

b) Sketch the spectrum of $X(\omega)$ (magnitude and phase) accurately labeling the axes and important points.

c) Sketch the spectrum of $H(\omega)$ (magnitude and phase) accurately labeling the axes and important points.

d) Determine $y(t)$, the output of the system.

Answer $y(t) = \dfrac{20}{\pi}\text{sinc}\left[\dfrac{2}{\pi}(t-3)\right] + \dfrac{10}{\pi}\text{sinc}^2\left[\dfrac{1}{\pi}(t-3)\right]$

5. Determine the transfer function $H(\omega)$ that would produce the following input /output relationships. Simplify your answers as much as possible.

a) $y(t) = a\dot{x}(t-b)$

b) $y(t) = ax(t+b) + ax(t-b)$

c) $\dot{y}(t) = x(t) * e^{-t}u(t-b)$

#1 Problem Set 9 ECE-300

$$x(t) = 5\,\text{sinc}\left(\frac{t}{0.002}\right) + 5\,\text{sinc}\left(\frac{t}{0.001}\right)$$

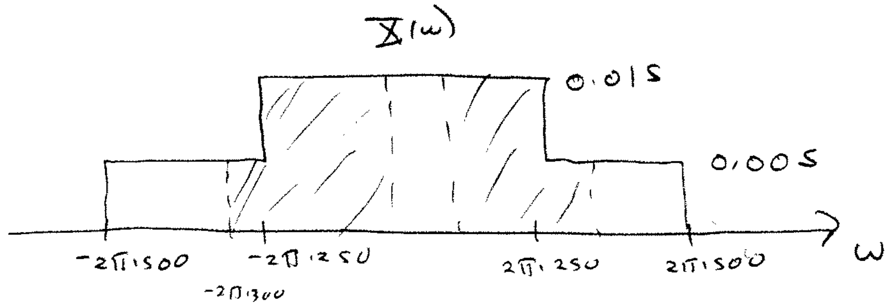
compute the % of energy between 100 and 300 Hz

for $\text{sinc}(Wt) \Longleftrightarrow \frac{1}{W}\,\text{rect}\left(\frac{\omega}{2\pi W}\right)$

here $W = \frac{1}{0.002} = 500$ or $W = \frac{1}{0.001} = 1000$

$$X(\omega) = \frac{5}{500}\,\text{rect}\left(\frac{\omega}{2\pi\cdot 500}\right) + \frac{5}{1000}\,\text{rect}\left(\frac{\omega}{2\pi\cdot 1000}\right)$$

$$= 0.01\,\text{rect}\left(\frac{\omega}{2\pi\cdot 500}\right) + 0.005\,\text{rect}\left(\frac{\omega}{2\pi\cdot 1000}\right)$$



$$E_{total} = \frac{1}{2\pi}\int_{-2\pi\cdot 500}^{2\pi\cdot 500} |X(\omega)|^2 d\omega = \frac{1}{2\pi}\left[2\cdot\int_{-2\pi\cdot 500}^{-2\pi\cdot 250}(0.005)^2 d\omega + 2\int_{-2\pi\cdot 250}^{0}(0.015)^2 d\omega\right]$$

$$= \frac{1}{\cancel{2\pi}}\left[2\cdot(\cancel{2\pi}\cdot 250)(0.005)^2 + 2\cdot(\cancel{2\pi}\cdot 250)(0.015)^2\right]$$

$$= 2\cdot 250\cdot(0.005)^2 + 2\cdot 250\cdot(0.015)^2 = \boxed{0.125 = E_{total}}$$

$$E_{Band} = \frac{1}{2\pi}\left[2\cdot\int_{-2\pi\cdot 300}^{-2\pi\cdot 250}(0.005)^2 d\omega + 2\int_{-2\pi\cdot 250}^{-2\pi\cdot 100}(0.015)^2 d\omega\right]$$

$$= \frac{1}{\cancel{2\pi}}\left[2\cdot(\cancel{2\pi}\cdot 50)(0.005)^2 + 2(\cancel{2\pi}\cdot 150)(0.015)^2\right]$$

$$= 2\cdot(50)(0.005)^2 + 2(150)(0.015)^2 = \boxed{0.070 = E_{Band}}$$

$$\text{ratio} = \frac{0.070}{0.125} = 0.560 \quad 56\%$$

#2

a) $g(t) = \text{sinc}^2(Bt)$

from the table,

$$g_1(t) = \Lambda\left(\frac{t}{W}\right) \longleftrightarrow G_1(\omega) = \frac{W}{2}\,\text{sinc}^2\left(\frac{W}{4\pi}\omega\right)$$

by duality

$$g_2(t) = G_1(t) = \frac{W}{2}\,\text{sinc}^2\left(\frac{W}{4\pi}t\right) \longleftrightarrow G_2(\omega) = 2\pi g_1(-\omega) = 2\pi\,\Lambda\left(\frac{\omega}{W}\right)$$

$$B = \frac{W}{4\pi} \quad \text{so} \quad W = 4\pi B$$

$$g_2(t) = 2\pi B\,\text{sinc}^2(Bt) \longleftrightarrow G_2(\omega) = 2\pi\,\Lambda\left(\frac{\omega}{4\pi B}\right)$$

or $$\boxed{\text{sinc}^2(Bt) \longleftrightarrow \frac{1}{B}\,\Lambda\left(\frac{\omega}{4\pi B}\right)}$$

b) $g(t) = \text{sinc}(Wt)$

from the table

$$g_1(t) = \text{rect}\left(\frac{t}{T}\right) \longleftrightarrow G_1(\omega) = T\,\text{sinc}\left(\frac{T}{2\pi}\omega\right)$$

by duality

$$g_2(t) = G_1(t) = T\,\text{sinc}\left(\frac{T}{2\pi}t\right) \longleftrightarrow G_2(\omega) = 2\pi g_1(-\omega) = 2\pi\,\text{rect}\left(\frac{\omega}{T}\right)$$

$$W = \frac{T}{2\pi} \quad \text{or} \quad T = 2\pi W$$

$$g_2(t) = 2\pi W\,\text{sinc}(Wt) \longleftrightarrow 2\pi\,\text{rect}\left(\frac{\omega}{2\pi W}\right)$$

$$\boxed{\text{sinc}(Wt) \longleftrightarrow \frac{1}{W}\,\text{rect}\left(\frac{\omega}{2\pi W}\right)}$$

(c) $g(t) = \delta(t)$

from the table

$$g_1(t) = 1 \Leftrightarrow G_1(\omega) = 2\pi\,\delta(\omega)$$

by duality

$$g_2(t) = G_1(t) = 2\pi\,\delta(t) \Leftrightarrow G_2(\omega) = 2\pi g_1(-\omega) = 2\pi$$

$$\boxed{\delta(t) \Leftrightarrow 1}$$

(d) $g(t) = \cos(\omega_0 t)$

for $G_1(\omega) = \cos(T\omega) = \frac{1}{2}e^{j\omega T} + \frac{1}{2}e^{-j\omega T} \Leftrightarrow g_1(t) = \frac{1}{2}\delta(t+T) + \frac{1}{2}\delta(t-T)$

$$\text{so } g_1(t) = \frac{1}{2}\left[\delta(t+T) + \delta(t-T)\right] \Leftrightarrow G_1(\omega) = \cos(T\omega)$$

by duality

$$g_2(t) = G_1(t) = \cos(Tt) \Leftrightarrow G_2(\omega) = 2\pi g_1(-\omega) = \pi\,\delta(-\omega+T) + \pi\,\delta(-\omega-T)$$

$T = \omega_0$

$$\boxed{\cos(\omega_0 t) \Leftrightarrow \pi\,\delta(\omega-\omega_0) + \pi\,\delta(\omega+\omega_0)}$$

$\delta()$ is an <u>even</u> function

#3 Sol 6 $H(\omega) = \begin{cases} 1+\cos(2\pi\omega) & -0.5<\omega<0.5 \\ 0 & \text{else} \end{cases}$

a) find $h(t)$

b) find $y(t)$ for $x(t) = \mathrm{sinc}\left(\frac{t}{2\pi}\right)$

c) find $y(t)$ for $x(t) = \mathrm{sinc}\left(\frac{t}{4\pi}\right)$

a) $H(\omega) = \left[1+\cos(2\pi\omega)\right]\mathrm{rect}\left(\frac{\omega}{1}\right)$

$$\frac{1}{W}\,\mathrm{rect}\left(\frac{\omega}{2\pi W}\right) \longleftrightarrow \mathrm{sinc}(Wt)$$

$$\mathrm{rect}\left(\frac{\omega}{2\pi W}\right) = \mathrm{rect}\left(\frac{\omega}{1}\right) \qquad W = \frac{1}{2\pi}$$

$$\mathrm{rect}\left(\frac{\omega}{1}\right) \Longleftrightarrow \frac{1}{2\pi}\,\mathrm{sinc}(t/2\pi)$$

$$\cos(2\pi\omega)\,\mathrm{rect}\left(\frac{\omega}{1}\right) = \frac{e^{j2\pi\omega}}{2}\mathrm{rect}\left(\frac{\omega}{1}\right) + \frac{e^{-j2\pi\omega}}{2}\mathrm{rect}\left(\frac{\omega}{1}\right)$$

$$\Longleftrightarrow \frac{1}{4\pi}\,\mathrm{sinc}\left(\frac{t+2\pi}{2\pi}\right) + \frac{1}{4\pi}\,\mathrm{sinc}\left(\frac{\omega-2\pi}{2\pi}\right)$$

so $$\boxed{h(t) = \frac{1}{2\pi}\,\mathrm{sinc}\left(\frac{t}{2\pi}\right) + \frac{1}{4\pi}\,\mathrm{sinc}\left(\frac{t+2\pi}{2\pi}\right) + \frac{1}{4\pi}\,\mathrm{sinc}\left(\frac{t-2\pi}{2\pi}\right)}$$

(b) $x(t) = \mathrm{sinc}\left(\frac{t}{2\pi}\right) \qquad X(\omega) = 2\pi\,\mathrm{rect}\left(\frac{\omega}{1}\right)$

$$Y(\omega) = H(\omega)X(\omega) = 2\pi\,H(\omega)$$

$$\boxed{y(t) = \mathrm{sinc}\left(\frac{t}{2\pi}\right) + \frac{1}{2}\,\mathrm{sinc}\left(\frac{t+2\pi}{2\pi}\right) + \frac{1}{2}\,\mathrm{sinc}\left(\frac{t-2\pi}{2\pi}\right)}$$

(c) $x(t) = \mathrm{sinc}\left(\frac{t}{4\pi}\right) \qquad X(\omega) = 4\pi\,\mathrm{rect}\left(\frac{\omega}{0.5}\right)$

$$Y(\omega) = H(\omega)X(\omega) = 4\pi\left[1+\cos(2\pi\omega)\right]\mathrm{rect}\left(\frac{\omega}{0.5}\right)$$

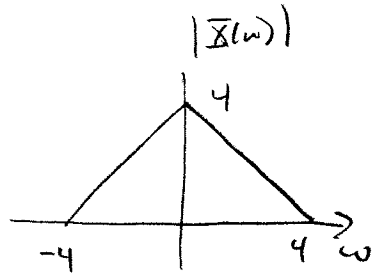
$$\boxed{y(t) = \mathrm{sinc}\left(\frac{t}{4\pi}\right) + \frac{1}{2}\,\mathrm{sinc}\left(\frac{t+2\pi}{4\pi}\right) + \frac{1}{2}\,\mathrm{sinc}\left(\frac{t-2\pi}{4\pi}\right)}$$

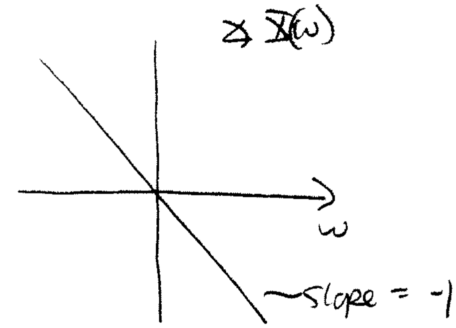
#4 $x(t) = \frac{8}{\pi}\,\text{sinc}^2\left(\frac{2(t-1)}{\pi}\right)$ $\qquad H(\omega) = \begin{cases} 5e^{-j2\omega} & |\omega| \le 2 \\ 0 & \text{else} \end{cases}$

a) for $x(t) = \text{sinc}^2\left(\frac{2t}{\pi}\right) \longleftrightarrow X(\omega) = \frac{\pi}{2}\Lambda\left(\frac{\omega}{8}\right)$

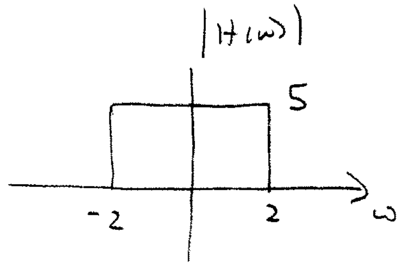
so $\boxed{X(\omega) = 4\Lambda\left(\frac{\omega}{8}\right)e^{-j\omega}}$

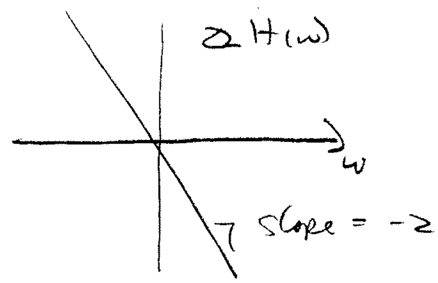
b)



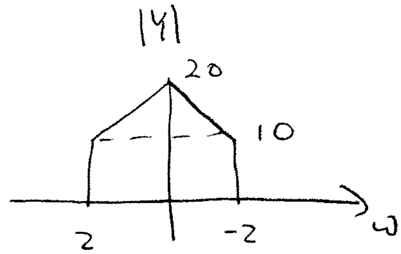


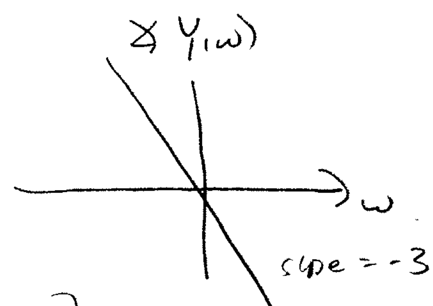
c)





d)





$$Y(\omega) = \left[10\,\text{rect}\left(\frac{\omega}{4}\right) + 10\,\Lambda\left(\frac{\omega}{4}\right)\right]e^{-j3\omega}$$

$$\text{rect}\left(\frac{\omega}{2\pi W}\right) \longleftrightarrow W\text{sinc}(Wt) \qquad W = \frac{2}{\pi}$$

$$\Lambda\left(\frac{\omega}{4\pi B}\right) \longleftrightarrow B\text{sinc}^2(Bt) \qquad B = 1/\pi$$

$$\boxed{y(t) = \frac{20}{\pi}\text{sinc}\left(\frac{2}{\pi}(t-3)\right) + \frac{10}{\pi}\text{sinc}^2\left(\frac{1}{\pi}(t-3)\right)}$$

#5

a) $y(t) = a\dot{x}(t-b)$

$$Y(\omega) = a(j\omega)e^{-j\omega b}X(\omega) \qquad \boxed{H(\omega) = j\omega a e^{-j\omega b}}$$

b) $y(t) = a x(t+b) + a x(t-b)$

$$Y(\omega) = a e^{j\omega b}X(\omega) + a e^{-j\omega b}X(\omega)$$

$$= 2a\left[\frac{e^{j\omega b} + e^{-j\omega b}}{2}\right]X(\omega)$$

$$= 2a\cos(\omega b)X(\omega)$$

$$\boxed{H(\omega) = 2a\cos(\omega b)}$$

c) $\dot{y}(t) = x(t) * e^{-t}u(t-b)$

rewrite $e^{-t}u(t-b) = e^{-(t-b+b)}u(t-b)$

$$= e^{-(t-b)}e^{-b}u(t-b)$$

$$\dot{y}(t) = x(t) * e^{-b}e^{-(t-b)}u(t-b)$$

$$j\omega Y(\omega) = X(\omega)\, e^{-b}\frac{e^{-j\omega b}}{1+j\omega}$$

$$\frac{Y(\omega)}{X(\omega)} = \boxed{\frac{e^{-b}e^{-j\omega b}}{j\omega(1+j\omega)} = H(\omega)}$$