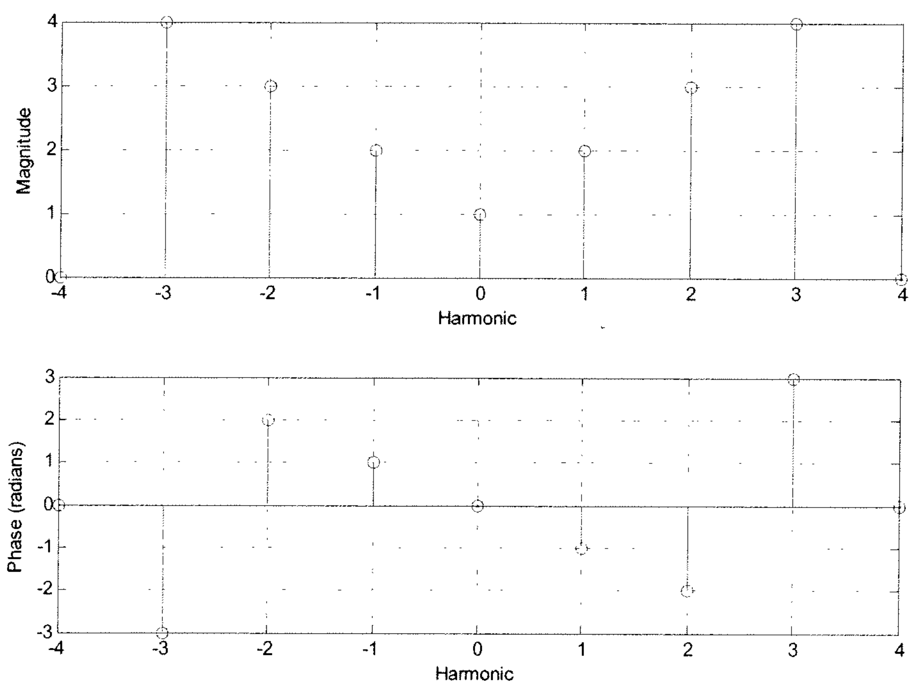
ECE 300
Signals and Systems
Homework 7

Due Date: Thursdayday October 22, 2009 at the beginning of class

Exam 2, Monday October 26, 2009

Problems:

1. Assume $x(t)$ has the spectrum shown below (the phase is shown in radians) and a fundamental frequency $\omega_o = 2\ \mathrm{rad/sec}$:



Assume $x(t)$ is the input to a system with the transfer function

$$H(\omega) = \begin{cases} e^{-j\omega} & 1 \le |\omega| < 3 \\ 2e^{-j2\omega} & 3 < |\omega| < 5 \\ 0 & else \end{cases}$$

Determine an expression for the steady state output $y(t)$. Be as specific as possible, simplifying all values and using actual numbers wherever possible.

2. A periodic signal $x(t)$ is the input to an LTI system with output $y(t)$. The signal $x(t)$ has period 2 seconds, and is given over one period as

$$x(t) = e^{-t} \quad 0 < t < 2$$

$x(t)$ has the Fourier series representation

$$x(t) = \sum_k \frac{0.4323}{1 + jk\pi} e^{jk\pi t}$$

The system is an ideal highpass filter that eliminates all signals with frequency content less than 0.75 Hz.

a) Find the average power in $x(t)$.

b) Determine an expression for the output, $y(t)$. Your expression for $y(t)$ must be real.

(*Answer:* $y(t) = e^{-t} - 0.4323 - 0.2622\cos(\pi t - 1.2626)$)

c) Determine the average power in $y(t)$.

d) What fraction of the average power in $x(t)$ is contained in the DC and fundamental frequency components?

3. Assume $x(t) = t^2 \quad -\pi \le t \le \pi$ with Fourier Series representation

$$x(t) = \sum_k c_k^x e^{jkt}$$

where

$$c_k^x = \begin{cases} \dfrac{\pi^2}{3} & k = 0 \\ \dfrac{2(-1)^k}{k^2} & k \neq 0 \end{cases}$$

a) Assume $x(t)$ is the input to a system that eliminates all signals with frequencies outside the range 0.5 to 0.7 Hz. What is the output of the system $y(t)$ and what fraction of the average power in $x(t)$ is in $y(t)$? (Note: your answers must be real, no e^{ja} terms.)

b) Assume $x(t)$ is the input to a system that eliminates all signals with frequencies in the range 0.5 to 0.7 Hz. What is the output of the system $y(t)$ and what fraction of the average power in $x(t)$ is in $y(t)$?

4. A periodic signal $x(t)$ with period T_0 has the constant component $c_0 = 2$. The signal $x(t)$ is applied to an LTI system with transfer function

$$H(j\omega) = \begin{cases} 10e^{-j5\omega} & |\omega| > \dfrac{\pi}{T_0} \\ 0 & otherwise \end{cases}$$

The output of the system $y(t)$ can be written

$$y(t) = ax(t-b) + c$$

Determine the constants $a, b,$ and c.

5. Assume the periodic signal $x(t)$ with the Fourier series representation $x(t) = \sum_k c_k^x e^{jk\omega_o t}$ is the input to system with output $y(t)$ which has the Fourier series representation $y(t) = \sum_k c_k^y e^{jk\omega_o t}$. If the input and output are given by the relationships below, determine (i) how c_k^x and c_k^y are related, and determine a transfer function between the input and output if possible.

a) $y(t) = bx(t)$ b) $y(t) = ax(t-b)$ c) $y(t) = \dfrac{d}{dt}x(t)$

d) the input and output are related through the transfer function

$$\dot{y}(t) + ay(t) = dx(t-b)$$

6. The periodic signal $x(t)$ has the Fourier series representation

$$x(t) = \sum_{k=-\infty}^{k=\infty} \frac{1}{1+kj} e^{jk2t}$$

$x(t)$ is the input to an LTI system (a band reject or notch filter) with the transfer function

$$H(j\omega) = \begin{cases} 2e^{-j3\omega} & |0 \le |\omega| \le 3.5 \; and \; 4.5 \le |\omega| < \infty \\ 0 & 3.5 < |\omega| < 4.5 \end{cases}$$

The steady state output of the system can be written as $y(t) = ax(t-b) + d\cos(et + f)$. Determine numerical values for the parameters a, b, d, e and f

7. (Matlab/Prelab) Microcontrollers often need to output an analog signal in order to control sensors or motors. Although it is possible to use analog to digital converters, they take up a lot of space on a chip. An alternative is to use a form of pulse width modulation (PWM) and a lowpass filter. For our purposes, the pulse width modulated signal is a periodic square wave which is either at 5 volts or 0 volts. We only change *period* of the pwm signal and the *duty cycle* of the pwm signal since this can easily be done with timers.

a) Construct a square wave signal $x(t)$ with an amplitude of 5 volts (it goes from a minimum of 0 volts to a maximum of 5 volt), a period of 1 microsecond (the signal should start at zero), and a duty cycle of 80%. Leave the duty cycle a variable in your program.

b) Plot the original signal and the Fourier series representation of the signal (using 10 terms) over one period to verify everything is working before going on.

c) Plot the original signal and the Fourier series representation over 5 periods. Note that you will probably need to use the *mod function* <u>in your original definition of</u> $x(t)$ but not in the Fourier series. Your figure should look like that in Figure 1 on the next page. *Print out your plots and code and turn them in.*

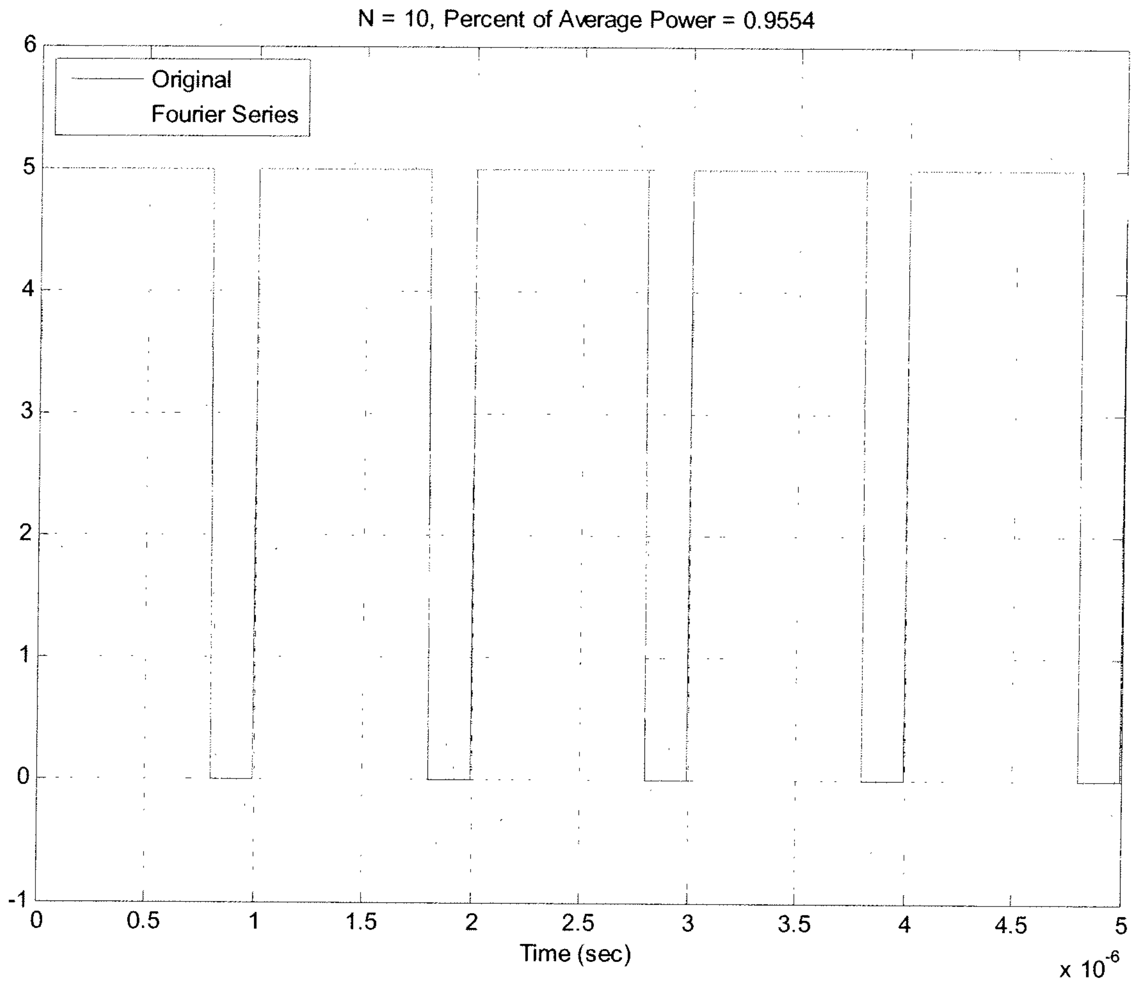
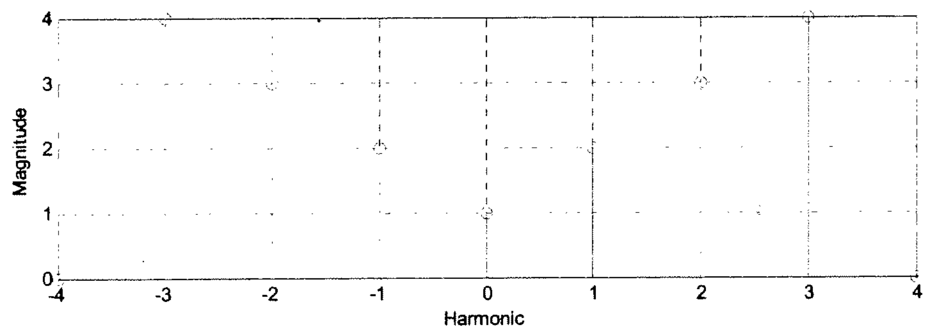
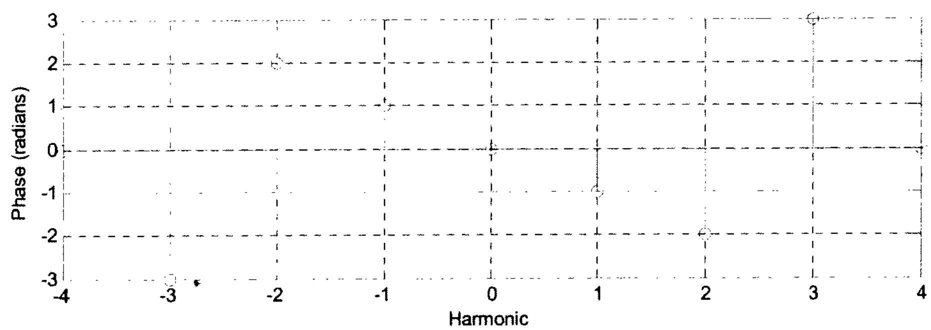


Figure 1: Pulse Width Modulated (PWM) signal with 80% duty cycle.

#1

$$H(\omega) = \begin{cases} e^{-j\omega} & 1 \le |\omega| < 3 \\ 2e^{-j2\omega} & 3 < |\omega| \le 5 \\ 0 & \text{else} \end{cases} \qquad \omega_0 = 2\,\text{rad/sec}$$





$$Y_0 = X_0 H(0) = 0$$

$$Y_1 = X_1 H(1\omega_0) = (2e^{-j1})(e^{-j2}) = 2e^{-j3} = 2 \angle -3 \text{ rad}$$

$$Y_2 = X_2 H(2\omega_0) = (3e^{-j2})(2e^{-j8}) = 6e^{-j10} = 2 \angle -10 \text{ rad}$$

$$Y_3 = X_3 H(3\omega_0) = 0$$

$$y(t) = Y_0 + 2|Y_1|\cos(\omega_0 t + \angle Y_1) + 2|Y_2|\cos(2\omega_0 t + \angle Y_2) + 0 + \cdots$$

$$\boxed{y(t) = 4\cos(2t-3) + 12\cos(4t-10)}$$

#2 $x(t) = e^{-t} \quad 0 \le t \le 2$ $\qquad T_0 = 2 \quad f_0 = \frac{1}{2} = 0.5\,Hz$

$$x(t) = \sum_{K=-\infty}^{\infty} \frac{0.4323}{1+jK\pi} e^{jK\pi t}$$

a) $P_{ave}^{x} = \frac{1}{T_0}\int_{T_0} |x(t)|^2 dt = \frac{1}{2}\int_0^2 e^{-2t} dt = \frac{1}{2} \left.\frac{e^{-2t}}{-2}\right|_0^2 = \frac{1}{4}\left(1-e^{-4}\right) = 0.2454$

$$\boxed{P_{ave}^{x} = 0.2454}$$

b) The high pass filter removes signals with frequency content below 0.75 Hz. Let's figure out what they are

It removes $K=0$, $K=\pm 1$

$c_0^x = 0.4323$

$c_1^x = \frac{0.4323}{1+j\pi} = 0.13112 \measuredangle 1.2626$ rad

$y(t) = e^{-t} - 0.4323 - 2(0.13112)\cos(\pi t - 1.2626)$

$$\boxed{y(t) = e^{-t} - 0.4323 - 0.26225\cos(\pi t - 1.2626)}$$

c) $P_{ave}^{y} = P_{ave}^{x} - |c_0^x|^2 - 2|c_1^x|^2 = 0.2454 - (0.4323)^2 - 2(0.13112)^2$

$$\boxed{P_{ave}^{y} = 0.02413}$$

d) $\frac{|c_0^x|^2 + 2|c_1^x|^2}{P_x^{ave}} = \frac{(0.4323)^2 + 2(0.13112)^2}{0.2454} = 0.90166 \approx 90\%$

#3 $x(t) = t^2 \quad -\pi \le t \le \pi$

$x(t) = \sum X_K e^{jKt} \qquad X_K = \begin{cases} \frac{\pi^2}{3} & K = 0 \\ \frac{2(-1)^K}{K^2} & K \neq 0 \end{cases}$

a) Bandpass filter, removes everything outside range 0.5 to 0.7 Hz

$\omega_0 = 1 \text{ rad/sec} = 2\pi f_0 \qquad f_0 = 1/2\pi = 0.159 \text{ Hz}$

K	$f = K f_0$
0	0
1	0.159
2	0.318
3	0.477
4	0.636 — only term
5	0.795

$y(t) = 2|c_4^x| \cos(4\omega_0 t + \measuredangle c_4^x)$

$c_4^x = \frac{2(-1)^4}{4^2} = \frac{2}{16} \measuredangle 0^\circ = \frac{1}{8} \measuredangle 0^\circ = 0.125 \measuredangle 0^\circ$

$\boxed{y(t) = 0.25 \cos(4t)}$

$P_{ave}^x = \frac{1}{2\pi} \int_{-\pi}^{\pi} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} t^4 dt = \frac{1}{2\pi} \frac{t^5}{5} \Big|_{-\pi}^{\pi} = \frac{2(\pi^5)}{5(2\pi)} = 19.48$

$\frac{2|c_4^x|^2}{P_{ave}^x} = \frac{2|0.125|^2}{19.48} \times 100\% = 0.16\%$ of total power

b) $y(t) = t^2 - 0.25\cos(4t)$

$\frac{P_{ave}^y}{P_{ave}^x} = 100\% - 0.16\% = \boxed{99.84\%}$

#4

$c_0 = 2$ $\quad H(j\omega) = \begin{cases} 10e^{-j5\omega} & |\omega| > \frac{\pi}{10} \\ 0 & \text{else} \end{cases}$

$x(t)$ is periodic with period T_0, $\omega_0 = \frac{2\pi}{T_0}$

$y(t)$ can be written $y(t) = a\,x(t-b) + c$

since $\omega_0 = \frac{2\pi}{T_0}$ we can write $H(j\omega) = \begin{cases} 10e^{-j5\omega} & |\omega| > \frac{\omega_0}{2} \\ 0 & \text{else} \end{cases}$

so the filter removes the dc value of $x(t)$, scales by 10, and delays by 5

so $\boxed{y(t) = 10\,x(t-5) - 20 \qquad a = 10,\ b = 5,\ c = -20}$

#5 $x(t) = \sum c_k^x e^{jk\omega_0 t}$

$y(t) = \sum c_k^y e^{jk\omega_0 t}$

a) $y(t) = b\,x(t)$ $\quad \sum c_k^y e^{jk\omega_0 t} = \sum b c_k^x e^{jk\omega_0 t}$

$$\boxed{\begin{aligned} c_k^y &= b c_k^x \\ H(j\omega) &= b \end{aligned}}$$

b) $y(t) = a x(t-b)$ $\quad \sum c_k^y = a \sum c_k^x e^{jk\omega_0 (t-b)}$

$= \sum c_k^x a e^{-jk\omega_0 b} e^{jk\omega_0 t}$

$$\boxed{\begin{aligned} c_k^y &= c_k^x a e^{-jk\omega_0 b} \\ H(j\omega) &= a e^{-j\omega b} \end{aligned}}$$

c) $y(t) = \frac{d}{dt} x(t)$ $\quad \sum c_k^y = \sum c_k^x (jk\omega_0) e^{jk\omega_0 t}$

$$\boxed{\begin{aligned} c_k^y &= jk\omega_0 c_k^x \\ H(j\omega) &= j\omega \end{aligned}}$$

d) $\dot{y}(t) + a y(t) = d\,x(t-b)$

$\sum c_k^y (jk\omega_0) e^{jk\omega_0 t} + \sum a c_k^y e^{jk\omega_0 t} = \sum d c_k^x e^{-jk\omega_0 b} e^{jk\omega_0 t}$

$c_k^y [a + jk\omega_0] = c_k^x [d e^{-jk\omega_0 b}]$

$c_k^y = c_k^x \dfrac{d e^{-jk\omega_0 b}}{a + jk\omega_0}$ $\quad \boxed{H(j\omega) = \dfrac{d e^{-j\omega b}}{a + j\omega}}$

6. The periodic signal $x(t)$ has the Fourier series representation

$$x(t) = \sum_{k=-\infty}^{k=\infty} \frac{1}{1+kj} e^{jk2t}$$

$x(t)$ is the input to an LTI system (a band reject or notch filter) with the transfer function

$$H(j\omega) = \begin{cases} 2e^{-j3\omega} & 0 \le |\omega| \le 3.5 \text{ and } 4.5 \le |\omega| < \infty \\ 0 & 3.5 < |\omega| < 4.5 \end{cases}$$

The steady state output of the system can be written as $y(t) = ax(t-b) + d\cos(et+f)$. Determine numerical values for the parameters a, b, d, e and f

The filter removes the second harmonic $(k=2)$

so we have

$x(t) - 2|C_2|\cos(2\omega_0 t + \angle C_2)$

This is the input to the system $H(j\omega) = 2e^{-j3\omega}$

$\boxed{x(t) - 2|C_2|\cos(2\omega_0 t + \angle C_2)} \rightarrow \boxed{2e^{-j3\omega}} \rightarrow y(t)$

so $y(t) = 2[x(t-3) - 2|C_2|\cos(2\omega_0(t-3)) + \angle C_2]$

$C_2 = \frac{1}{1+j2} = \frac{1}{\sqrt{5}}\angle -63.4^\circ = 0.447\angle -63.4^\circ$

$$\boxed{y(t) = 2x(t-3) - 1.789\cos(4(t-3) - 63.4^\circ)}$$