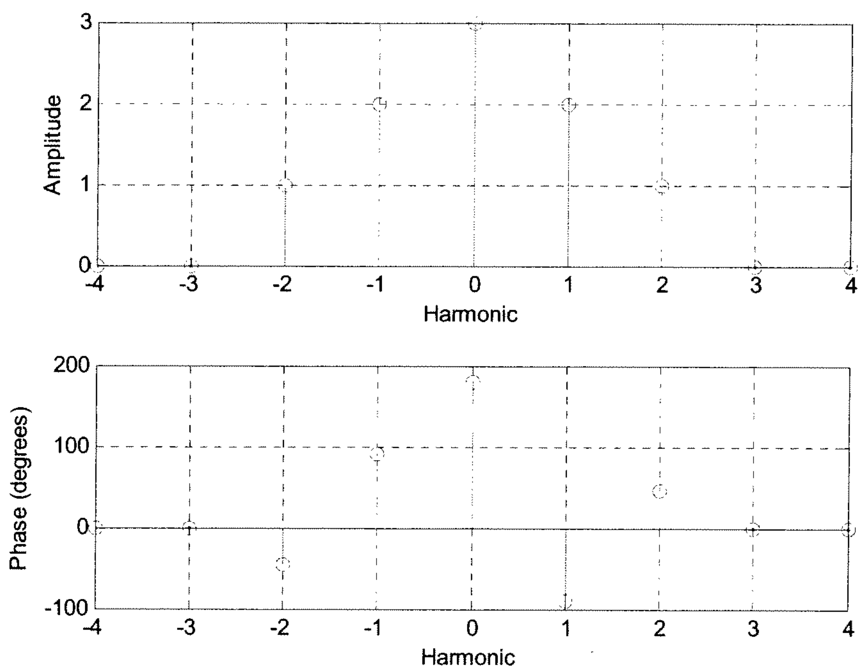
ECE 300
Signals and Systems
Homework 7

<u>Due Date:</u> Tuesday October 21, 2008 at the beginning of class

<u>Exam 2, Thursday October 23, 2008</u>

<u>Problems:</u>

1. Assume $x(t)$, which has a fundamental period of 2 seconds, has the following spectrum (all phases are multiples of 45 degrees)

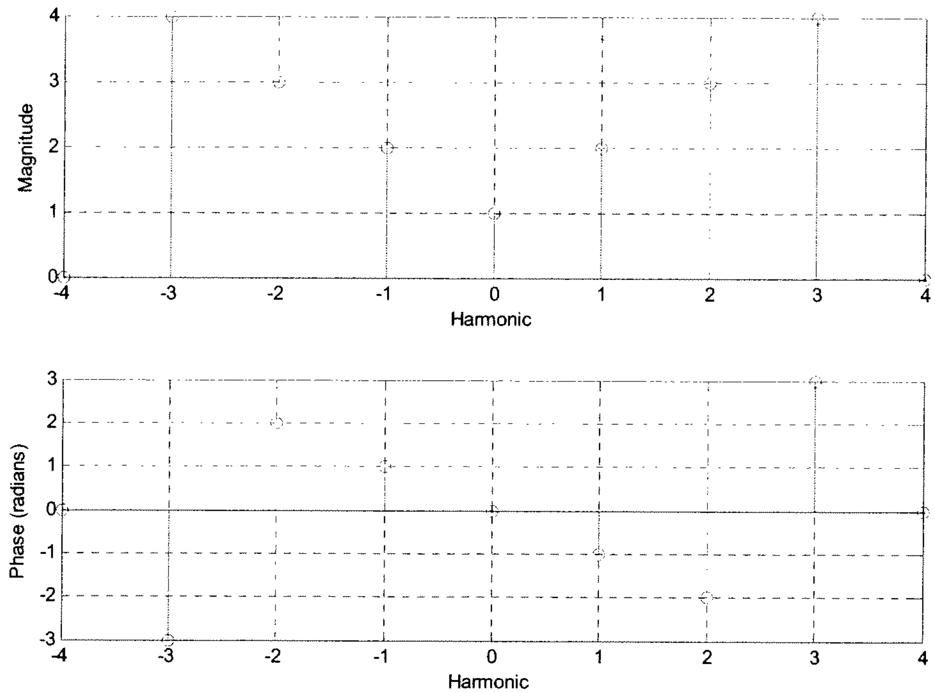


a) What is $x(t)$? Your expression must be real.

b) What is the average value of $x(t)$?

c) What is the average power in $x(t)$?

2. Assume $x(t)$ has the spectrum shown below (the phase is shown in radians) and a fundamental frequency $\omega_o = 2 \text{ rad/sec}$:



Assume $x(t)$ is the input to a system with the transfer function

$$H(\omega) = \begin{cases} e^{-j\omega} & 1 \le |\omega| < 3 \\ 2e^{-j2\omega} & 3 < |\omega| < 5 \\ 0 & else \end{cases}$$

Determine an expression for the steady state output $y(t)$. Be as specific as possible, simplifying all values and using actual numbers wherever possible.

3. A periodic signal $x(t)$ is the input to an LTI system with output $y(t)$. The signal $x(t)$ has period 2 seconds, and is given over one period as

$$x(t) = e^{-t} \quad 0 < t < 2$$

$x(t)$ has the Fourier series representation

$$x(t) = \sum_k \frac{0.4323}{1 + jk\pi} e^{jk\pi t}$$

The system is an ideal highpass filter that eliminates all signals with frequency content less than 0.75 Hz.

a) Find the average power in $x(t)$.

b) Determine an expression for the output, $y(t)$. Your expression for $y(t)$ must be real.

(*Answer:* $y(t) = e^{-t} - 0.4323 - 0.2622\cos(\pi t - 1.2626)$)

c) Determine the average power in $y(t)$.

d) What fraction of the average power in $x(t)$ is contained in the DC and fundamental frequency components?

4. Assume $x(t) = t^2 \quad -\pi \le t \le \pi$ with Fourier Series representation

$$x(t) = \sum_k X_k e^{jkt}$$

where

$$X_k = \begin{cases} \dfrac{\pi^2}{3} & k = 0 \\ \dfrac{2(-1)^k}{k^2} & k \neq 0 \end{cases}$$

a) Assume $x(t)$ is the input to a system that eliminates all signals with frequencies outside the range 0.5 to 0.7 Hz. What is the output of the system $y(t)$ and what fraction of the average power in $x(t)$ is in $y(t)$? (Note: your answers must be real, no e^{ja} terms.)

b) Assume $x(t)$ is the input to a system that eliminates all signals with frequencies in the range 0.5 to 0.7 Hz. What is the output of the system $y(t)$ and what fraction of the average power in $x(t)$ is in $y(t)$?

5. Assume two periodic signals have the Fourier series representations

$$x(t) = \sum X_k e^{jk\omega_o t} \quad y(t) = \sum Y_k e^{jk\omega_o t}$$

For the following system (input/output) relationships:

a) $y(t) = bx(t-a)$
b) $y(t) = b\dot{x}(t-a)$
c) $y(t) = bx(t)\cos(\omega_o t)$ (Answer: $Y_n = \frac{b}{2}(X_{n-1} + X_{n+1})$)
d) $\ddot{y}(t) + 2\zeta\omega_n\dot{y}(t) + \omega_n^2 y(t) = K\omega_n^2 x(t)$

i) write Y_k in terms of the X_k
ii) If possible, determine the system transfer function $H(j\omega)$
iii) A system must be both linear and time-invariant to have a transfer function. If you cannot determine the transfer function, indicate which system property is not satisfied (**L** or **TI**).

6. Consider an LTI system with impulse response $h(t)$ and input $x(t) = A\cos(\omega_0 t)$

a) Using the convolution integral, show that

$$y(t) = \frac{A}{2}e^{j\omega_0 t}H(\omega_0) + \frac{A}{2}e^{-j\omega_0 t}H(-\omega_0)$$

Where

$$H(\omega_0) = \int_{-\infty}^{\infty} h(\lambda)e^{-j\omega_0\lambda}d\lambda$$

(This is the Fourier transform of the impulse response evaluated at the input frequency.)

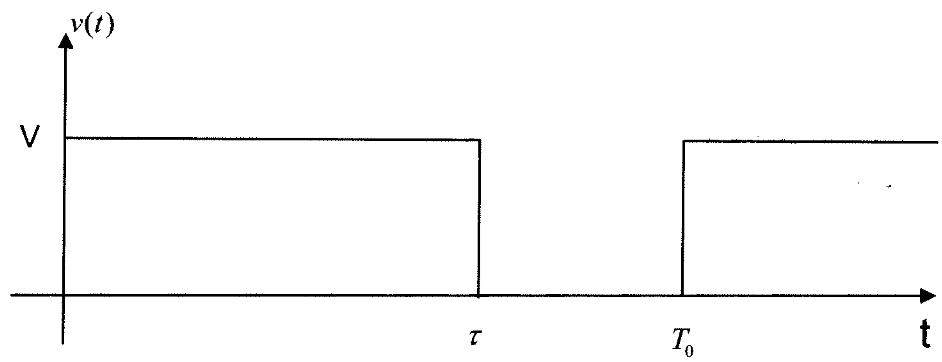
b) Show that for a real impulse response, $|H(\omega_0)| = |H(-\omega_0)|$ (the magnitude is even), and $\measuredangle H(-\omega_0) = -\measuredangle H(\omega_0)$ (the phase is odd).

c) Using the results from parts **a** and **b**, show that

$$y(t) = A|H(\omega_0)|\cos(\omega_0 t + \measuredangle \mathrm{H}(\omega_0))$$

Cultural Information: This problem shows that $e^{j\omega_0 t}$ is an *eigenfunction* of LTI systems.

7. Most microcontrollers are capable of generating pulse width modulation (PWM) signals on one or more output pins. These signals are square waves where both the period and the duty cycle can be programmed in the microcontroller by the use of timers and different reference clocks. These PWM output signals can then be used in conjunction with lowpass filters to produce reasonable approximations to analog output signals. In this problem we will use what we have learned in the course to investigate how to do this. The signal $v(t)$ below is a PWM signal, shown for about one and a half periods. The signal has period T_0, amplitude V (usually fixed at 5 or 3.3 volts), pulse duration τ, and duty cycle $D = \frac{\tau}{T_0}$.



The Fourier series representation is $v(t) = \sum_{k=-\infty}^{k=\infty} e^{-\frac{jk\pi\tau}{T_0}} \left(\frac{V\tau}{T_0}\right) \text{sinc}\left(\frac{k\tau}{T_0}\right) e^{jk\omega_0 t}$

a) For the periodic signal $v(t)$, determine an expression for the **average power** in the periodic signal in terms of T_0, τ, and V. Your answer must contain no sums or integrals.

b) Determine an expression for the **average value** of $v(t)$ in terms of T_0, τ, and V.

c) It is the average value of $v(t)$ that we want to use as our analog output. Hence we need to design a lowpass filter that allows us to keep our DC term, and, ideally, remove all of the other harmonics. Let's assume we want to use a simple first order RC lowpass filter with transfer function

$$H(j\omega) = \frac{1}{j\omega RC + 1} = \frac{1}{j\omega \frac{\sqrt{\alpha}}{\omega_0} + 1}$$

where we have set $RC = \frac{\sqrt{\alpha}}{\omega_0}$ for convenience. Determine the value of α so that the average power in the first harmonic of the output signal is 20 dB lower than the average power in the DC component the output signal. Assume here that the fundamental frequency is $f_0 = 100\,Hz$, the duty cycle is $\frac{\tau}{T_0} = 0.8$, and $V = 5.0$ volts.

d) For your value of α determined in part **c** and the parameter values given in part **c**, determine an expression for the first two terms (the DC and first harmonic) in the Fourier series representation of the output signal.
(Answer: $y(t) \approx 4 + 0.566\cos(2\pi 100t - 3.77)$)

e) For your value of α determined in part **c**, and the parameter values given in part **c**, determine the **bandwidth** of the filter you designed. Be sure to include units!

#1 See Spectrum Below

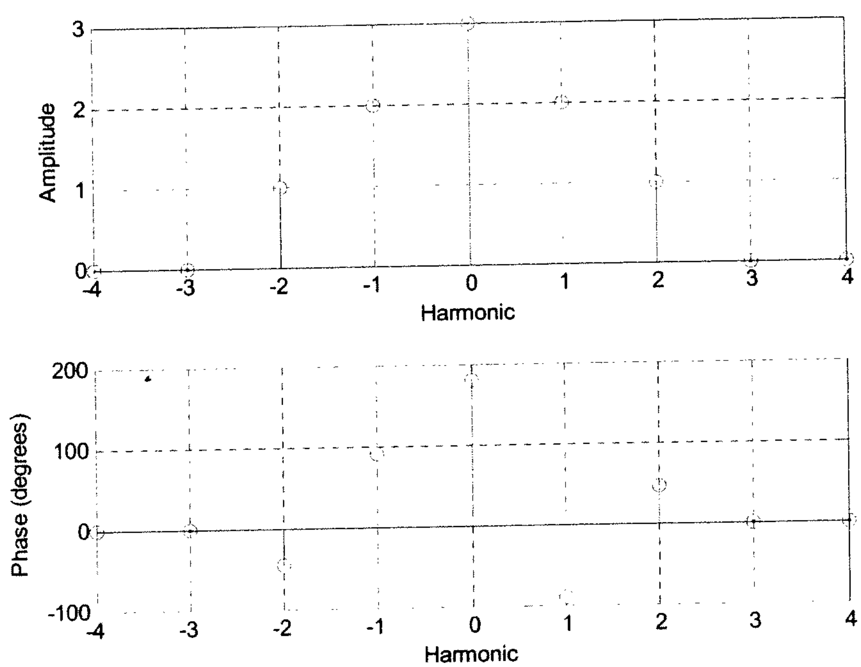
(a) $x(t) = c_0^x + 2|c_1^x| \cos(\omega_0 t + \measuredangle c_1^x) + 2|c_2^x| \cos(2\omega_0 t + \measuredangle c_2^x)$

$$\boxed{x(t) = -3 + 4\cos(\pi t - 90^\circ) + 2\cos(2\pi t + 45^\circ)}$$

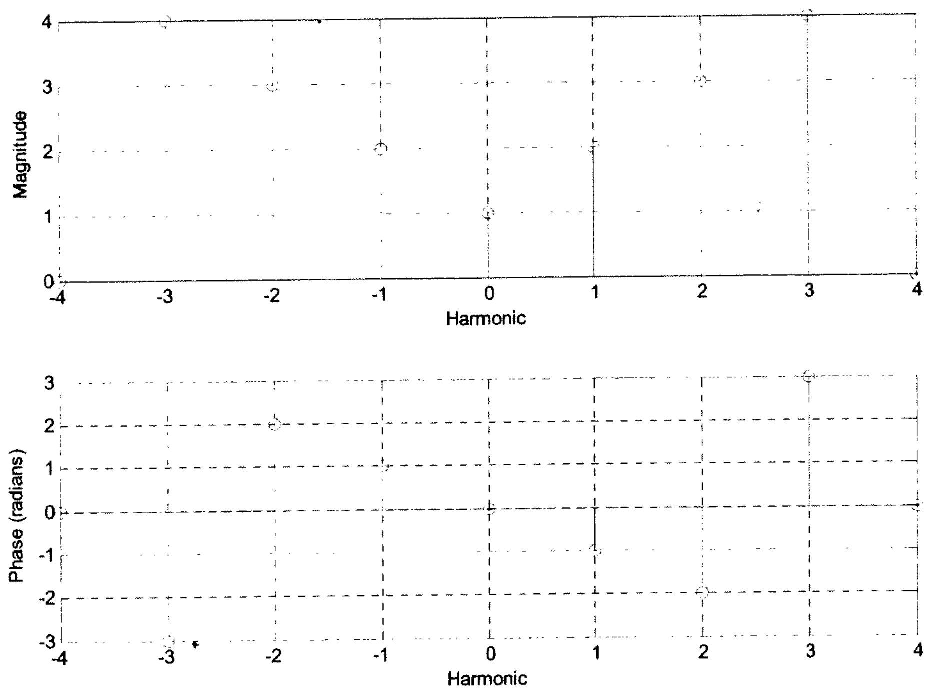
$T_0 = 2 \quad \omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{2} = \pi$

(b) $\boxed{\bar{x} = c_0^x = -3}$

(c) $P_{ave}^x = \Sigma |c_k|^2 = 1^2 + 2^2 + 3^2 + 2^2 + 1^2 = 2 + 8 + 9 = 19$



#2 $H(\omega) = \begin{cases} e^{-j\omega} & 1 \le |\omega| < 3 \\ 2e^{-j2\omega} & 3 < |\omega| 5 \\ 0 & \text{else} \end{cases}$ $\omega_0 = 2\,\text{rad/sec}$



$Y_0 = X_0 H(0) = 0$

$Y_1 = X_1 H(1\omega_0) = (2e^{-j1})(e^{-j2}) = 2e^{-j3} = 2 \measuredangle -3\ \text{rad}$

$Y_2 = X_2 H(2\omega_0) = (3e^{-j2})(2e^{-j8}) = 6e^{-j10} = 2 \measuredangle -10\ \text{rad}$

$Y_3 = X_3 H(3\omega_0) = 0$

$y(t) = Y_0 + 2|Y_1|\cos(\omega_0 t + \measuredangle Y_1) + 2|Y_2|\cos(2\omega_0 t + \measuredangle Y_2) + 0 + \cdots$

$\boxed{y(t) = 4\cos(2t-3) + 12\cos(4t-10)}$

#3 $x(t) = e^{-t} \quad 0 \le t \le 2$ $\qquad T_0 = 2 \quad f_0 = 1/2 = 0.5\,Hz$

$$x(t) = \sum_{K=-\infty}^{\infty} \frac{0.4323}{1+jK\pi} e^{jK\pi t}$$

a) $$P^x_{ave} = \frac{1}{T_0}\int_{T_0} |x(t)|^2 dt = \frac{1}{2}\int_0^2 e^{-2t} dt = \frac{1}{2} \left.\frac{e^{-2t}}{-2}\right|_0^2 = \frac{1}{4}(1-e^{-4}) = 0.2454$$

$$\boxed{P^x_{ave} = 0.2454}$$

b) The high pass filter removes signals with frequency content below 0.75 Hz. Let's figure out what they are

It removes $K=0$, $K=\pm 1$

$c_0^x = 0.4323$

$c_1^x = \frac{0.4323}{1+j\pi} = 0.13112 \angle 1.2626$ rad

$$y(t) = e^{-t} - 0.4323 - 2(0.13112)\cos(\pi t - 1.2626)$$

$$\boxed{y(t) = e^{-t} - 0.4323 - 0.26225\cos(\pi t - 1.2626)}$$

c) $$P^y_{ave} = P^x_{ave} - |c_0^x|^2 - 2|c_1^x|^2 = 0.2454 - (0.4323)^2 - 2(0.13112)^2$$

$$\boxed{P^y_{ave} = 0.02413}$$

d) $$\frac{|c_0^x|^2 + 2|c_1^x|^2}{P_x^{ave}} = \frac{(0.4323)^2 + 2(0.13112)^2}{0.2454} = 0.90166 \approx 90\%$$

#4

$x(t) = t^2 \quad -\pi \le t \le \pi$

$x(t) = \sum X_k e^{jkt} \qquad X_k = \begin{cases} \frac{\pi^2}{3} & k = 0 \\ \frac{2(-1)^k}{k^2} & k \neq 0 \end{cases}$

a) Bandpass filter, removes everything outside range 0.5 to 0.7 Hz

$w_0 = 1 \text{ rad/sec} = 2\pi f_0 \qquad f_0 = 1/2\pi = 0.159 \text{ Hz}$

k	$f = kf_0$
0	0
1	0.159
2	0.318
3	0.477
4	0.636 — only term
5	0.795

$y(t) = 2|c_4^x| \cos(4w_0 t + \measuredangle c_4^x)$

$c_4^x = \frac{2(-1)^4}{4^2} = \frac{2}{16} \measuredangle 0^\circ = \frac{1}{8} \measuredangle 0^\circ = 0.125 \measuredangle 0^\circ$

$\boxed{y(t) = 0.25 \cos(4t)}$

$P_{ave}^x = \frac{1}{2\pi} \int_{-\pi}^{\pi} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} t^4 dt = \frac{1}{2\pi} \frac{t^5}{5} \Big|_{-\pi}^{\pi} = \frac{2(\pi^5)}{5(2\pi)} = 19.48$

$\frac{2|c_4^x|^2}{P_{ave}^x} = \frac{2|0.125|^2}{19.48} \times 100\% = 0.16\%$ of total power

b) $y(t) = t^2 - 0.25\cos(4t)$

$\frac{P_{ave}^y}{P_{ave}^x} = 100\% - 0.16\% = \boxed{99.84\%}$

#5

$$x(t) = \sum X_K e^{jKw_0t} \qquad y(t) = \sum Y_K e^{jKw_0t}$$

a) $y(t) = b\,x(t-a)$

$$\sum Y_K e^{jKw_0t} = b \sum X_K e^{jKw_0(t-a)} = \sum b X_K e^{-jKw_0a} e^{jKw_0t}$$

$$\boxed{Y_K = X_K b e^{-jKw_0a} \qquad H(jw) = b e^{-jwa}}$$

b) $y(t) = b\,\dot{x}(t-a)$

$$\sum Y_K e^{jKw_0t} = \frac{d}{dt} \sum b e^{-jKw_0a} X_K e^{jKw_0t}$$

$$= \sum b e^{-jKw_0a}\, jKw_0 X_K e^{jKw_0t}$$

$$\boxed{Y_K = b e^{-jKw_0a}\, jKw_0 X_K \qquad H(jw) = b\,jw\, e^{-jwa}}$$

c) $y(t) = b\,x(t)\cos(w_0t)$

$$Y_K = \frac{1}{T_0}\int_{T_0} y(t) e^{-jKw_0t}\,dt$$

$$= \frac{1}{T_0}\int_{T_0} b\,x(t)\cos(w_0t) e^{-jKw_0t}\,dt$$

$$= \frac{1}{T_0}\int_{T_0} b\,x(t)\left[\frac{e^{jw_0t}}{2} + \frac{e^{-jw_0t}}{2}\right] e^{-jKw_0t}\,dt$$

$$= \frac{b}{2}\left[\frac{1}{T_0}\int_{T_0} x(t) e^{-j(K-1)w_0t}\,dt + \frac{1}{T_0}\int_{T_0} x(t) e^{-j(K+1)w_0t}\,dt\right]$$

$$\boxed{Y_K = \frac{b}{2}\left[X_{K-1} + X_{K+1}\right]}$$

not TI

d) $\ddot{y}(t) + \frac{2\zeta}{\omega_n}\dot{y}(t) + \frac{1}{\omega_n^2} y(t) = K x(t)$

$$\sum Y_K (jKw_0)^2 e^{jKw_0t} + \sum Y_K \frac{2\zeta}{\omega_n}(jKw_0) e^{jKw_0t} + \sum Y_K \frac{1}{\omega_n^2} e^{jKw_0t} = \sum X_n K e^{jKw_0t}$$

$$\boxed{Y_K = \frac{K}{(jKw_0)^2 + \frac{2\zeta}{\omega_n}(jKw_0) + \frac{1}{\omega_n^2}} X_K \qquad H(jw) = \frac{K}{(jw)^2 + \frac{2\zeta}{\omega_n}(jw) + \frac{1}{\omega_n^2}}}$$

#6 $x(t) = A\cos(\omega_0 t) = \frac{A}{2}e^{j\omega_0 t} + \frac{A}{2}e^{-j\omega_0 t}$

a) $y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\lambda)\, x(t-\lambda)\, d\lambda$

$= \frac{A}{2}\int_{-\infty}^{\infty} h(\lambda) e^{j\omega_0(t-\lambda)} d\lambda + \frac{A}{2}\int_{-\infty}^{\infty} h(\lambda)\, e^{-j\omega_0(t-\lambda)} d\lambda$

$= \frac{A}{2}e^{j\omega_0 t}\int_{-\infty}^{\infty} h(\lambda) e^{-j\omega_0\lambda} d\lambda + \frac{A}{2}e^{-j\omega_0 t}\int_{-\infty}^{\infty} h(\lambda) e^{+j\omega_0\lambda} d\lambda$

$= \boxed{\frac{A}{2}e^{j\omega_0 t}\, H(\omega_0) + \frac{A}{2}e^{-j\omega_0 t}\, H(-\omega_0) = y(t)}$

b) $H(\omega_0) = \int_{-\infty}^{\infty} h(\lambda) e^{-j\omega_0\lambda} d\lambda$

$H^*(\omega_0) = \int_{-\infty}^{\infty} h(\lambda) e^{j\omega_0\lambda} d\lambda = H(-\omega_0)$

$\boxed{|H(\omega_0)| = |H^*(\omega_0)| = |H(-\omega_0)| \text{ so magnitude is even}}$

$H(\omega_0) = |H(\omega_0)| e^{j\measuredangle H(\omega_0)}$

$H(-\omega_0) = |H(-\omega_0)| e^{j\measuredangle H(-\omega_0)} = |H(\omega_0)| e^{j\measuredangle H(-\omega_0)}$

$= H^*(\omega_0) = |H(\omega_0)| e^{-j\measuredangle H(\omega_0)}$

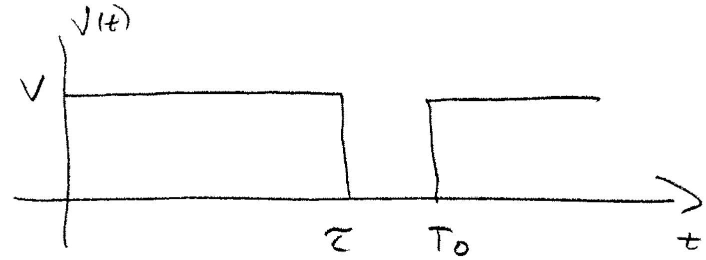
so $\boxed{\measuredangle H(-\omega_0) = -\measuredangle H(\omega_0)}$ and phase is odd

c) $y(t) = \frac{Ae^{j\omega_0 t}}{2}|H(\omega_0)| e^{j\measuredangle H(\omega_0)} + \frac{Ae^{-j\omega_0 t}}{2}\underbrace{|H(-\omega_0)|}_{=|H(\omega_0)|} e^{j\overbrace{\measuredangle H(-\omega_0)}^{=-\measuredangle H(\omega_0)}}$

$= A|H(\omega_0)|\left\{\frac{e^{j(\omega_0 t + \measuredangle H(\omega_0))} + e^{-j(\omega_0 t + \measuredangle H(\omega_0))}}{2}\right\}$

$\boxed{y(t) = A|H(\omega_0)|\cos(\omega_0 t + \measuredangle H(\omega_0))}$

#7



$$v(t) = \sum_{k=-\infty}^{\infty} e^{-jk\pi\tau/T_0}\left(\frac{V\tau}{T_0}\right)\operatorname{sinc}\left(\frac{k\tau}{T_0}\right)e^{jk\omega_0 t}$$

a) $$P_{ave}^{v} = \frac{1}{T_0}\int_{T_0}|v(t)|^2 dt = \frac{1}{T_0}\int_0^{\tau} V^2 dt = \boxed{V^2\frac{\tau}{T_0} = P_{ave}^{v}}$$

b) $$c_0^v = e^{-jk\pi\tau/T_0}\left(\frac{V\tau}{T_0}\right)\operatorname{sinc}\left(\frac{k\tau}{T_0}\right)\Bigg|_{k=0} = \boxed{\frac{V\tau}{T_0} = c_0^v = V_{ave}}$$

c) $v(t) \longrightarrow \boxed{H(j\omega)} \longrightarrow y(t)$

$$P_y^1 = 2|c_1^y|^2 \qquad P_y^0 = |c_0^y|^2$$

$$20\,dB = 10\log_{10}\left(\frac{|c_0^y|^2}{2|c_1^y|^2}\right)$$

$$2 = \log_{10}\left(\frac{|c_0^y|^2}{2|c_1^y|^2}\right)$$

$$\frac{|c_0^y|^2}{2|c_1^y|^2} = 10^2 = 100$$

$$|c_0^y|^2 = 200|c_1^y|^2 \quad \text{or} \quad |c_1^y|^2 = \frac{1}{200}|c_0^y|^2$$

$$c_0^y = c_0^v H(0) = \left(\frac{V\tau}{T_0}\right)(1) = 5(0.8)(1) = 4.0$$

$$|c_1^y| = |c_1^v H(j\omega_0)| = |c_1^v||H(j\omega_0)|$$

$$|H(j\omega_0)| = \left|\frac{1}{1+j\sqrt{\alpha}}\right| = \frac{1}{\sqrt{1+\alpha}}$$

$$|c_1^v| = \left|\frac{V\tau}{T_0}\right|\left|\operatorname{sinc}\left(\frac{\tau}{T_0}\right)\right| = (4)(\operatorname{sinc}(0.8)) = 4\,\frac{\sin(\pi(0.8))}{\pi(0.8)} = 0.9355$$

$$|c_1^y| = \frac{0.9355}{\sqrt{1+\alpha}}$$

#7 (continued)

$$|c_1^y|^2 = \frac{1}{200}|c_0^y|^2$$

$$\frac{(0.9355)^2}{(1+\alpha)} = \frac{1}{200}(4)^2 = \frac{16}{200} \qquad \left(\frac{200}{16}\right)(0.9355)^2 = 1+\alpha$$

$$\boxed{\alpha = 9.94}$$

(d) $c_0^y = 4$

$$c_1^y = c_1^v H(jw_0) = e^{-j\pi\tau/T_0}\left(\frac{V\tau}{T_0}\right)\mathrm{sinc}\left(\frac{\tau}{T_0}\right)\frac{1}{1+j\sqrt{\alpha}}$$

$$= e^{-j0.8\pi}(0.9355)\frac{1}{1+j\sqrt{9.94}}$$

$|c_1^y| = 0.2828$

$2|c_1^y| = 0.566$

$\measuredangle c_1^y = -0.8\pi - 1.26365 = -3.777$

$y(t) \approx c_0^y + 2|c_1^y|\cos(\omega_0 t + \measuredangle c_1^y)$

$$\boxed{y(t) \approx 4 + 0.566\cos(2\pi\cdot 100t - 3.777)}$$

(e) $\alpha = 9.94 \quad \sqrt{\alpha} = 3.153$

$$H(jw) = \frac{1}{jw\frac{\sqrt{\alpha}}{w_0}+1} = \frac{1}{jw\frac{(3.153)}{2\pi\cdot 100}+1} = \frac{1}{jw(0.00502)+1}$$

$$\approx \frac{1}{jw\frac{1}{199.3}+1}$$

Bandwidth $\approx$ $\boxed{199.3 \text{ rad/sec or } 31.7 \text{ Hz}}$