ECE 300
Signals and Systems
Homework 6

Due Date: Tuesday October 14. 2008 *at the beginning of class*

1. Show that any function $x(t)$ can be written in terms of and even function and an odd function, i.e. $x(t) = x_e(t) + x_o(t)$, where $x_e(t)$ is an even function, and $x_o(t)$ is an odd function. Determine expressions for $x_e(t)$ and $x_o(t)$ in terms of $x(t)$ (if you can do this than you have shown that $x(t) = x_e(t) + x_o(t)$).

2. Find the Fourier series representation for the signal indicated using hand analysis. Clearly indicate the values of ω_0 and the c_k. *Hints:* (1) *Draw the signal, and then use the sifting property to calculate the* c_k. (2) *If you understand how to do this, there is very little work involved.*

$$x(t) = \sum_{p=-\infty}^{\infty} \delta(t-3p)$$

3. Simplify each of the following into the form $c_k = \alpha(k) e^{-j\beta(k)} \mathrm{sinc}(\lambda k)$

a) $c_k = \dfrac{e^{j7k\pi} - e^{-j2k\pi}}{k\pi j}$

b) $c_k = \dfrac{e^{-j2\pi k} - e^{-j5\pi k}}{jk}$

c) $c_k = \dfrac{e^{j5k} - e^{j2k}}{k}$

Scrambled Answers $c_k = 3\pi e^{-j\frac{7\pi k}{2}} \mathrm{sinc}\left(\frac{3k}{2}\right)$, $c_k = 3e^{j(\frac{7}{2}k+\frac{\pi}{2})} \mathrm{sinc}\left(\frac{3k}{2\pi}\right)$, $c_k = 9e^{j\frac{5}{2}k\pi} \mathrm{sinc}\left(k\frac{9}{2}\right)$

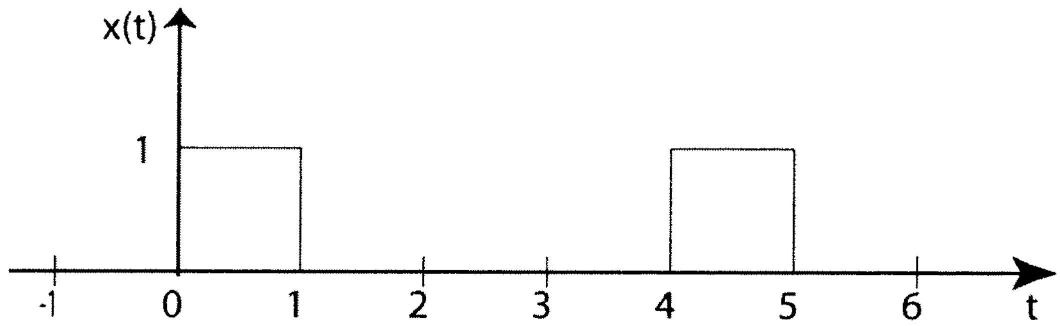
4. For the periodic square wave $x(t)$ with period $T_o = 0.5$ and

$$x(t)\begin{cases} 1 & 0 \le t < 0.25 \\ -1 & 0.25 \le t < 0.5 \end{cases}$$

show that the Fourier series coefficients are given by

$$c_k = \begin{cases} \dfrac{-2j}{k\pi} & k \quad odd \\ 0 & k \quad even \end{cases}$$

5. For the periodic signal shown below, with period $T = 4$



a) Determine the fundamental frequency ω_0.

b) Determine the average value.

c) Determine the average power in the DC component of the signal.

d) Determine an expression for the expansion coefficients, c_k. You must write your expression in terms of the **sinc** function, and possibly a leading phase term.

Matlab Problems

6. (Write a Matlab script for this problem, and turn it in along with your plots.)

Consider the following functions:

$$v_1(t) = 0.7071$$
$$v_2(t) = 1.0754e^t - 1.2638$$
$$v_3(t) = 4.9632t + 4.9632 - 4.2232e^t$$

a) Verify that these functions are (approximately) orthogonal over the interval [-1,1].

b) Assume we want to approximate $x(t) = t^4 - t$ in this interval using all three functions, $x(t) \approx c_1 v_1(t) + c_2 v_2(t) + c_3 v_3(t)$ Determine the expansion coefficients, then plot the approximation and the real function on the same graph. *Be sure to use different line types and a legend! Write down the expansion coefficients on the graph!*

c) A signal $x(t)$ is approximated in terms of a signal $v(t)$ over the interval $[t_1, t_2]$,

$$x(t) \approx cv(t)$$

where c is chosen to minimize the squared error (the magnitude of the error signal). Show that $v(t)$ and $e(t) = x(t) - cv(t)$ are orthogonal over the interval $[t_1, t_2]$.

This can be generalized to a very important result, called the *<u>principle of orthogonality</u>.,* *which is widely used in estimation theory. In this context it states that the error signal will always be orthogonal to the vectors (functions) used to make the approximation.*

d) Assume we want to approximate the function $x(t) = t^4 - t$ using the functions, $w_1(t) = 1$, $w_2(t) = t$, and $w_3(t) = e^t$ over the interval [-1,1]. Note that these functions are not orthogonal! Use the principle of orthogonality to determine the coefficients in the expansion

$$x(t) \approx d_1 w_1(t) + d_2 w_2(t) + d_3 w_3(t)$$

and compare this approximation with that in part **b** by plotting the original function, the results from part **b**, and the results from part **d** on one graph. (Note that the coefficients will be different because the basis functions are different. However, the approximation functions should be the same!). *Be sure to plot all three on one graph using different line types (you may need to use discrete points and only a few time instances....) and a legend. Write down the expansion coefficients on the graph!*

<u>Hints:</u>

(1) look at homework 2 for determining how to use **quadl** to integrate the product of functions

(2) In order to implement the constant function, use

```
v1 = @(t) 0.7071*ones(1,length(t));
```

(3) For part **d**, you will end up with a matrix equation (to be solved in Matlab), of the form

A d = b

The A matrix will be symmetric, and the first few elements will be

```
A(1,1) = quadl( @(t) w1(t).*w1(t), low, high);
A(1,2) = quadl( @(t) w2(t).*w1(t), low, high);
A(1,3) = quadl( @(t) w3(t).*w1(t), low, high);
```

b will be a vector of the form

```
b = [quadl( @(t) x(t).*w1(t), low, high);
     quadl...;
     quadl...];
```

To get the d's you need to use d = inv(A)*b;

You can then access the d's by using d(1), d(2), and d(3)

7. Read the **Appendix** and then do the following:

a) Copy the file **Trigonometric_Fourier_Series.m** (you wrote this for homework 4) to file **Complex_Fourier_Series.m**.

b) Modify **Complex_Fourier_Series.m** so it computes the average value c_o.

c) Modify **Complex_Fourier_Series.m** so it also computes c_k for $k=1$ to $k=N$

d) Modify **Complex_Fourier_Series.m** so it also computes the Fourier series estimate using the formula

$$x(t) \approx c_o + \sum_{k=1}^{N} 2\,|c_k|\cos(k\omega_o t + \measuredangle c_k)$$

You will probably need to use the Matlab functions **abs** and **angle** for this.

e) Using the code you wrote in part **d**, find the complex Fourier series representation for the following functions (defined over a single period)

$$f_1(t) = e^{-t}u(t) \quad 0 \le t < 3$$

$$f_2(t) = \begin{cases} t & 0 \le t < 2 \\ 3 & 2 \le t < 3 \\ 0 & 3 \le t < 4 \end{cases}$$

$$f_3(t) = \begin{cases} 0 & -2 \le t < -1 \\ 1 & -1 \le t < 2 \\ 3 & 2 \le t < 3 \\ 0 & 3 \le t < 4 \end{cases}$$

Use N = 10 and turn in your plots for each of these functions. Also, turn in your Matlab program for one of these. *Note that the values of **low** and **high** will be different for each of these functions!*

Appendix

In the majority of this course we will be using the complex (or exponential) form of the Fourier series, since it is really easier to do various mathematical things with it once you get used to it.

<u>Exponential Fourier Series</u> If $x(t)$ is a periodic function with fundamental period T, then we can represent $x(t)$ as a Fourier series

$$x(t)=\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_o t}=\sum_{k=-\infty}^{\infty} X_k e^{jk\omega_o t}$$

where $\omega_o=\frac{2\pi}{T}$ is the fundamental period, c_o is the average (or DC, i.e. zero frequency) value, and

$$c_o=\frac{1}{T}\int_T x(t)dt$$

$$c_k=\frac{1}{T}\int_T x(t)e^{-jk\omega_o t}dt$$

If $x(t)$ is a real function, then we have the relationships $|c_k|=|c_{-k}|$ (the magnitude is even) and $\measuredangle c_{-k}=-\measuredangle c_k$ (the phase is odd). Using these relationships we can then write

$$x(t)=c_o+\sum_{k=1}^{\infty}2|c_k|\cos(k\omega_o t+\measuredangle c_k)$$

This is usually a much easier form to deal with, since it lends itself easily to thinking of a phasor representation of $x(t)$. This will be particularly useful when we starting filtering periodic signals.

(#1) $x(t) = x_e(t) + x_o(t)$

$x_e(t)$ is an even function, so $x_e(-t) = x_e(t)$

$x_o(t)$ is and odd function, so $x_o(-t) = -x_o(t)$

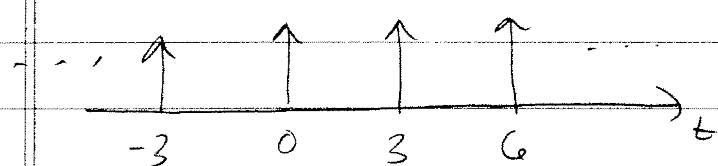
$x(t) = x_e(t) + x_o(t)$

$x(-t) = x_e(t) - x_o(t)$

adding $x(t) + x(-t) = 2x_e(t)$ $\boxed{x_e(t) = \frac{x(t) + x(-t)}{2}}$

subtracting $x(t) - x(-t) = 2x_o(t)$ $\boxed{x_o(t) = \frac{x(t) - x(-t)}{2}}$

(#2) $x(t) = \sum_{p=-\infty}^{\infty} \delta(t-3p)$



$T_0 = 3 \quad \omega_0 = \frac{2\pi}{3}$

$c_K = \frac{1}{T_0}\int_{-1.5}^{1.5} \delta(t) e^{-jK\omega_0 t} dt = \frac{1}{3}\int_{-1.5}^{1.5} \delta(t) dt = \frac{1}{3}$

$$\boxed{\begin{array}{l} c_K = \frac{1}{3} \\ x(K) = \sum_{K=-\infty}^{\infty} \frac{1}{3} e^{jK\frac{2\pi}{3}t} \end{array}}$$

#3

(a) $$c_k = \frac{e^{j7k\pi} - e^{-j2k\pi}}{jk\pi} = \frac{e^{j\frac{5}{2}\pi k}\left[e^{j\frac{9}{2}\pi k} - e^{-j\frac{9}{2}\pi k}\right]}{jk\pi}$$

$$= \frac{e^{j\frac{5}{2}\pi k}}{k\pi}\, 2\left[\frac{e^{j\frac{9}{2}\pi k} - e^{-j\frac{9}{2}\pi k}}{2j}\right]$$

$$= \frac{2e^{j\frac{5}{2}\pi k}}{k\pi}\sin\left(\frac{9}{2}\pi k\right)$$

$$= 2e^{j\frac{5}{2}\pi k}\,\frac{\sin\left(\frac{9}{2}\pi k\right)}{\pi k\left(\frac{9}{2}\right)\left(\frac{2}{9}\right)} = \boxed{9e^{j\frac{5}{2}\pi k}\,\mathrm{sinc}\left(k\frac{9}{2}\right) = c_k}$$

(b) $$c_k = \frac{e^{-j2k\pi} - e^{-j5\pi k}}{jk} = \frac{e^{-j\frac{7}{2}\pi k}\left[e^{+j\frac{3}{2}\pi k} - e^{-j\frac{3}{2}\pi k}\right]}{jk}$$

$$= \frac{2e^{-j\frac{7}{2}\pi k}}{k}\sin\left(\frac{3}{2}\pi k\right) = 2e^{-j\frac{7}{2}\pi k}\,\frac{\sin\left(\frac{3}{2}\pi k\right)}{k\cdot\left(\frac{3}{2}\pi\right)\left(\frac{2}{3\pi}\right)}$$

$$= 2e^{-j\frac{7}{2}\pi k}\,\mathrm{sinc}\left(\frac{3k}{2}\right)\cdot\frac{3\pi}{2}$$

$$\boxed{c_k = 3\pi e^{-j\frac{7}{2}\pi k}\,\mathrm{sinc}\left(\frac{3k}{2}\right)}$$

(c) $$c_k = \frac{e^{j5k} - e^{j2k}}{k} = \frac{e^{j\frac{7}{2}k}\left(e^{j\frac{3}{2}k} - e^{-j\frac{3}{2}k}\right)}{k}$$

$$= \frac{2j}{k}e^{j\frac{7}{2}k}\left(\frac{e^{j\frac{3}{2}k} - e^{-j\frac{3}{2}k}}{2j}\right) = \frac{2j}{k}e^{j\frac{7}{2}k}\sin\left(\frac{3}{2}k\right)$$

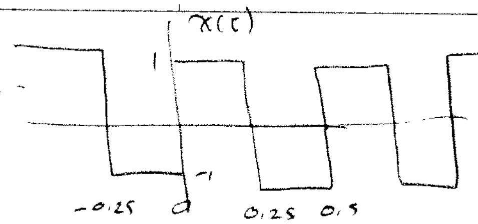
$$= \frac{2e^{j\left(\frac{7}{2}k + \frac{\pi}{2}\right)}}{k}\sin\left(\frac{3}{2}k\frac{\pi}{\pi}\right) = \frac{2}{k}e^{j\left(\frac{7}{2}k+\frac{\pi}{2}\right)}\sin\left(\pi\cdot\frac{3k}{2\pi}\right)$$

$$= \frac{2e^{j\left(\frac{7}{2}k+\frac{\pi}{2}\right)}\sin\left(\pi\cdot\frac{3k}{2\pi}\right)}{k\cdot\frac{\pi}{\pi}\,\frac{3}{3}\,\frac{2\pi}{2\pi}} = \frac{2e^{j\left(\frac{7}{2}k+\frac{\pi}{2}\right)}\sin\left(\pi\frac{3k}{2\pi}\right)}{\frac{\pi 3k}{2\pi}\cdot\frac{2\pi}{3\pi}}$$

$$\boxed{c_k = 3e^{j\left(\frac{7}{2}k+\frac{\pi}{2}\right)}\,\mathrm{sinc}\left(\frac{3k}{2\pi}\right)}$$

#4

$$x(t) = \begin{cases} 1 & 0 \le t \le 0.25 \\ -1 & 0.25 \le t \le 0.5 \end{cases}$$



$T_0 = \frac{1}{2}$ $\quad \omega_0 = \frac{2\pi}{T_0} = 4\pi$

$C_0 = 0$ (average value is zero)

$$C_K = \frac{1}{T_0}\int_{-0.25}^{0} (-1)\, e^{-jK\omega_0 t}\, dt + \frac{1}{T_0}\int_{0}^{0.25} (1)\, e^{-jK\omega_0 t}\, dt$$

$$= 2\int_{-0.25}^{0} (-1)\, e^{-jK4\pi t}\, dt + 2\int_{0}^{0.25} (1)\, e^{-jK4\pi t}\, dt$$

$$= -2\, \frac{e^{-jK4\pi t}}{-jK4\pi}\Bigg|_{-0.25}^{0} + 2\, \frac{e^{-jK4\pi t}}{-jK4\pi}\Bigg|_{0}^{0.25}$$

$$= \frac{2}{jK4\pi}\left[1 - e^{-jK\pi}\right] - \frac{2}{jK4\pi}\left[e^{+jK\pi} - 1\right]$$

$$= \frac{4}{jK4\pi}\left[1 - e^{jK\pi}\right] = \frac{1}{jK\pi}\left[1 - (-1)^K\right]$$

$$= = \frac{-j}{K\pi}\left[1 - (-1)^K\right]$$

$$a_K = \begin{cases} -\frac{2j}{K\pi} & K \text{ odd} \\ 0 & K \text{ even} \end{cases}$$

#5

(a) $T_0 = 4$ so $\omega_0 = 2\pi/4 = \boxed{\pi/2 = \omega_0}$

(b) $c_0 = \frac{1}{T_0}\int_{T_0} x(t)\,dt = \frac{1}{4}\int_0^1 1\,dt = \boxed{1/4 = c_0}$

(c) $P_0 = c_0^2 = \boxed{1/16 = P_0}$

(d) $c_K = \frac{1}{T_0}\int_{T_0} x(t)\,e^{-jK\omega_0 t}\,dt$

$= \frac{1}{T_0}\int_0^1 1\,e^{-jK\omega_0 t}\,dt = \frac{1}{T_0}\left[\frac{e^{-jK\omega_0 t}}{-jK\omega_0}\Big|_0^1\right]$

$= \frac{e^{-jK\omega_0} - 1}{-jK\omega_0 T_0} = \frac{1 - e^{-jK\omega_0}}{2\pi K j} = \frac{e^{-jK\frac{\omega_0}{2}}}{\pi K(2j)}\left[e^{jK\frac{\omega_0}{2}} - e^{-jK\frac{\omega_0}{2}}\right]$

$= \frac{e^{-jK\frac{\omega_0}{2}}}{\pi K}\sin\left(K\frac{\omega_0}{2}\right) = e^{-jK\pi/4}\,\frac{\sin\left(K\,\pi/4\right)}{\pi K\,(4/4)}$

$$\boxed{c_K = \frac{1}{4}\,e^{-jK\pi/4}\,\text{sinc}\left(K/4\right)}$$

#6

(a) see code $\langle V_i(t), V_k(t) \rangle \approx 0$ for $i \neq K$

(b) see code and graph $c_1 = 0.2828 \quad c_2 = -0.7028 \quad c_3 = -0.5490$

(c) $e(t) = x(t) - cV(t) \qquad c = \dfrac{\langle x(t), V(t) \rangle}{\langle V(t), V(t) \rangle}$

so
$$\langle e(t), V(t) \rangle = \langle x(t) - cV(t), V(t) \rangle$$
$$= \langle x(t), V(t) \rangle - c \langle V(t), V(t) \rangle$$
$$= \langle x(t), V(t) \rangle - \frac{\langle x(t), V(t) \rangle}{\langle V(t), V(t) \rangle} \langle V(t), V(t) \rangle = 0$$

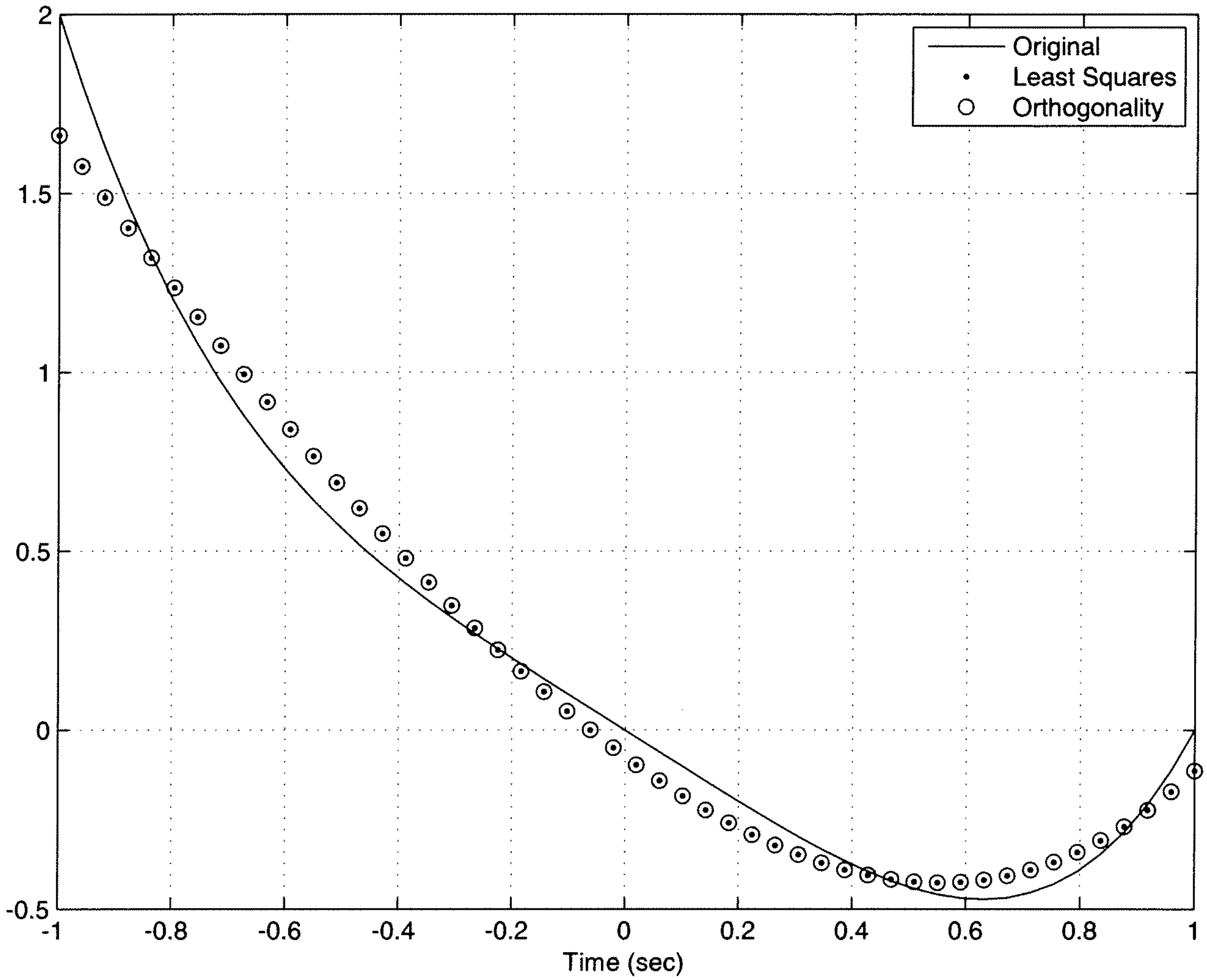
(d) $e(t) = x(t) - d_1 w_1(t) - d_2 w_2(t) - d_3 w_3(t)$

$$\langle e(t), w_1(t) \rangle = 0 = \langle x(t) - d_1 w_1(t) - d_2 w_2(t) - d_3 w_3(t), w_1(t) \rangle$$
or
$$d_1 \langle w_1(t), w_1(t) \rangle + d_2 \langle w_2(t) w_1(t) \rangle + d_3 \langle w_3(t), w_1(t) \rangle = \langle x(t), w_1(t) \rangle$$

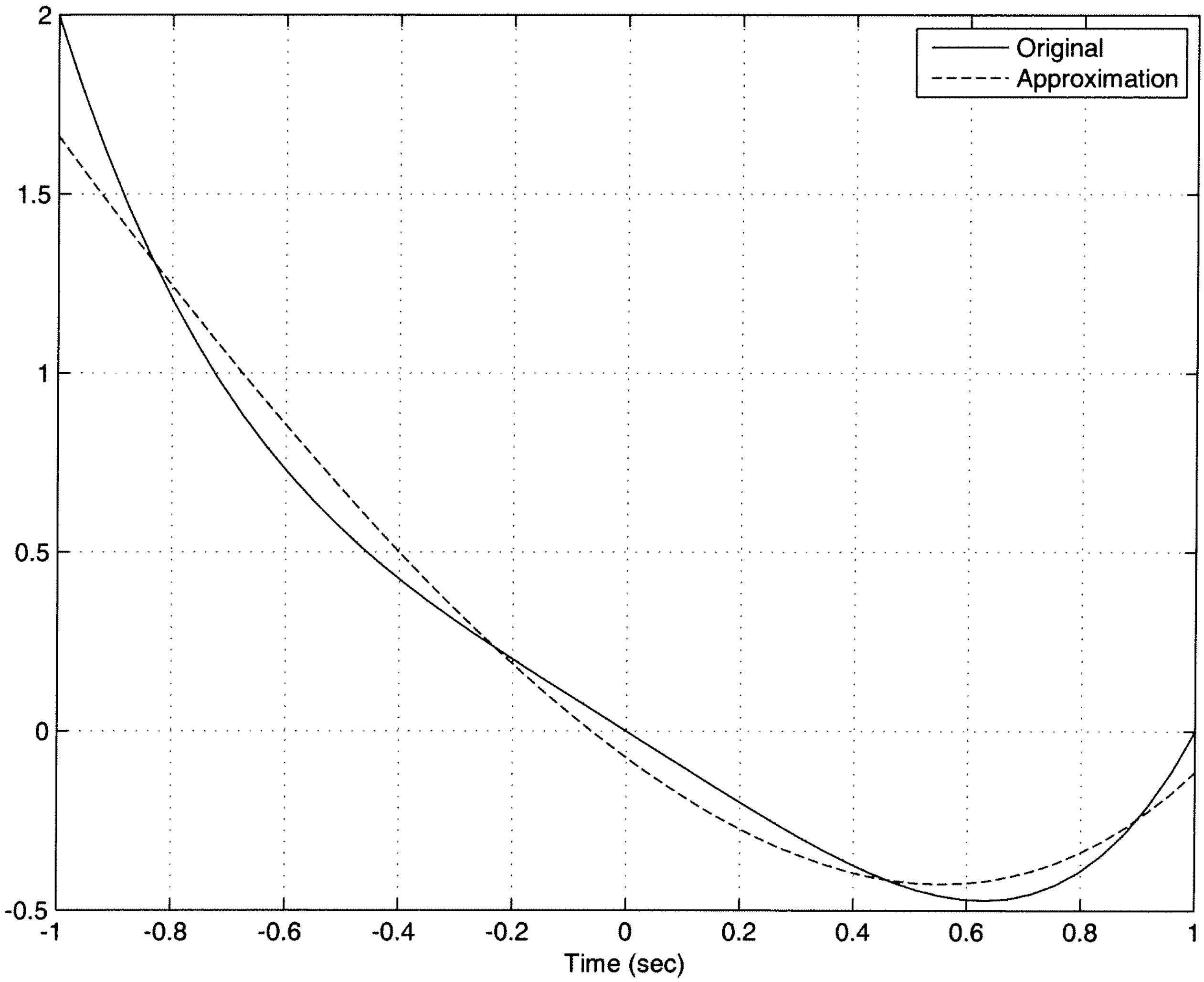
putting all 3 components together

$$\underbrace{\begin{bmatrix} \langle w_1, w_1 \rangle & \langle w_2, w_1 \rangle & \langle w_3, w_1 \rangle \\ \langle w_1, w_2 \rangle & \langle w_2, w_2 \rangle & \langle w_3, w_2 \rangle \\ \langle w_1, w_3 \rangle & \langle w_2, w_3 \rangle & \langle w_3, w_3 \rangle \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}}_{d} = \underbrace{\begin{bmatrix} \langle x, w_1 \rangle \\ \langle x, w_2 \rangle \\ \langle x, w_3 \rangle \end{bmatrix}}_{b}$$

$d_1 = -1.6368 \quad d_2 = -2.7249 \quad d_3 = 1.5629$



Original
Least Squares
Orthogonality
2
1.5
1
0.5
0
-0.5
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
Time (sec)



Original
Approximation
2
1.5
1
0.5
0
-0.5
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
Time (sec)

```
%
%  othogonal functions problems
%
 low = -1; high = 1;
%
%  get our original function
%
 x = @(t) t.^4 - t;
%
%  get our othogonal functions
%
 v1 = @(t) 0.7071*ones(1,length(t));
 v2 = @(t) 1.0754*exp(t)-1.2638;
 v3 = @(t) 4.9632*t+4.9632-4.2232*exp(t);
%
%  check for orthogonality
%
 value = quadl( @(t) v1(t).*v2(t), low, high )
 vaue = quadl( @(t) v1(t).*v3(t), low, high )
 value = quadl (@(t) v2(t).*v3(t), low, high )
%
%  determine approximation
%
 c1 = quadl( @(t) x(t).*v1(t), low, high ) / quadl( @(t) v1(t).*v1(t), low, high)
 c2 = quadl( @(t) x(t).*v2(t), low, high ) / quadl( @(t) v2(t).*v2(t), low, high)
 c3 = quadl( @(t) x(t).*v3(t), low, high ) / quadl( @(t) v3(t).*v3(t), low, high)
%
 xhat = @(t) c1*v1(t)+c2*v2(t)+c3*v3(t);
%
%  get the time vector
%
 t = linspace(low,high,50);
%
 plot(t,x(t),'-',t,xhat(t),'--'); grid;
 legend('Original','Approximation');
 xlabel('Time (sec)');
%
%  Now principle of orthogonality
%
 w1 = @(t) ones(1,length(t));
 w2 = @(t) t;
 w3 = @(t) exp(t);
%
%  set up the matrix
%
 A(1,1) = quadl( @(t) w1(t).*w1(t), low, high );
 A(1,2) = quadl( @(t) w2(t).*w1(t), low, high );
 A(1,3) = quadl( @(t) w3(t).*w1(t), low, high );
 A(2,1) = A(1,2);
 A(2,2) = quadl( @(t) w2(t).*w2(t), low, high );
 A(2,3) = quadl( @(t) w3(t).*w2(t), low, high );
 A(3,1) = A(1,3);
```

```
  A(3,2) = A(2,3);
  A(3,3) = quadl( @(t) w3(t).*w3(t), low, high );
%
% set up b
%
  b =  [ quadl( @(t) x(t).*w1(t), low, high );
         quadl( @(t) x(t).*w2(t), low, high );
         quadl( @(t) x(t).*w3(t), low, high )]
%
%  solve for the coefficents
%
  d = inv(A)*b
%
  yhat = @(t) d(1)*w1(t) + d(2)*w2(t) + d(3)*w3(t);
%
  figure;
  plot(t,x(t),'-',t,xhat(t),'.',t,yhat(t),'o'); grid;
  legend('Original','Least Squares','Orthogonality');
  xlabel('Time (sec)');
```