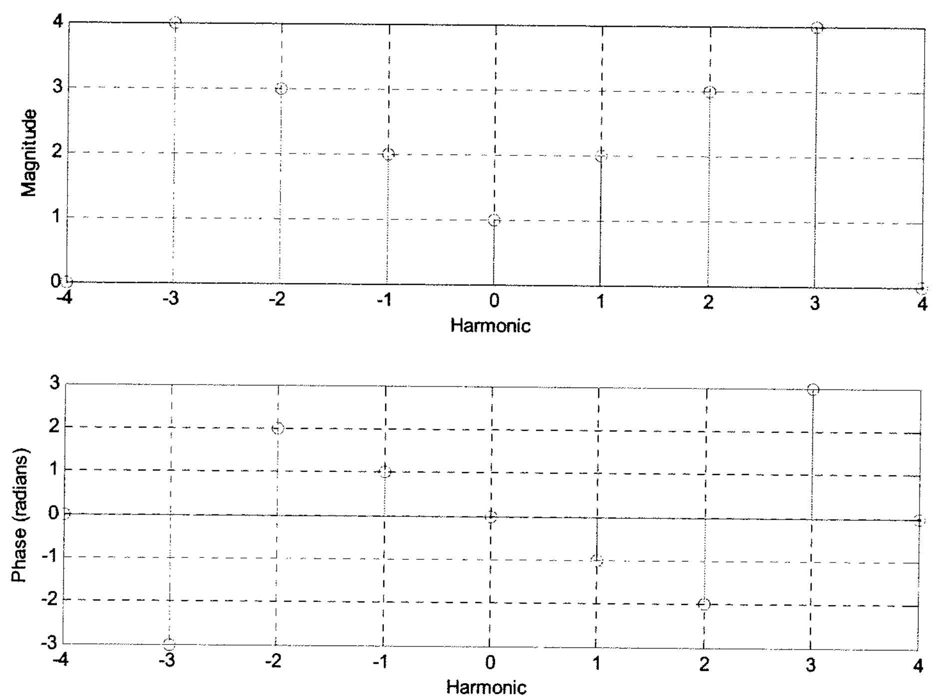
ECE 300
Signals and Systems
Homework 6

Due Date: Tuesday October 16, 2007 at the beginning of class

Exam 2, Thursday October 18, 2007

Problems:

1. Assume $x(t)$ has the spectrum shown below (the phase is shown in radians) and a fundamental frequency $\omega_o = 2 \text{ rad/sec}$:



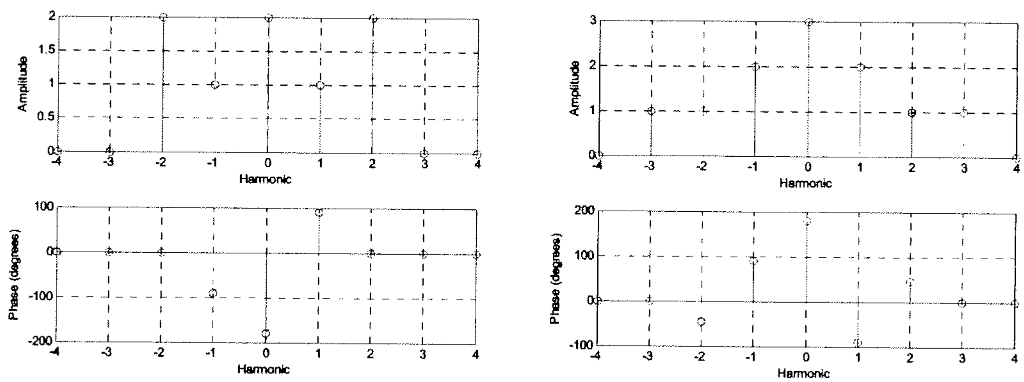
Assume $x(t)$ is the input to a system with the transfer function

$$H(\omega) = \begin{cases} e^{-j\omega} & 1 \le |\omega| < 3 \\ 2e^{-j2\omega} & 3 < |\omega| < 5 \\ 0 & else \end{cases}$$

Determine an expression for the steady state output $y(t)$. Be as specific as possible, simplifying all values and using actual numbers wherever possible.

2. ZTM, Problem 3-16.

3. The output of a LTI system, $y(t)$, has the following spectrum shown on the left, while the system transfer function, $H(k\omega_o)$, has the spectrum shown on the right. Assume all angles are multiples of 45 degrees.



a) Determine (sketch) the spectrum (magnitude and phase) of the input to the system, $x(t)$.

b) If $x(t)$ has the fundamental period $T = 2$ seconds, determine an analytical expression for $x(t)$ in terms of sine, cosines, and constants.

4. Assume two periodic signals have the Fourier series representations

$$x(t) = \sum X_k e^{jk\omega_o t} \quad y(t) = \sum Y_k e^{jk\omega_o t}$$

For the following system (input/output) relationships:

a) $y(t) = bx(t-a)$

b) $y(t) = b\dot{x}(t-a)$

c) $y(t) = bx(t)\cos(\omega_o t)$ (Answer: $Y_n = \frac{b}{2}(X_{n-1} + X_{n+1})$)

d) $\ddot{y}(t) + \frac{2\varsigma}{\omega_n}\dot{y}(t) + \frac{1}{\omega_n^2}y(t) = Kx(t)$

i) write Y_k in terms of the X_k

ii) If possible, determine the system transfer function $H(j\omega)$

iii) A system must be both linear and time-invariant to have a transfer function. If you cannot determine the transfer function, indicate which system property is not satisfied (**L** or **TI**).

5. Assume $x(t)$ has the Fourier series representation $x(t) = \sum X_k e^{jk\omega_o t}$ and fundamental period T_o. The function $y(t)$ is related to $x(t)$ through the relationship $y(t) = x\left(\frac{t}{b}\right)$.

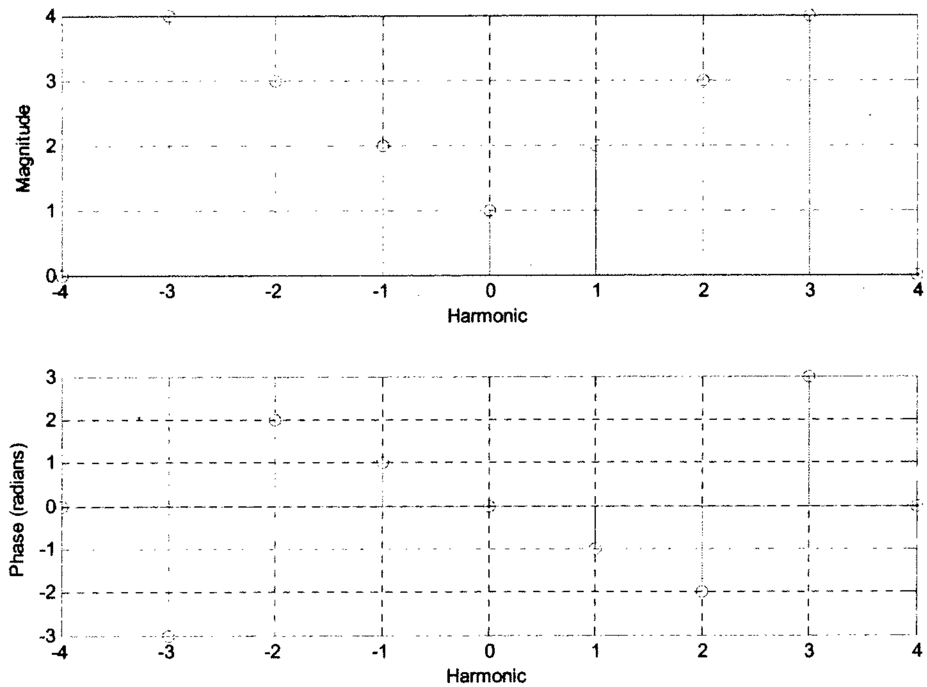
a) Determine the period of $y(t)$ in terms of T_o (the period of $x(t)$) and fundamental frequency for $y(t)$ in terms of ω_o (the fundamental frequency for $x(t)$)

b) Set up the integral to determine the Fourier series coefficients Y_k in terms of the parameters determined in part a (the integral should be centered at 0), and determine how Y_k is related to X_k

c) Starting from the relationship $x(t) = \sum X_k e^{jk\omega_o t}$ and making a simple substitution, show how we can determine the results from part b.

This problem demonstrates that compression or expansion of a signal does not change the Fourier series coefficients, it only changes the fundamental frequency.

#1 $H(\omega) = \begin{cases} e^{-j\omega} & 1 \le |\omega| < 3 \\ 2e^{-j2\omega} & 3 < |\omega| 5 \\ 0 & \text{else} \end{cases}$ $\omega_0 = 2\text{rad/sec}$



$Y_0 = X_0 H(0) = 0$

$Y_1 = X_1 H(1\omega_0) = (2e^{-j1})(e^{-j2}) = 2e^{-j3} = 2 \angle -3 \text{ rad}$

$Y_2 = X_2 H(2\omega_0) = (3e^{-j2})(2e^{-j8}) = 6e^{-j10} = 2 \angle -10 \text{ rad}$

$Y_3 = X_3 H(3\omega_0) = 0$

$y(t) = Y_0 + 2|Y_1|\cos(\omega_0 t + \angle Y_1) + 2|Y_2|\cos(2\omega_0 t + \angle Y_2) + 0 + \cdots$

$\boxed{y(t) = 4\cos(2t-3) + 12\cos(4t-10)}$

#2 ZTF Problem 3-16 $x(t) = \sum x_n e^{jn\omega_0 t}$ $X_n = \frac{1}{T_0}\int_{T_0} x(t) e^{-jn\omega_0 t}dt$

a) assume $x(t)$ is real and even

$$X_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\left[\cos(n\omega_0 t) - j\sin(n\omega_0 t)\right]dt$$

$$= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\cos(n\omega_0 t)dt - j\,\underbrace{\frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \underbrace{\underbrace{x(t)}_{\text{even}}\,\underbrace{\sin(n\omega_0 t)}_{\text{odd}}}_{\text{product is odd}}\,dt}_{\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \text{odd function} = 0}$$

so $$X_n = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\cos(n\omega_0 t)dt$$

$$X_{-n} = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\cos(-n\omega_0 t)dt = X_n \quad \text{(cosine is even)}$$

since $x(t)$ is real and even we have $\boxed{X_n \text{ is real and even}}$

b) assume $x(t)$ is real and odd

Then by the above arguments, $X_n = \frac{-j}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\sin(n\omega_0 t)dt$

$$X_{-n} = \frac{-j}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\sin(-n\omega_0 t)dt = \frac{j}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\sin(n\omega_0 t)dt = -X_n$$

since $x(t)$ is real and odd, $\boxed{X_n \text{ is purely imaginary and odd}}$

c) assume $x(t) = -x(t \pm \frac{T_0}{2})$ (half wave symmetry)

$$x(t) = \sum X_n e^{jn\omega_0 t} = -x(t+\tfrac{T_0}{2}) = -\sum X_n e^{jn\omega_0(t+T_0/2)}$$

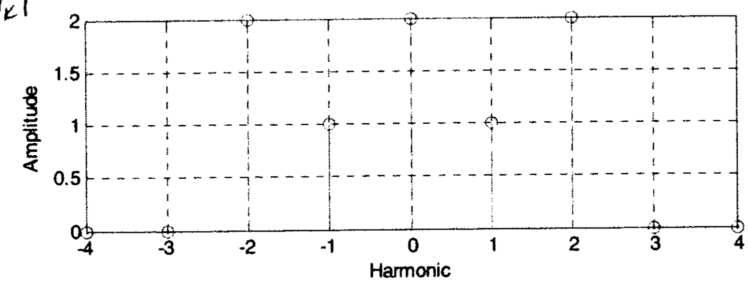
$$= -\sum X_n e^{jn\omega_0 t} e^{jn(\omega_0 T_0)/2} = e^{j\pi}\sum X_n e^{jn\omega_0 t} e^{jn\pi}$$

$$= \sum X_n e^{jn\omega_0 t} e^{j(n+1)\pi} = \sum X_n e^{jn\omega_0 t} = x(t)$$

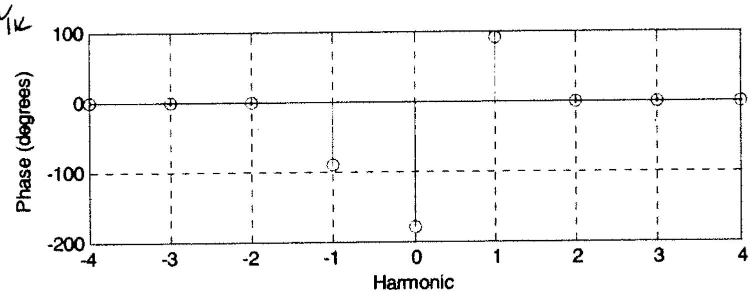
This will only happen if $X_n = 0$ for n even

#3

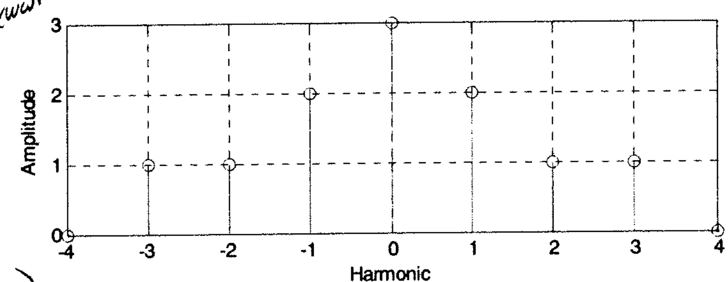
$|Y_k|$



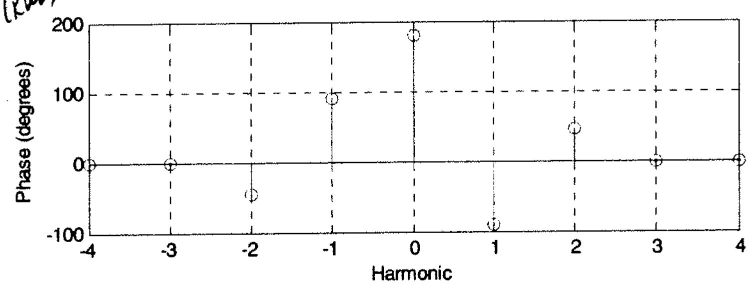
$\measuredangle Y_k$



$|H(k\omega_0)|$



$\measuredangle H(k\omega_0)$



$T_0 = 2 \qquad \omega_0 = 2\pi/T_0 = \pi$ rad/sec

$Y_K = X_K\, H(K\omega_0)$ or $X_K = \dfrac{Y_K}{H(K\omega_0)}$

$|X_K| = \dfrac{|Y_K|}{|H(K\omega_0)|}$

$\measuredangle X_K = \measuredangle Y_K - \measuredangle H(K\omega_0)$

$|X_0| = \frac{2}{3} \quad \measuredangle X_0 = -180^\circ - 180^\circ = -360^\circ = 0^\circ$

$|X_1| = \frac{1}{2} \quad \measuredangle X_1 = 90^\circ - (-90^\circ) = 180^\circ$

$|X_2| = \frac{2}{1} \quad \measuredangle X_2 = 0^\circ - 45^\circ = -45^\circ$

$$\boxed{x(t) = \tfrac{2}{3} + \cos(\pi t + 180^\circ) + 4\cos(2\pi t - 45^\circ)}$$

#4 $x(t) = \sum X_K e^{jKw_0 t}$ $\quad y(t) = \sum Y_K e^{jKw_0 t}$

a) $y(t) = b\,x(t-a)$

$$\sum Y_K e^{jKw_0 t} = b \sum X_K e^{jKw_0(t-a)} = \sum b X_K e^{-jKw_0 a} e^{jKw_0 t}$$

$$\boxed{Y_K = X_K b e^{-jKw_0 a} \qquad H(jw) = b e^{-jwa}}$$

b) $y(t) = b\,\dot{x}(t-a)$

$$\sum Y_K e^{jKw_0 t} = \frac{d}{dt} \sum b e^{-jKw_0 a} X_K e^{jKw_0 t}$$

$$= \sum b e^{-jKw_0 a} jKw_0 X_K e^{jKw_0 t}$$

$$\boxed{Y_K = b e^{-jKw_0 a} jKw_0 X_K \qquad H(jw) = b jw e^{-jwa}}$$

c) $y(t) = b\,x(t)\cos(w_0 t)$

$$Y_K = \frac{1}{T_0} \int_{T_0} y(t) e^{-jKw_0 t} dt$$

$$= \frac{1}{T_0} \int_{T_0} b\,x(t) \cos(w_0 t) e^{-jKw_0 t} dt$$

$$= \frac{1}{T_0} \int_{T_0} b\,x(t) \left[\frac{e^{jw_0 t}}{2} + \frac{e^{-jw_0 t}}{2}\right] e^{-jKw_0 t} dt$$

$$= \frac{b}{2}\left[\frac{1}{T_0}\int_{T_0} x(t) e^{-j(K-1)w_0 t} dt + \frac{1}{T_0}\int_{T_0} x(t) e^{-j(K+1)w_0 t} dt\right]$$

$$\boxed{Y_K = \frac{b}{2}\left[X_{K-1} + X_{K+1}\right]}$$ not TI

d) $\ddot{y}(t) + \frac{2\zeta}{\omega_n}\dot{y}(t) + \frac{1}{\omega_n^2} y(t) = K x(t)$

$$\sum Y_K (jKw_0)^2 e^{jKw_0 t} + \sum Y_K \frac{2\zeta}{\omega_n}(jKw_0) e^{jKw_0 t} + \sum Y_K \frac{1}{\omega_n^2} e^{jKw_0 t} = \sum X_n K e^{jKw_0 t}$$

$$\boxed{Y_K = \frac{K}{(jKw_0)^2 + \frac{2\zeta}{\omega_n}(jKw_0) + \frac{1}{\omega_n^2}} X_K \qquad H(jw) = \frac{K}{(jw)^2 + \frac{2\zeta}{\omega_n}(jw) + \frac{1}{\omega_n^2}}}$$

#5 $x(t) = \sum x_k e^{jk\omega_0 t}$ $\quad X_k = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t) e^{-jk\omega_0 t} dt$

$y(t) = x\left(\frac{t}{b}\right)$

a) $y(bT_0) = x(T_0)$ so the period of $y(t)$ is bT_0 and the fundamental frequency is

$$\frac{2\pi}{bT_0} = \frac{\omega_0}{b}$$

b) $$Y_n = \frac{1}{bT_0}\int_{-\frac{bT_0}{2}}^{\frac{bT_0}{2}} x\left(\frac{t}{b}\right) e^{-jk\frac{\omega_0}{b}t} dt$$

let $\sigma = \frac{t}{b}$ $\quad b\sigma = t$ $\quad b\,d\sigma = dt$

$$Y_n = \frac{1}{bT_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(\sigma) e^{-jk\frac{\omega_0}{b}b\sigma} b\,d\sigma$$

$$= \frac{1}{T_0}\int_{-T_0}^{\frac{T_0}{2}} x(\sigma) e^{-jk\omega_0\sigma} d\sigma = X_n$$

so $\boxed{Y_n = X_n}$

c)
$x(t) = \sum X_k e^{jk\omega_0 t}$

$y(t) = x\left(\frac{t}{b}\right) = \sum X_k e^{jk\omega_0\frac{t}{b}} = \sum x_k e^{jk\left(\frac{\omega_0}{b}\right)t}$

so $Y_k = X_k$