ECE 300
Signals and Systems
Homework 3

Due Date: Tuesday September 25. 2007 *at the beginning of class*

EXAM #1, Thursday September 27

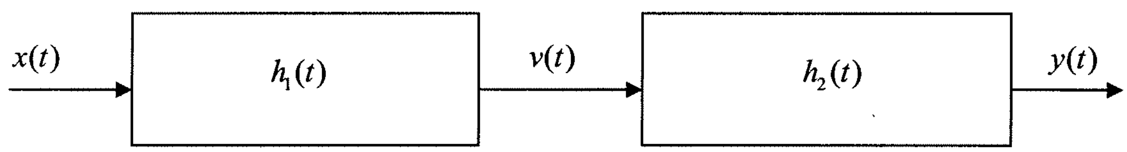
Problems

1. ZTF Problem 2-19 (use the method from class, solve the DE).

2. ZTF Problem 2-20.

3. ZTF Problem 2-24.

4. Consider the following two subsystems, connected together to form a single LTI system.



Determine the impulse response $h(t)$ of the entire system if the impulse responses of the subsystems are given as:

a) $h_1(t) = \delta(t) \quad h_2(t) = 2e^{-t}u(t)$

b) $h_1(t) = e^{-t}u(t) \quad h_2(t) = 2\delta(t-1)$

c) $h_1(t) = e^{-t}u(t) \quad h_2(t) = e^{-t}u(t)$

d) $h_1(t) = 2\delta(t-1) \quad h_2(t) = 3\delta(t-2)$

e) $h_1(t) = 2\delta(t-1) \quad h_2(t) = u(t)$

Simplify your answers as much as possible.

5. Consider a causal linear time invariant system with impulse response given by

$$h(t) = e^{-(t-1)}u(t-1)$$

The input to the system is given by

$$x(t) = u(t) - u(t-1) + u(t-3)$$

Using ***graphical convolution***, determine the output $y(t)$ **for** $2 \le t \le 5$. ***Note the limited range of t we are interested in !***

Specifically, you must

a) Flip and slide $h(t)$
b) Show graphs displaying both $h(t-\lambda)$ and $x(\lambda)$ for each region of interest
c) Determine the range of t for which each part of your solution is valid
d) Set up any necessary integrals to compute $y(t)$
e) Evaluate the integrals

You should get (in unsimplified form)

$$y(t) = \begin{cases} e^{-(t-1)}[e^1 - 1] & 2 \le t \le 4 \\ e^{-(t-1)}[e^1 - 1] + e^{-(t-1)}[e^{t-1} - e^3] & 4 \le t \le 5 \end{cases}$$

6. Consider a linear time invariant system with impulse response given by

$$h(t) = e^{-(t+1)}u(t+1)$$

The input to the system is given by

$$x(t) = 2[u(t) - u(t-1)] + 3[u(t-3) - u(t-4)]$$

Using ***graphical convolution***, determine the output $y(t)$. Specifically, you must

a) Flip and slide $h(t)$
b) Show graphs displaying both $h(t-\lambda)$ and $x(\lambda)$ for each region of interest
c) Determine the range of t for which each part of your solution is valid
d) Set up any necessary integrals to compute $y(t)$ ***Do Not Evaluate the Integrals***

7. Pre-Lab Exercises (to be done by all students. Turn this in with your homework and bring a copy of this with you to lab!)

a) Calculate the impulse response of the RC lowpass filter shown in Figure 2, in terms of unspecified components R and C. Determine the time constant for the circuit.

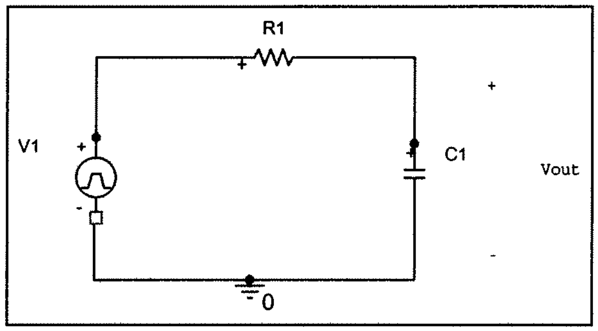


Figure 2. Simple RC lowpass filter circuit.

b) Find the **step response** of the circuit (the response of the system when the input is a unit step), and determine the 10-90% rise time. t_r, as shown below in Figure 3. The rise time is simply the amount of time necessary for the output to rise from 10% to 90% of its final value. Specifically, show that the rise time is given by $t_r = \tau \ln(9)$

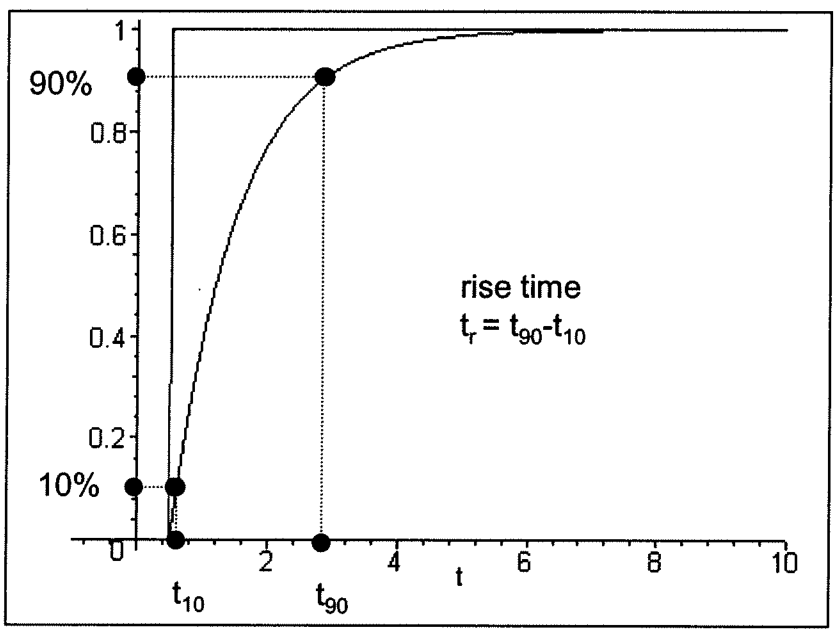


Figure 3. Step response of the RC lowpass filter circuit of Figure 1, showing the definition of the 10-90% risetime.

c) Specify values R and C which will produce a time constant of approximately 1 msec. Be sure to consider the fact that the capacitor will be asked to charge and discharge quickly in these measurements.

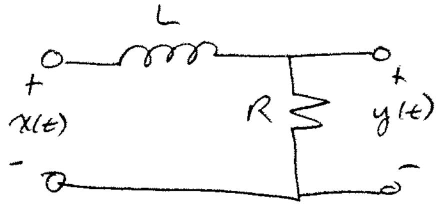
d) Using linearity and time-invariance, show that the response of the circuit to a unit pulse of length T (, i.e. a pulse of amplitude 1 starting at 0 and ending at T) is given by

$$y_{pulse}(t) = (1 - e^{-t/\tau})u(t) - (1 - e^{-(t-T)/\tau})u(t-T)$$

e) Plot the response to a unit pulse (in Matlab) for $\tau = 0.001$ and $T = 0.003$, 0.001, and 0.0001 from 0 to 0.008 seconds. Note on the plots the times the capacitor is charging and discharging. Use the **subplot** command to make three separate plots, one on top of another (i.e., use subplot(3,1,1), subplot(3,1,2), subplot(3,1,3)).

f) If the input is a pulse of amplitude A and width T, determine an expression for the amplitude of the output at the end of the pulse, $y_{pulse}(T)$. Assume that $\frac{T}{\tau} \ll 1$ (the duration for the pulse is much small than the time constant of the circuit) and use Taylor series approximations for the exponentials. Under these assumptions, show that the amplitude of $y_{pulse}(t)$ at time T is approximately the area of the pulse divided by the time constant, that is $y_{pulse}(T) = \frac{AT}{\tau}$

ZTF Problem 2-19



$$V_L(t) = L\frac{di_L(t)}{dt}$$

$$V_L(t) = x(t) - y(t)$$

$$i_L(t) = \frac{y(t)}{R}$$

so $x(t) - y(t) = L\frac{\dot{y}(t)}{R}$

$$\frac{L}{R}\dot{y}(t) + y(t) = x(t)$$

$$\dot{y}(t) + \frac{R}{L}y(t) = \frac{R}{L}x(t)$$

for impulse response $x(t) \to \delta(t)$
$y(t) \to h(t)$
initial conditions are zero

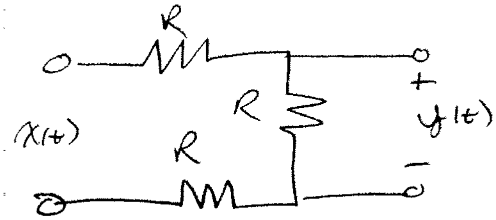
$$\dot{h}(t) + \frac{R}{L}h(t) = \frac{R}{L}\delta(t)$$

$$\frac{d}{dt}\left(h(t)e^{\frac{R}{L}t}\right) = e^{\frac{R}{L}t}\frac{R}{L}\delta(t)$$

$$\int_{-\infty}^{t}\frac{d}{d\lambda}\left(h(\lambda)e^{\frac{R}{L}\lambda}\right)d\lambda = h(t)e^{\frac{R}{L}t} = \int_{-\infty}^{t}e^{\frac{R}{L}\lambda}\frac{R}{L}\delta(\lambda)d\lambda = \frac{R}{L}u(t)$$

$$\boxed{h(t) = \frac{R}{L}e^{-\frac{R}{L}t}u(t)}$$

ZTF Problem 2-20.



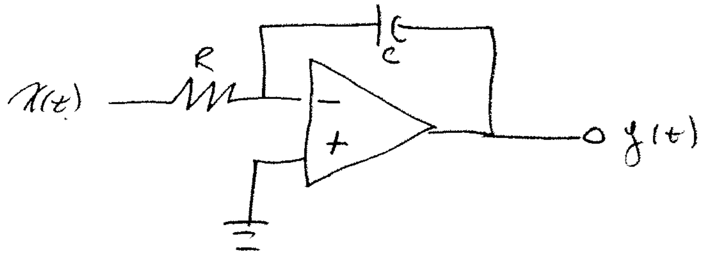
$$y(t) = x(t)\,\frac{R}{3R} = \frac{x(t)}{3}$$

impulse response $x(t) \to \delta(t)$

$y(t) \to h(t)$

$$\boxed{h(t) = \frac{1}{3}\delta(t)}$$

ZTF Problem 2-24



$$\frac{x(t)}{R} + C\frac{dV_c}{dt} = 0 \quad V_c(t) = y(t)$$

$$C\frac{dy(t)}{dt} + \frac{x(t)}{R} = 0$$

$$\frac{dy(t)}{dt} + \frac{1}{RC}x(t) = 0$$

for impulse response $x(t) \rightarrow \delta(t)$
$y(t) \rightarrow h(t)$
initial conditions are zero

$$\dot{h}(t) + \frac{1}{RC}\delta(t) = 0$$

$$\frac{dh}{dt} = \frac{-1}{RC}\delta(t)$$

$$\int_{-\infty}^{t} \frac{dh(\lambda)}{d\lambda}\,d\lambda = \int_{-\infty}^{t} -\frac{1}{RC}\delta(\lambda)\,d\lambda$$

$$\boxed{h(t) = -\frac{1}{RC}u(t)}$$

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(t-\lambda)\,x(\lambda)\,d\lambda = \int_{-\infty}^{\infty} -\frac{1}{RC}u(t-\lambda)\,x(\lambda)\,d\lambda$$

$$\boxed{y(t) = -\frac{1}{RC}\int_{-\infty}^{t} x(\lambda)\,d\lambda}$$

#4

$$x(t) \longrightarrow \boxed{h_1(t)} \xrightarrow{v(t)} \boxed{h_2(t)} \longrightarrow y(t)$$

Note $v(t) = h_1(t) * x(t)$

$$y(t) = h_2(t) * v(t) = h_2(t) * \big(h_1(t) * x(t)\big) = \big(h_1(t) * h_2(t)\big) * x(t)$$

so $h(t) = h_1(t) * h_2(t)$

a) $h_1(t) = \delta(t) \quad h_2(t) = 2e^{-t}u(t)$

$$h(t) = \int_{-\infty}^{\infty} h_1(t-\lambda)\, h_2(\lambda)\, d\lambda = \int_{-\infty}^{\infty} \delta(t-\lambda)\, 2e^{-\lambda} u(\lambda)\, d\lambda$$

$$= \boxed{2e^{-t}u(t) = h(t)}$$

note $h(t) * \delta(t) = h(t)$ (no need to do the math!)

b) $h_1(t) = e^{-t}u(t) \quad h_2(t) = 2\delta(t-1)$

$$h(t) = h_1(t) * h_2(t) = \boxed{2e^{-(t-1)}u(t-1) = h(t)}$$

using LTI property

c) $h_1(t) = e^{-t}u(t) \quad h_2(t) = e^{-t}u(t)$

$$h(t) = \int_{-\infty}^{\infty} h_1(t-\lambda)\, h_2(\lambda)\, d\lambda = \int_{-\infty}^{\infty} e^{-(t-\lambda)} u(t-\lambda)\, e^{-\lambda} u(\lambda)\, d\lambda$$

$$= e^{-t} \int_{-\infty}^{\infty} u(t-\lambda)\, u(\lambda)\, d\lambda = e^{-t} \int_0^t d\lambda = \boxed{te^{-t}u(t) = h(t)}$$

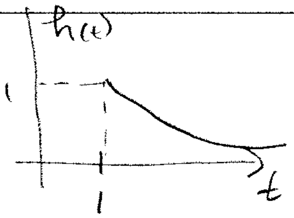
d) $h_1(t) = 2\delta(t-1) \quad h_2(t) = 3\delta(t-2)$

$$h(t) = h_1(t) * h_2(t) = \boxed{6\,\delta(t-3) = h(t)}$$ – using LTI

e) $h_1(t) = 2\delta(t-1) \quad h_2(t) = u(t)$

$$h(t) = h_1(t) * h_2(t) = \boxed{2u(t-1) = h(t)}$$ using LTI

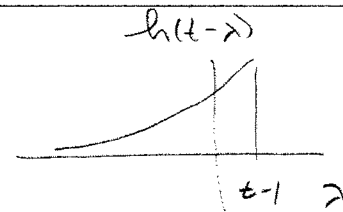
#5 $h(t) = e^{-(t-1)}u(t-1) \quad x(t) = u(t) - u(t-1) + u(t-3)$

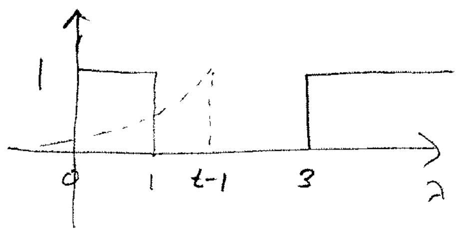


$h(1) = h(t-\lambda)$

$1 = t - \lambda$

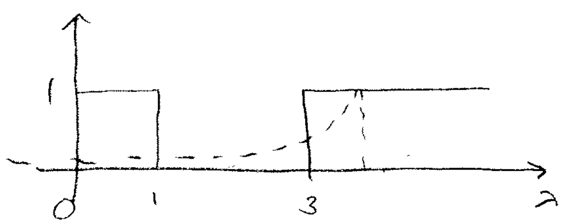
$\lambda = t - 1$





$2 \le t \le 4$

$$y(t) = \int_0^1 e^{-(t-\lambda-1)}d\lambda = e^{-(t-1)}\int_0^1 e^{\lambda}d\lambda = e^{-(t-1)}\left[e^1 - 1\right]$$



$t \ge 4$

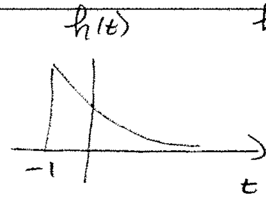
$$y(t) = \int_0^1 e^{-(t-\lambda-1)}d\lambda + \int_3^{t-1} e^{-(t-\lambda-1)}d\lambda$$

$$= e^{-(t-1)}\left[e^1 - 1\right] + e^{-(t-1)}\left[e^{\lambda}\right]_3^{t-1}$$

$$= e^{-(t-1)}\left[e^1 - 1\right] + e^{-(t-1)}\left[e^{t-1} - e^3\right]$$

$$\boxed{y(t) = \begin{cases} e^{-(t-1)}\left[e^1 - 1\right] & 2 \le t \le 4 \\ e^{-(t-1)}\left[e^1 - 1\right] + e^{-(t-1)}\left[e^{t-1} - e^3\right] & 4 \le t \end{cases}}$$

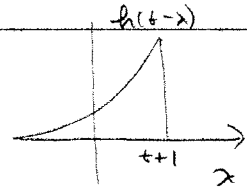
#4 $h(t) = e^{-(t+1)} u(t+1) \qquad x(t) = 2[u(t) - u(t-1)] + 3[u(t-3) - u(t-4)]$

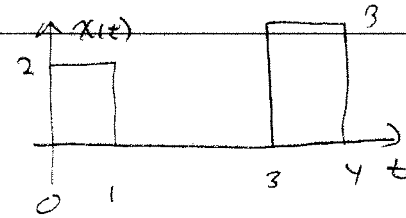


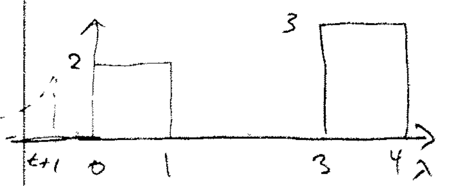
$h(-1) = h(t-\lambda)$

$-1 = t - \lambda$

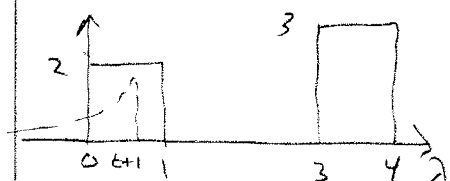
$\lambda = t+1$



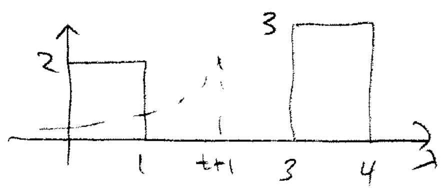




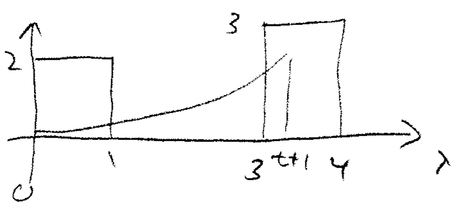
$t \le -1 \qquad y(t) = 0$



$-1 \le t \le 0 \qquad y(t) = \int_0^{t+1} e^{-(t-\lambda+1)}\, 2\, d\lambda$

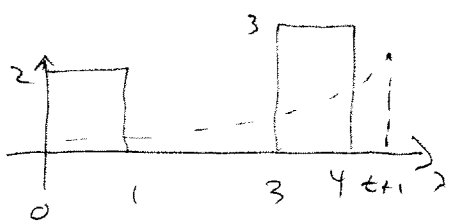


$0 \le t \le 2 \qquad y(t) = \int_0^{1} e^{-(t-\lambda+1)}\, 2\, d\lambda$



$2 \le t \le 3$

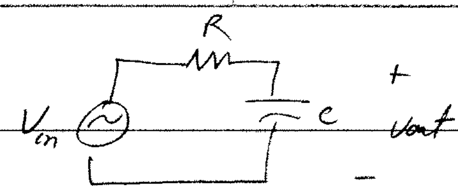
$$y(t) = \int_0^1 e^{-(t-\lambda+1)}\, 2\, d\lambda + \int_3^{t+1} e^{-(t-\lambda+1)}\, 3\, d\lambda$$



$t \ge 3$

$$y(t) = \int_0^1 e^{-(t-\lambda+1)}\, 2\, d\lambda + \int_3^{4} e^{-(t-\lambda+1)}\, 3\, d\lambda$$

#7



(a)

$$\frac{V_{in} - V_{out}}{R} = C\frac{dV_{out}}{dt} \qquad CR\frac{dV_{out}}{dt} + V_{out} = V_{in}$$

$$\frac{dV_{out}}{dt} + \frac{1}{RC}V_{out} = \frac{1}{RC}V_{in} \qquad \frac{d}{dt}\left(e^{t/RC}V_{out}\right) = \frac{1}{RC}V_{in}\,e^{t/RC}$$

$$\int_{-\infty}^{t}\frac{d}{dt}\left(e^{t/RC}V_{out}(t)\right)dt = e^{t/RC}V_{out} = \int_{-\infty}^{t}\frac{1}{RC}e^{\lambda/RC}V_{in}(\lambda)\,d\lambda$$

for impulse response $V_{in}(t) = \delta(t)$ $V_{out}(t) = h(t)$

$$e^{+t/RC}h(t) = \int_{-\infty}^{t}\frac{1}{RC}e^{\lambda/RC}\delta(\lambda)\,d\lambda = \frac{1}{RC}u(t)$$

$$\boxed{h(t) = \frac{1}{RC}e^{-t/RC}u(t)} \qquad \boxed{\tau = RC}$$

or $\dfrac{V_{out}(s)}{V_{in}(s)} = \dfrac{1/Cs}{R + 1/Cs} = \dfrac{1}{RCs+1} = \dfrac{1}{RC}\dfrac{1}{s + 1/RC} \Rightarrow h(t) = \dfrac{1}{RC}e^{-t/RC}u(t)$

(b) $h(t) = \frac{1}{\tau}e^{-t/\tau}u(t)$

$$y(t) = h(t) * u(t) = \int_{-\infty}^{\infty} h(\lambda)\,u(t-\lambda) = \int_0^t \frac{1}{\tau}e^{-t/\tau}d\tau = -e^{-t/\tau}\Big|_0^t$$

$$\boxed{y(t) = \left[1 - e^{-t/\tau}\right]u(t)}$$

For rise time $\quad 0.9 = 1 - e^{-t_9/\tau} \quad$ or $\quad 0.1 = e^{-t_9/\tau}$

$\qquad\qquad\qquad 0.1 = 1 - e^{-t_1/\tau} \quad$ or $\quad 0.9 = e^{-t_1/\tau}$

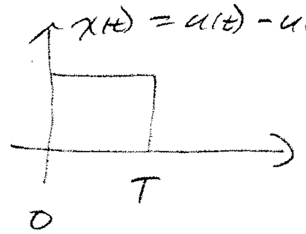
taking ratios $\quad 9 = e^{-(t_1 - t_9)/\tau} = e^{(t_9 - t_1)/\tau} = e^{t_r/\tau}$

$$\ln 9 = t_r/\tau \qquad \boxed{t_r = \tau \ln 9}$$

(#7 continued)

(c) up to you, but c should be small

(d) $x(t) = u(t) - u(t-T)$



for $x(t) = u(t)$ $\quad y(t) = [1 - e^{-t/\tau}]u(t)$

since LTI for $x(t) = u(t) - u(t-T)$, $\boxed{y(t) = [1 - e^{-t/\tau}]u(t) - [1 - e^{-(t-T)/\tau}]u(t-T)}$

(e) See plots

(f) For $x(t) = A[u(t) - u(t-T)]$ the output is

$$y(t) = A[1 - e^{-t/\tau}]u(t) - A[1 - e^{-(t-T)/\tau}]u(t-T)$$

$$y(T) = A[1 - e^{-T/\tau}]$$

For $T/\tau \ll 1$ $\quad e^{-T/\tau} \approx 1 - T/\tau$

$$y(T) \approx A[1 - (1 - T/\tau)] = \boxed{\frac{AT}{\tau} = y(T)}$$

```
%
% step response plot for homework 3 (prelab for impulse response lab)
%
t = linspace(0,0.008,10000);
tau = 0.001;
%
  T = 0.003;
  y1 = (1-exp(-t/tau)).*unit_step(t,0)-(1-exp(-(t-T)/tau)).*unit_step(t,T);
  T = 0.001;
  y2 = (1-exp(-t/tau)).*unit_step(t,0)-(1-exp(-(t-T)/tau)).*unit_step(t,T);
  T = 0.0001;
  y3 = (1-exp(-t/tau)).*unit_step(t,0)-(1-exp(-(t-T)/tau)).*unit_step(t,T);
  orient tall

  plot(t*1000,y1,':',t*1000,y2,'--',t*1000,y3,'-'); grid;
  legend('T=3 ms','T=1 ms','T=0.1 ms');
  xlabel('Time (ms)'); title('Step Responses for Problem 8e');

  figure;
  orient tall
  subplot(3,1,1); plot(t*1000,y1); grid; ylabel('T = 3 ms'); xlabel('Time (ms)');
  title('Results for Problem 8e');
  subplot(3,1,2); plot(t*1000,y2); grid; ylabel('T = 1 ms'); xlabel('Time (ms)');
  subplot(3,1,3); plot(t*1000,y3); grid; ylabel('T = 0.1 ms'); xlabel('Time (ms)');
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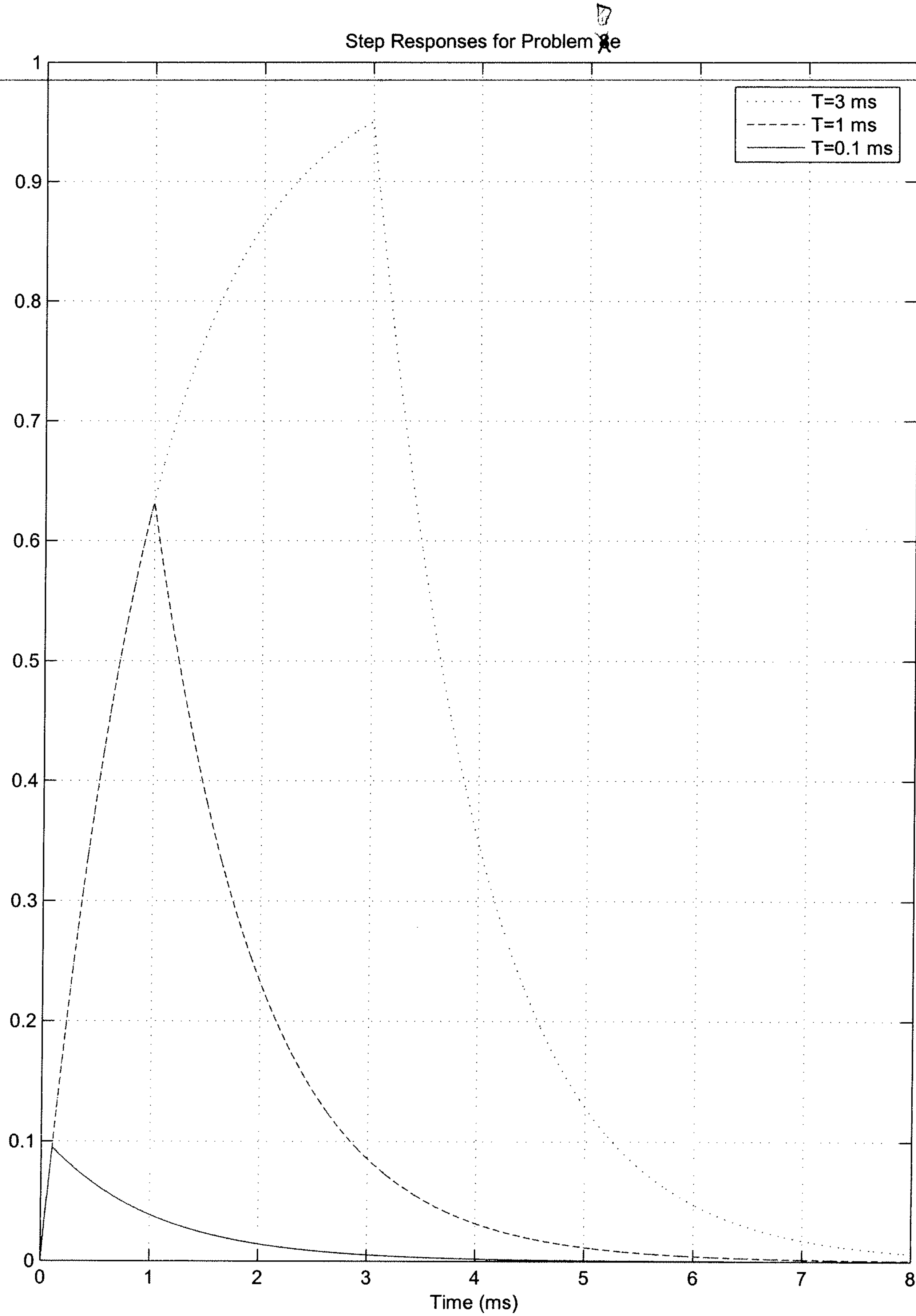
Step Responses for Problem ~~8~~7e
T=3 ms
T=1 ms
T=0.1 ms
Time (ms)
0
1
2
3
4
5
6
7
8
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9