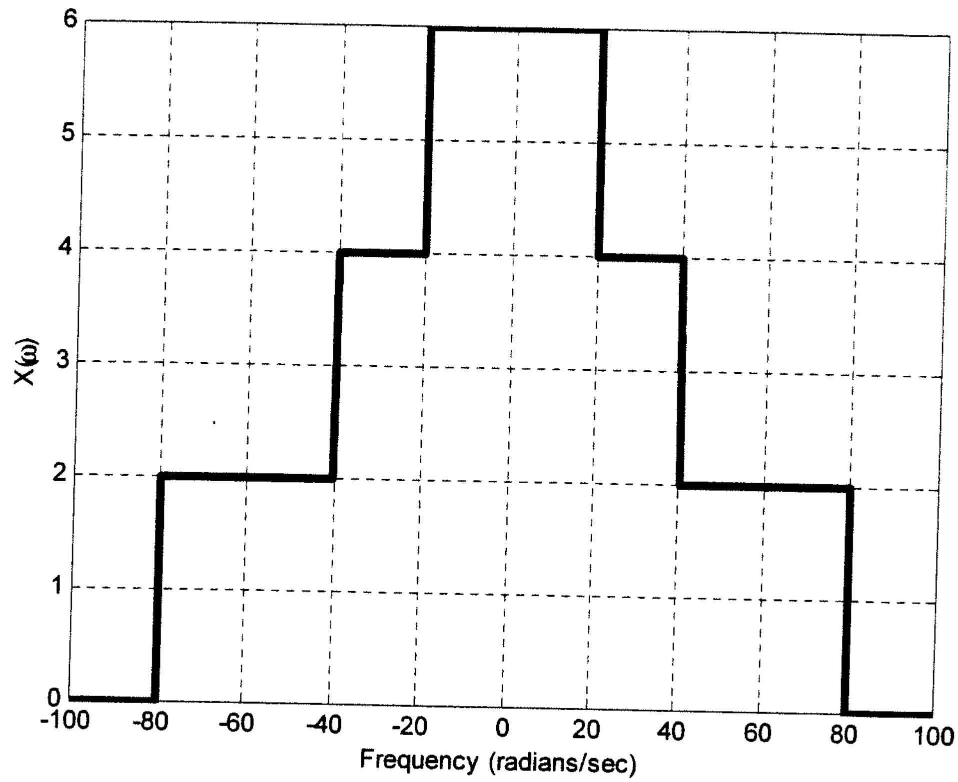
Name Exam 3 Solns CM

1. **(25 points) Finding the energy in a given bandwidth**

For the signal $x(t)$ with spectrum shown below:



Determine the percentage of the total energy in $x(t)$ between 20 and 60 radians/sec.

$$E_T = \frac{1}{2\pi}\left[\int_{-80}^{-40} 2^2 d\omega + \int_{-40}^{-20} 4^2 d\omega + \int_{-20}^{20} 6^2 d\omega + \int_{20}^{40} 4^2 d\omega + \int_{40}^{80} 2^2 d\omega\right]$$

$$= \frac{1}{2\pi}\left[4\cdot 40 + 16\cdot 20 + 36\cdot 40 + 16\cdot 20 + 4\cdot 40\right] = \frac{2400}{2\pi}$$

$$E_{20,60} = \frac{1}{2\pi}\left[\int_{20}^{40} 4^2 d\omega + \int_{40}^{60} 2^2 d\omega\right] 2 = \frac{2}{2\pi}\left[16\cdot 20 + 4\cdot 20\right] = \frac{800}{2\pi}$$

$$\text{Fraction} = 1/3 = 33.3\%$$

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2. (30 points) System analysis with the Fourier Transform

Consider a linear time invariant system with impulse response given by

$$h(t) = \frac{3}{2\pi}\operatorname{sinc}\left(\frac{t-2}{2\pi}\right)$$

with input

$$x(t) = \frac{2}{\pi}\operatorname{sinc}^2\left(\frac{t-3}{\pi}\right)\cos(t-3)$$

The output of the system is $y(t)$. Show all of your work and draw a ***BOX*** around your final answer.

a) Determine $X(\omega)$.

b) Sketch the spectrum of $X(\omega)$ (magnitude and phase) accurately labeling the axes and important points.

c) Determine $H(\omega)$.

d) Sketch the spectrum of $H(\omega)$ (magnitude and phase) accurately labeling the axes and important points.

e) Determine $y(t)$, the output of the system.

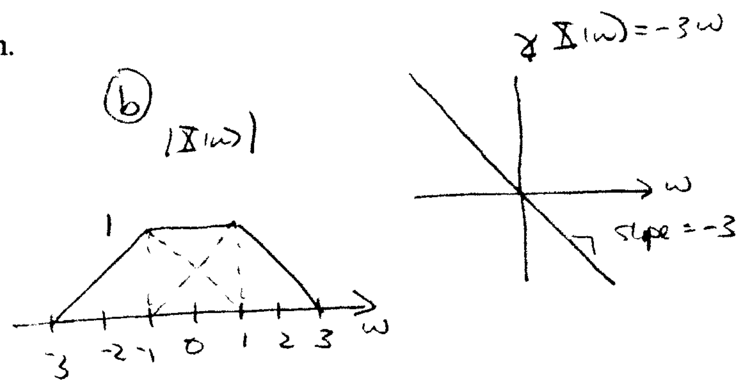
(a) For $x_1(t) = \operatorname{sinc}^2\left(\frac{t}{\pi}\right)$ $X_1(\omega) = \pi\Lambda\left(\frac{\omega}{4}\right)$

$B = 1/\pi$

$x_2(t) = \frac{2}{\pi}x_1(t)$ $X_2(\omega) = 2\Lambda\left(\frac{\omega}{4}\right)$

$$\boxed{X(\omega) = \left[\Lambda\left(\frac{\omega+1}{4}\right) + \Lambda\left(\frac{\omega-1}{4}\right)\right]e^{-j3\omega}}$$

(b)

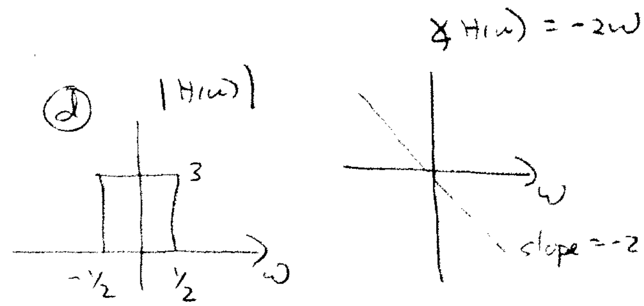


(c) For $h_1(t) = \frac{1}{2\pi}\operatorname{sinc}\left(\frac{t}{2\pi}\right)$ $H_1(\omega) = \frac{1}{2\pi}\cdot 2\pi \operatorname{rect}\left(\frac{\omega}{1}\right)$

$= \operatorname{rect}\left(\frac{\omega}{1}\right)$

$$\boxed{H(\omega) = 3\operatorname{rect}\left(\frac{\omega}{1}\right)e^{-j2\omega}}$$

(d)



(e) $Y(\omega) = H(\omega)X(\omega) = 3\operatorname{rect}\left(\frac{\omega}{1}\right)e^{-j5\omega}$

$$\Rightarrow \boxed{y(t) = \frac{3}{2\pi}\operatorname{sinc}\left(\frac{t-5}{2\pi}\right)}$$

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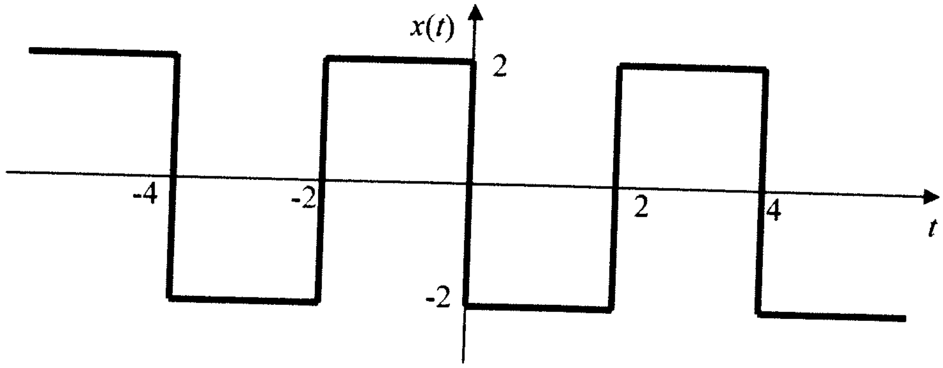
3. (**25 points**) **Fourier Series of a Periodic Signal**
The following set of questions refer to the signal below



(a) What is the fundamental frequency of x(t) in (rad/s)?

(b) Find an expression for the Fourier Series Coefficients, c_k, of x(t). Simplify your answer as much as possible.

(a) $T_0 = 4\ sec \quad \omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{4} = \frac{\pi}{2}$

(b) $c_K = \frac{1}{4}\left[\int_{-2}^{0} 2e^{-jK\omega_0 t}\,dt + \int_{0}^{2} -2\, e^{-jK\omega_0 t}\,dt\right]$

$= \frac{1}{2}\left[\left.\frac{e^{-jK\omega_0 t}}{-jK\omega_0}\right|_{-2}^{0} + \left.\frac{e^{-jK\omega_0 t}}{jK\omega_0}\right|_{0}^{2}\right] = \frac{1}{2}\left[\frac{e^{jK\omega_0 2} - 1 + e^{-jK\omega_0 2} - 1}{jK\omega_0}\right]$

$= \frac{1}{2}\left[\frac{e^{jK\omega_0 2} + e^{-jK\omega_0 2} - 2}{jK\omega_0}\right] = \frac{1}{2}\left[\frac{2\cos(K\omega_0 2) - 2}{jK\omega_0}\right]$

$= \frac{1}{jK\frac{\pi}{2}}\left[\cos(K\pi) - 1\right] = \boxed{\frac{2}{jK\pi}\left[(-1)^K - 1\right] = c_K}$

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4. **(20 points) Properties of the Fourier Transform**

Show that if a signal, x(t), is real and even, then the Fourier Transform of the signal, X(ω), is also real and even.

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} x(t)\cos(\omega t)dt - j\underbrace{\int_{-\infty}^{\infty} \overbrace{x(t)}^{\text{even}} \overbrace{\sin(\omega t)}^{\text{odd}} dt}_{\text{integrating over} \pm T}$$

$= \text{odd}$ zero

$$= \int_{-\infty}^{\infty} x(t)\cos(\omega t)dt$$

$$X(-\omega) = \int_{-\infty}^{\infty} x(t)\cos(-\omega t)dt = \int_{-\infty}^{\infty} x(t)\cos(\omega t)dt = X(\omega)$$

so $X(\omega)$ is real and even