ECE 300
Signals and Systems
Homework 6

Due Date: Tuesday October 10 at 2:30 PM   **Exam 2, Thursday October 19**

Problems:

1. A periodic signal $x(t)$ is the input to an LTI system with output $y(t)$. The signal $x(t)$ has period 2 seconds, and is given over one period as

$$x(t) = e^{-t} \quad 0 < t < 2$$

$x(t)$ has the Fourier series representation

$$x(t) = \sum_k \frac{0.4323}{1 + jk\pi} e^{jk\pi t}$$

The system is an ideal lowpass filter that eliminates all signals with frequency content higher than 1.25 Hz.

a) Find the average power in $x(t)$.

b) Determine an expression for the output, $y(t)$. Your expression for $y(t)$ must be real.

c) Determine the average power in $y(t)$.

d) Plot the spectrum (magnitude and phase) for $x(t)$. Include the DC through second harmonic. Accurately label your plot.

2. Assume $x(t) = t^2 \quad -\pi \le t \le \pi$ with Fourier Series representation

$$x(t) = \sum_k a_k e^{jkt}$$

where

$$a_k = \begin{cases} \dfrac{\pi^2}{3} & k = 0 \\ \dfrac{2(-1)^k}{k^2} & k \ne 0 \end{cases}$$

a) Assume $x(t)$ is the input to a system that eliminates all signals with frequencies outside the range 0.5 to 0.7 Hz. What is the output of the system $y(t)$ and what fraction of the average power in $x(t)$ is in $y(t)$? (Note: your answers must be real, no e^{ja} terms.)

b) Assume $x(t)$ is the input to a system that eliminates all signals with frequencies in the range 0.5 to 0.7 Hz. What is the output of the system $y(t)$ and what fraction of the average power in $x(t)$ is in $y(t)$?

3. K & H, Problem 5.1. Use the example we did in class to get the Fourier series coefficients for part c.

4. K & H, Problem 5.3 (very easy)

5. K & H, Problem 5.12. Note that $y(t) = x(t) - x(t-1)$. You need to write c_k^y in terms of c_k^x.

6. K & H, Problme 5.13 (very easy)

7. The output of a LTI system, $y(t)$, has the following spectrum shown on the left, while the system transfer function, $H(k\omega_o)$, has the spectrum shown on the right. Assume all angles are multiples of 45 degrees.
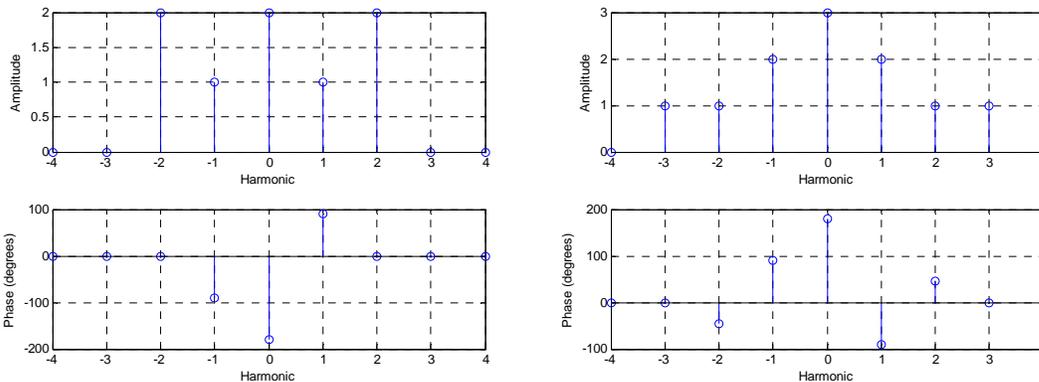
a) Determine (sketch) the spectrum (magnitude and phase) of the input to the system, $x(t)$.

b) If $x(t)$ has the fundamental period $T = 2$ seconds, determine an analytical expression for $x(t)$ in terms of sine, cosines, and constants.

8. **Pre-Lab/Matlab Exercises (to be done by all students. Turn this in with your homework and bring a copy of this with you to lab! Yes, it counts towards both grades!)**

Note: The function we use for numerical integration, **quadl**, has a default tolerance. This tolerance level is used so the function can determine when its estimate of the integral is "good enough". When dealing with either small valued functions, or very short intervals, you may need to change this tolerance. In the following problems, you will need to change the tolerance to get the proper results (use help **quadl**).

Read the **Appendix** and then do the following:

a) Plot the Fourier series representation and the single sided power spectrum in dBmV for each of the waveforms listed below. Use 9 terms for each function. It is probably easiest to use your program **Complex_Fourier_Series.m**. You should use the **log10** command to compute the base 10 logarithm. You should also use the **subplot** command so you can plot both a signal and its Fourier series representation in one graph, and its single-sided power spectrum in a subsequent graph within the same window. Utilize the **stem** command in Matlab to do the plotting. For the power spectrum plot, the y-axis should be labeled *Power(dBmV)*, the x-axis labeled *Harmonic* .Your plots for function $x_2(t)$ should look like those in Figure 1. (Note the axis on the top graph is limited using the command **axis('tight')**, and is limited on the second plot using the command **axis([0 N 0 40])**;)
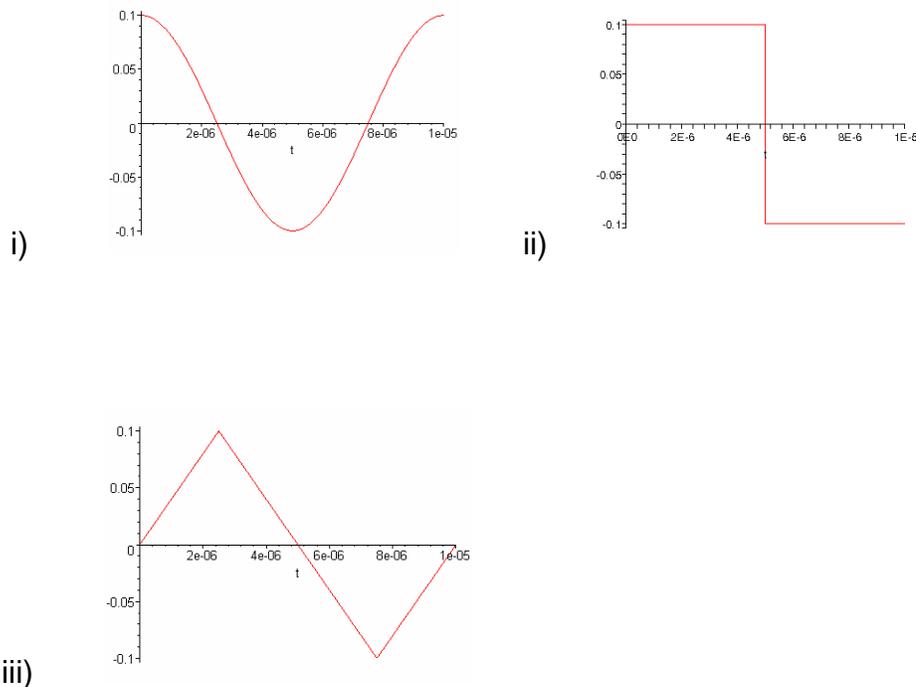
The waveforms, each having zero DC offset, are:

i) $x_1(t) = 0.1\cos\left(2\pi 100 \times 10^3 t\right)$ V

II) $x_2(t)$ is a square wave of period 10 μs and peak-to-peak amplitude 0.2 V.

III) $x_3(t)$ is a triangle wave of period 10 μs and peak-to-peak amplitude 0.2 V.

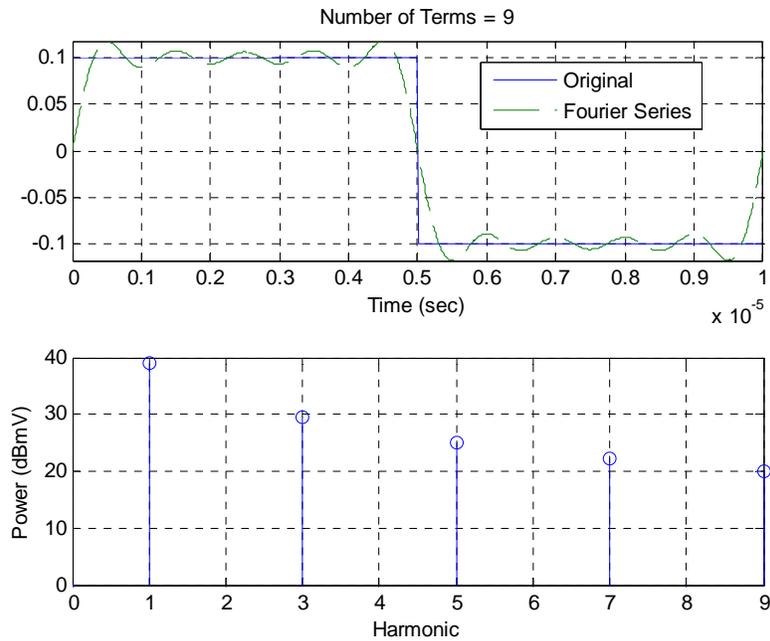Just to be sure there is no confusion regarding the waveforms, they are displayed below.

i)

ii)

iii)

Figure 1: Results for problem 8a for function $x_2(t)$.

b) For each waveform, create a table containing a column of values of c_k and a column of values of predicted decibel levels (dBmV). Do not read these from your plots, but get them directly from you program. Note that if you do not put a semicolon after a statement in Matlab, the values in the array will be printed to the screen.

c) If the periodic signal is a pulse train, we know that the magnitude of the coefficients will be given by

$$|c_k| = A\frac{\tau}{T}\left|\text{sinc}\left(k\frac{\tau}{T}\right)\right|$$

where A is the amplitude of the pulse, τ is the duration of the pulse, and T is the period. The ratio $\frac{\tau}{T}$ is the *duty cycle* of the pulse. Using this expression, and our knowledge of the properties of the sinc function, we can determine that the c_k will be zero whenever $k\frac{\tau}{T} = 1,2,3,...$ or whenever $k = \frac{T}{\tau}, 2\frac{T}{\tau}, 3\frac{T}{\tau},...$ These zeros, or *nulls*, will occur at frequencies $k\omega_o = k\frac{2\pi}{T} = \frac{2\pi}{\tau}, \frac{4\pi}{\tau}, \frac{6\pi}{\tau},...$

In order to help see what is happening, in the following three problems we will fix the axes so they always plot over the same range of frequencies. Use the **axis** command to limit the frequencies from 0 to 2×10^7 radians/sec and the power from 0 to 40 dBmV.

i) We will first look at the case of a fixed length period with a changing pulse width. We will assume the pulse has an amplitude of 0.2 V with a fixed period of $T = 10\,\mu s$. Mathematically our pulse is defined as

$$x(t) = \begin{cases} 0.2 & 0 \le t \le \tau \\ 0 & \tau < t < T \end{cases}$$

Plot the Fourier series representation of $x(t)$ and the single sided power spectrum for $\tau = 2\,\mu s$, $\tau = 1\,\mu s$, and $\tau = 0.5\,\mu s$. Figure 2 shows how your plot should look (to get the τ to display, use \tau in the title). Use 40 terms in the Fourier series representation. Do the locations of the nulls occur where you expect them to occur? Does the spacing between spectral components change as the pulse width (τ) changes?
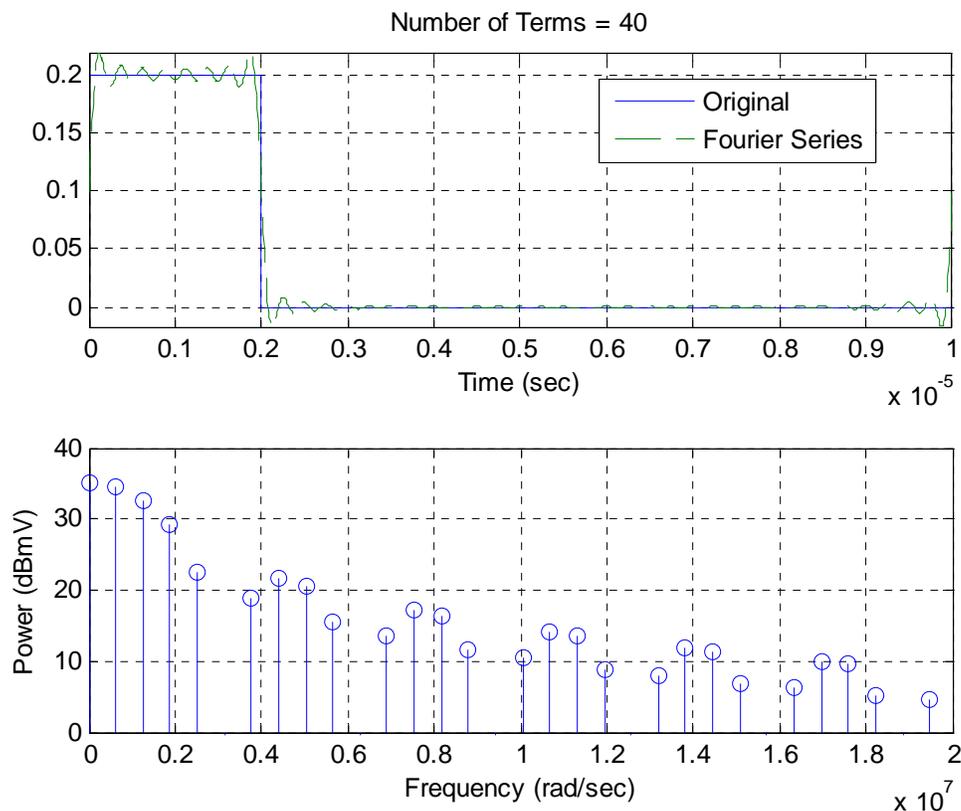

Figure 2: Plot for 8c-(i), Amplitude = 0.2V, $\tau = 2\,\mu s$, and $T = 10\,\mu s$.

ii) Now we will look at the case of a fixed pulse length with a varying period. We will assume the pulse has an amplitude of 0.2 V with a fixed pulse width of $\tau = 2\,\mu s$. Plot the Fourier series representation of $x(t)$ and the single sided power spectrum for $T = 10\,\mu s$, $T = 20\,\mu s$, and $T = 40\,\mu s$. Use 120 terms in the Fourier series representation. Do the locations of the nulls occur where you expect them to occur? Does the spacing between spectral components change as the period (T) changes?

iii) Now we will look at the case of a fixed duty cycle with a varying period and pulse width. We will assume the pulse has an amplitude of 0.2 V and a fixed duty cycle of 20%. Plot the Fourier series representation of $x(t)$ and the single sided power spectrum for ($\tau = 1\mu s, T = 5\mu s$), ($\tau = 2\mu s, T = 10\mu s$), and ($\tau = 4\mu s, T = 20\mu s$). Use 120 terms in the Fourier series representation. Do the locations of the nulls occur where you expect them to occur? Does the spacing between spectral components change as the period (T) changes?

Appendix

There are a number of different ways of graphically presenting the information in the Fourier series representation of a periodic function. The most common methods of presenting this information are by plotting the spectrum of the signal, or the power spectrum of the signal. In the following examples we will use the function

$$f(t) = \begin{cases} 0 & -2 \le t < -1 \\ t & -1 \le t < 1 \\ 1 & 1 \le t < 2 \end{cases}$$

Spectrum of a Periodic Signal The spectrum of a periodic signal is a plot of the magnitude $|c_k|$ against the corresponding frequency $k\omega_o$, and the phase $\measuredangle c_k$ against the corresponding frequency. Since our signal is periodic we only have values at *discrete* frequencies ($k\omega_o$), we do not know what happens in between these discrete frequencies so we <u>***do not***</u> "connect the dots". Since ω_o is common to all of the discrete frequencies, we often plot against the *harmonic* k, rather than the frequency $k\omega_o$. Figure 3 shows the spectrum of $f(t)$.

Power Spectrum of a Periodic Signal The power spectrum of a signal tells us how the power in the signal is distributed in frequency. To plot the power spectrum of a signal, we consider each term of the complex Fourier series, $c_k e^{jk\omega_o t}$. The average power in this term is $|c_k|^2$. The power spectrum indicates the power associated with each frequency, and is thus a plot of $|c_k|^2$ against frequency $k\omega_o$ or just the harmonic *k*. In this plot we need to include both positive and negative frequencies. Figure 4 shows the power spectrum of $f(t)$.

Single-Sided Power Spectrum of a Periodic Signal Because the magnitudes of the coefficients are even ($|c_k| = |c_{-k}|$), the powers associated with negative frequencies are the same as those associated with positive frequencies. Since only positive frequencies are realizable, we often just want to plot the power against positive frequencies. This gives us a single sided power spectrum, which is a plot of

$|c_0|^2 \quad 2|c_1|^2 \quad 2|c_2|^2 \quad ... \quad 2|c_N|^2$ versus the corresponding frequency $0 \quad \omega_0 \quad 2\omega_0 \quad ... \quad N\omega_0$. Since the fundamental frequency ω_0 is common to all of the frequency terms, we often just plot against the harmonics $0 \quad 1 \quad 2 \quad ... \quad N$. Figure 5 shows the single sided power spectrum of $f(t)$.

Spectrum Analyzer (SA) Display The SA displays a one-sided spectrum, but instead of showing the value of $2|c_k|^2$ at each frequency, the spectrum analyzer shows average power in decibels with respect to a one millivolt RMS reference. For the sinusoid at frequency kf_0, the average power in decibels is given by

$$P_k\big|_{dB} = 10\log_{10}\frac{P_k}{P_{ref}},$$

where the power P_k represents the power spectrum coefficient $2|c_k|^2$, and the power P_{ref} is the average power delivered to a one-ohm resistor by a one millivolt RMS sinusoid. We have

$$P_k\big|_{dBmV} = 10\log_{10}\left(\frac{2|c_k|^2}{(0.001)^2}\right) \; \mathrm{dBmV} \quad .$$

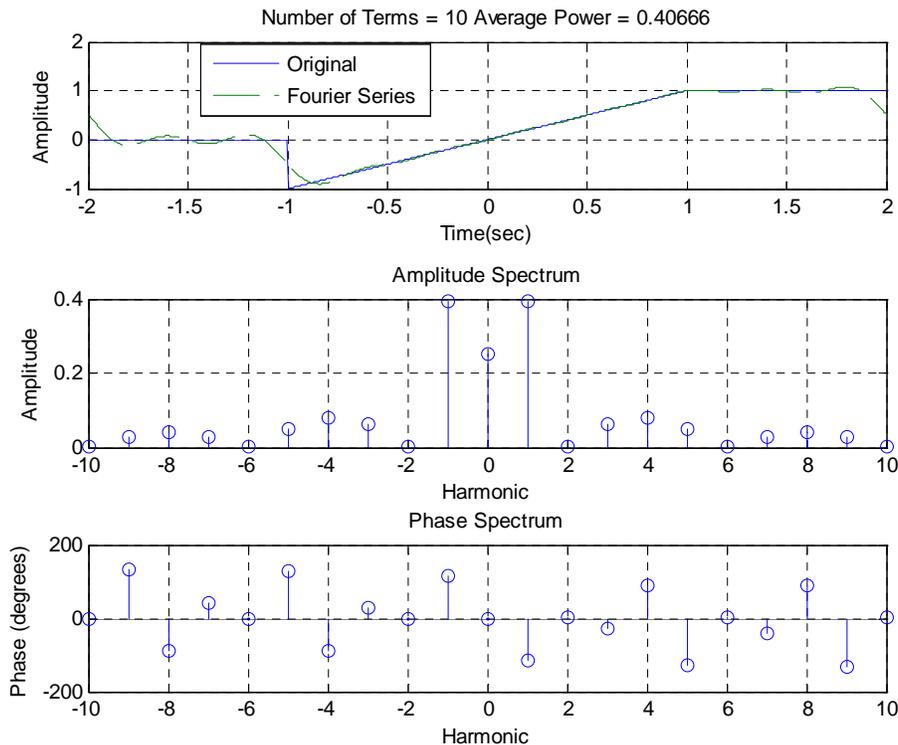The units "dBmV" indicate that the reference for the decibels is a one millivolt RMSsinusoid.
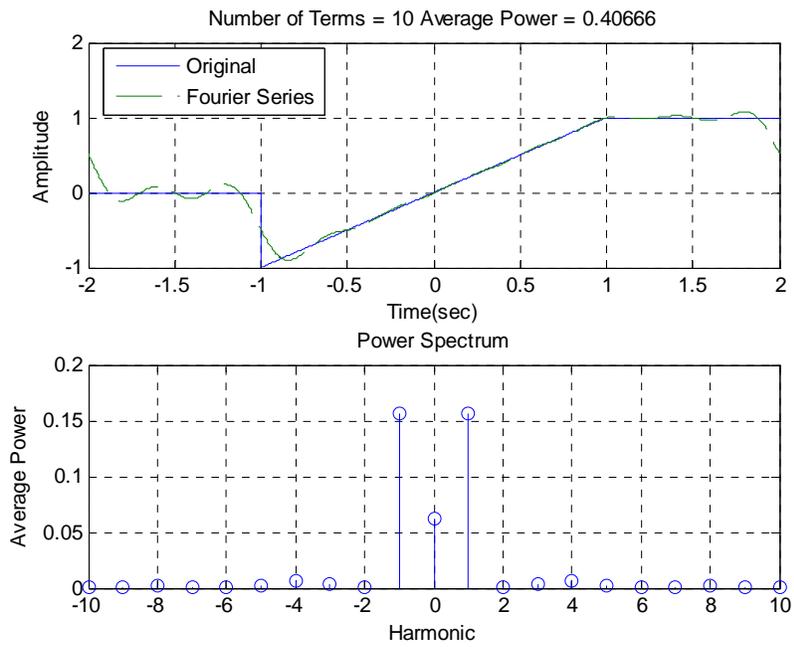
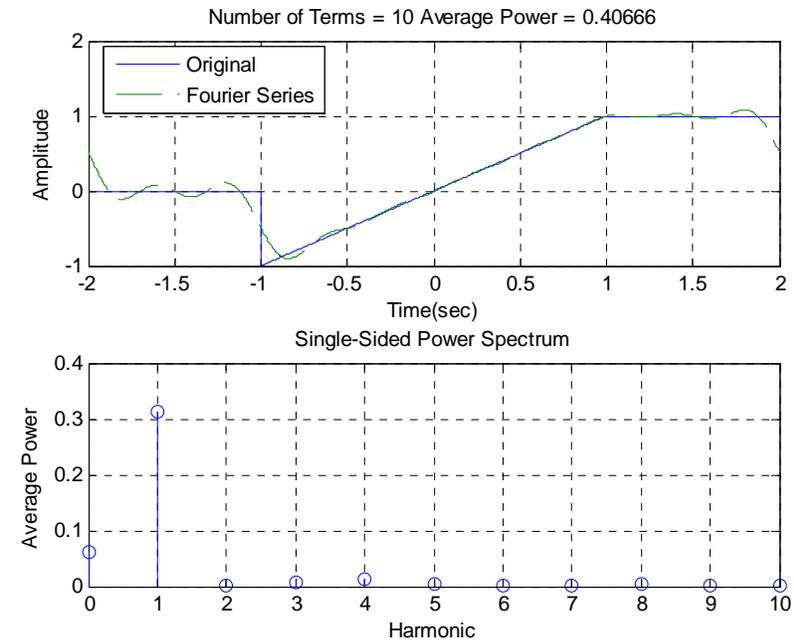Figure 3: The spectrum of $f(t)$.

Figure 4: The power spectrum of $f(t)$.

Figure 5: The single-sided power spectrum of $f(t)$.