Differential Equation Review

In this course you will be expected to be able to solve two types of first order differential equations without Maple, just as you are expected to be able to do basic calculus without Maple. The first type of differential equation to be reviewed is a ***separable*** differential equation, and the second is a differential equation that can be solved using an ***integrating factor***. Once you understand how to deal with these simple types of equations, and how they affect system properties, generalizing to other types of differential equations will not be difficult.

Separable Equations A separable differential equation is generally one where we can rewrite the equation so all of the variables on one side of the equation are the same. In what follows we will assume the system starts at time t_o and ends at time *t.*

Example 1: Consider the separable differential equation $\dot{x}(t) = t^2$. We first rewrite the derivative as $\dot{x}(t) = \frac{dx}{dt}$, so we have $\frac{dx}{dt} = t^2$. Next we put all of the x's on one side of the equation, and all of the t's on the other side of the equation, so we have $dx = t^2 dt$. Now we want to integrate both sides of the equation. At time t_o, x has value $x(t_o)$, while at time t, x has value $x(t)$. Hence we have

$$\int_{x(t_o)}^{x(t)} dx = \int_{t_o}^{t} \lambda^2 d\lambda$$

Note that we are using a dummy variable of integration (λ) so there is no confusion with the end points of the definite integral. Specifically, if we were to use t as the dummy variable of integration and as one of the endpoints of the integral we would most likely make mistakes.

Performing the integration we get

$$x(t) - x(t_o) = \frac{t^3}{3} - \frac{t_o^3}{3}$$

Finally we get the solution

$$x(t) = x(t_o) + \frac{t^3}{3} - \frac{t_o^3}{3}$$

Example 2: Consider the separable differential equation $\dot{y}(t) = -2t\, y(t)$. We first rewrite the derivative as $\dot{y}(t) = \frac{dy}{dt}$, so we have $\frac{dy}{dt} = -2ty(t)$. Next we put all of the y's on one side of the equation, and all of the t's on the other side of the equation, so we have $\frac{dy}{y} = -2tdt$. Now we want to integrate both sides of the equation. At time t_o, y has value $y(t_o)$, while at time t, y has value $y(t)$. Hence we have

$$\int_{y(t_o)}^{y(t)} \frac{dy}{y} = \int_{t_o}^{t} -2\lambda d\lambda$$

or

$$\ln[y(t)] - \ln[y(t_o)] = \ln\left[\frac{y(t)}{y(t_o)}\right] = -t^2 + t_o^2 = -(t^2 - t_o^2)$$

Finally we get the solution

$$y(t) = y(t_o)e^{-(t^2 - t_o^2)}$$

Example 3: Consider the separable differential equation $\dot{y}(t) = 3\sqrt{y(t)}$. We first rewrite the derivative as $\dot{y}(t) = \frac{dy}{dt}$, so we have $\frac{dy}{dt} = 3\sqrt{y(t)}$. Next we put all of the y's on one side of the equation, and all of the t's on the other side of the equation, so we have $\frac{dy}{\sqrt{y}} = 3dt$. Now we want to integrate both sides of the equation. At time t_o, y has value $y(t_o)$, while at time t, y has value $y(t)$. Hence we have

$$\int_{y(t_o)}^{y(t)} \frac{dy}{\sqrt{y}} = \int_{t_o}^{t} 3d\lambda$$

or

$$2\sqrt{y(t)} - 2\sqrt{y(t_o)} = 3(t - t_o)$$

Finally we get the solution

$$y(t) = \left[\sqrt{y(t_o)} + \frac{3}{2}(t - t_o)\right]^2$$

Differential Equations with Integrating Factors An integrating factor allows us to write one half of a first order differential equation as an exact derivative (something easy to integrate), and the other part as a function with no derivatives. In what follows we will assume the system starts at time t_o and ends at time *t.* In the first example we go over all of the details, but in the final two examples we just use results from **Example 4**.

<u>Example 4:</u> Consider the differential equation $\dot{y}(t) - y(t) = 2$. We first rewrite the derivative as $\dot{y}(t) = \frac{dy}{dt}$, so we have $\frac{dy}{dt} - y(t) = 2$. Next we want to write the left hand side of the equation as $\frac{d}{dt}\left[y(t)e^{a(t)}\right]$. Using basic properties from calculus we have

$$\frac{d}{dt}\left[y(t)e^{a(t)}\right] = e^{a(t)}\frac{dy(t)}{dt} + \frac{da(t)}{dt}e^{a(t)}y(t) = e^{a(t)}\left[\frac{dy(t)}{dt} + \frac{da(t)}{dt}y(t)\right]$$

We want the term in the brackets to look like our original equation, that is, we want

$$\left[\frac{dy(t)}{dt} + \frac{da(t)}{dt}y(t)\right] = \frac{dy(t)}{dt} - y(t)$$

Equating the two sides we get

$$\frac{da(t)}{dt} = -1$$

which gives us

$$a(t) = -t$$

At this point we have

$$\frac{d}{dt}[y(t)e^{-t}] = e^{-t}\left[\frac{dy(t)}{dt} - y(t)\right]$$

Now since we have (from our original differential equation)

$$\frac{dy}{dt} - y(t) = 2$$

we can multiple both sides of this equation by e^{-t} to get

$$e^{-t}\left[\frac{dy(t)}{dt} - y(t)\right] = e^{-t}[2]$$

or

$$e^{-t}\left[\frac{dy(t)}{dt}-y(t)\right]=\frac{d}{dt}\left[y(t)e^{-t}\right]=e^{-t}[2]$$

At this point we have the left hand side as an exact derivative

$$\frac{d}{dt}\left[y(t)e^{-t}\right]=2e^{-t}$$

Now we want to integrate both sides of the equation

$$\int_{t_o}^{t}\frac{d}{dt}\left[y(t)e^{-t}\right]dt=\int_{t_o}^{t}2e^{-\lambda}d\lambda$$

Integrating we have

$$y(t)e^{-t}-y(t_o)e^{-t_o}=-2e^{-t}+2e^{-t_0}$$

Finally we get the solution

$$y(t)=y(t_o)e^{(t-t_o)}-2+2e^{(t-t_o)}$$

Note: *We can also solve this equation in the same way we solved the separable equation, by going through the following steps:*

$$\frac{dy}{2+y}=dt$$

$$\int_{y(t_o)}^{y(t)}\frac{dy}{2+y}=\int_{t_o}^{t}d\lambda$$

$$\ln\left[2+y(t)\right]-\ln\left[2+y(t_o)\right]=\ln\left[\frac{2+y(t)}{2+y(t_o)}\right]=t-t_o$$

$$\frac{2+y(t)}{2+y(t_o)}=e^{(t-t_o)}$$

$$2+y(t)=\left[2+y(t_o)\right]e^{(t-t_o)}=y(t_o)e^{(t-t_o)}+2e^{(t-t_o)}$$

$$y(t)=y(t_o)e^{(t-t_o)}-2+2e^{(t-t_o)}$$

Example 5: Consider the differential equation $\dot{y}(t)-2ty(t)=x(t)$. From **Example 4**, we need $\frac{da(t)}{dt}=-2t$, or $a(t)=-t^2$. We then have $\frac{d}{dt}\left[y(t)e^{-t^2}\right]=e^{-t^2}x(t)$.
Integrating both sides we have

$$\int_{t_o}^{t} \frac{d}{dt}\left[y(t)e^{-t^2} \right] dt = \int_{t_o}^{t} x(\lambda)e^{-\lambda^2} d\lambda$$

or

$$y(t)e^{-t^2} - y(t_o)e^{-t_o^2} = \int_{t_o}^{t} x(\lambda)e^{-\lambda^2} d\lambda$$

Finally we have the solution

$$y(t) = y(t_o)e^{(t^2 - t_o^2)} + \int_{t_o}^{t} x(\lambda)e^{t^2 - \lambda^2} d\lambda$$

Note that we cannot go any further in the solution until we know $x(t)$.

Example 6: Consider the differential equation $\dot{y}(t) + \frac{3}{2}\sqrt{t}\,y(t) = e^t x(t)$. From **Example 4**, we need $\frac{da(t)}{dt} = \frac{3}{2}\sqrt{t}$, or. $a(t) = t^{\frac{3}{2}}$ We then have $\frac{d}{dt}\left[y(t)e^{t^{\frac{3}{2}}} \right] = e^{t^{\frac{3}{2}}} e^t x(t)$.
Integrating both sides we have

$$\int_{t_o}^{t} \frac{d}{dt}\left[y(t)e^{t^{\frac{3}{2}}} \right] dt = \int_{t_o}^{t} x(\lambda)e^{\lambda} e^{\lambda^{\frac{3}{2}}} d\lambda$$

or

$$y(t)e^{t^{\frac{3}{2}}} - y(t_o)e^{t_o^{\frac{3}{2}}} = \int_{t_o}^{t} x(\lambda)e^{\lambda} e^{\lambda^{\frac{3}{2}}} d\lambda$$

Finally we have the solution

$$y(t) = y(t_o)e^{-(t^{\frac{3}{2}} - t_o^{\frac{3}{2}})} + \int_{t_o}^{t} x(\lambda)e^{\lambda} e^{-(t^{\frac{3}{2}} - \lambda^{\frac{3}{2}})} d\lambda$$

Note that we cannot go any further in the solution until we know $x(t)$.