# ECE-205

# Exam 1

# Winter 2013

Calculators can only be used for simple calculations. Solving integrals, differential equations, systems of equations, etc. does not count as a simple calculation.

You must show your work to receive credit.

Problem 1 _____/11

Problem 2 _____/20

Problem 3 _____/20

Problem 4 _____/20

Problem 5 _____/20

Problems 6-8 _____/9

Total _____

**1) (11 points)** For a first order system described by the differential equation

$$\dot{y}(t) + 2y(t) = e^{-2t}x(t-1)$$

with $t_0 = 0$ and $y(t_0) = 1$, use integrating factors to solve the differential equation. Include the initial conditions in your solution.

$$\frac{d}{dt}\left(y(t)e^{\phi(t)}\right) = e^{\phi(t)}\left(\frac{dy(t)}{dt} + \frac{d\phi(t)}{dt}y(t)\right)$$

$$\text{let } \frac{d\phi(t)}{dt} = 2 \implies \phi(t) = 2t$$

$$\implies \frac{d}{dt}\left(y(t)e^{2t}\right) = e^{2t}\left(e^{-2t}x(t-1)\right) = x(t-1)$$

$$\implies \text{Integrating both sides:}$$

$$\int_0^t \frac{d}{d\lambda}\left(y(\lambda)e^{2\lambda}\right)d\lambda = \int_0^t x(\lambda-1)d\lambda$$

$$\implies y(t)e^{2t} - y(0)e^0 = \int_0^t x(\lambda-1)d\lambda$$
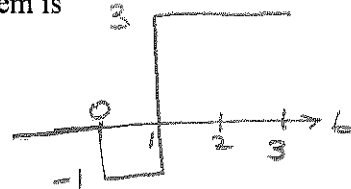
$$\implies y(t) = (1)e^{-2t} + e^{-2t}\int_0^t x(\lambda-1)d\lambda$$

**2) (20 points)** Assume we have a first order system with the governing differential equation

$$0.5\dot{y}(t) + y(t) = 2x(t)$$

The system has the initial value of 1, so $y(0) = 1$. The input to this system is

$$x(t) = \begin{cases} 0 & t < 0 \\ -1 & 0 \le t < 1 \\ 3 & t \ge 1 \end{cases}$$

Determine the output of the system in each of the above time intervals. *Simplify your final answer as much as possible and box it. Be sure to include the correct initial value in the first interval!*

$$y(t) = (y(t_0) - y(\infty)) e^{-(t-t_0)/\tau} + y(\infty)$$

Case 1: $0 \le t < 1$

$t_0 = 0$

$\tau = 1/2$

$A = -1$

$y(\infty) = 2(-1) = -2$

$\Rightarrow y(t) = (y(0) - y(\infty)) e^{-(t-0)/\tau} + y(\infty)$

$= (1 - (-2)) e^{-t/0.5} + (-2)$

$$\boxed{y(t) = 3e^{-2t} - 2} \quad \text{for } 0 \le t < 1$$

Case 2: $t \ge 1$

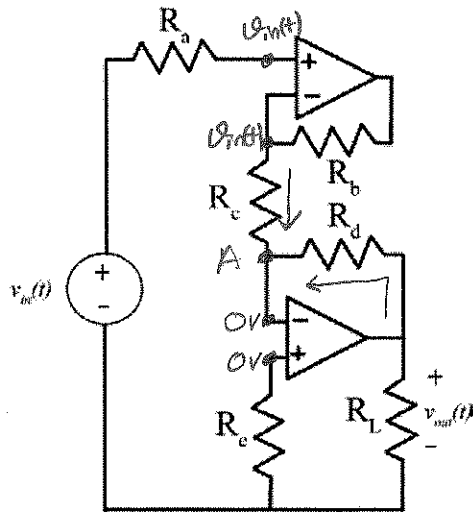$t_0 = 1$, $y(t_0) = y(1) = 3e^{-2(1)} - 2 \cong -1.594$

$\tau = 1/2$

$A = 3$

$y(\infty) = 2(3) = 6$

$\Rightarrow y(t) = (y(1) - y(\infty)) e^{-(t-1)/\tau} + y(\infty)$

$\cong (-1.594 - 6) e^{-2(t-1)} + 6$

$$\boxed{y(t) = -7.594 e^{-2(t-1)} + 6} \quad \text{for } t \ge 1$$

3

**3) (20 points)** For the following op-amp circuits we can write $v_{out}(t) = Gv_{in}(t)$. Determine an expression for $G$.
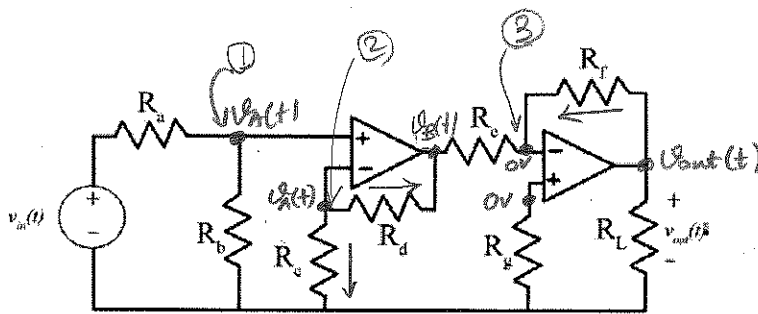


At node A:

$$\frac{v_{out}(t)}{R_d} + \frac{v_{in}(t)}{R_c} = 0$$

$$\Rightarrow v_{out} = -\frac{R_d}{R_c} v_{in}(t)$$

$$\Rightarrow \boxed{G = -R_d/R_c}$$



At node ①: We have a voltage divider, thus

① $v_A(t) = \dfrac{R_b}{R_a + R_b} v_{in}(t)$

At node ②: apply KCL

② $\dfrac{v_A(t)}{R_c} + \dfrac{v_A(t) - v_B(t)}{R_d} = 0$

At node ③: apply KCL

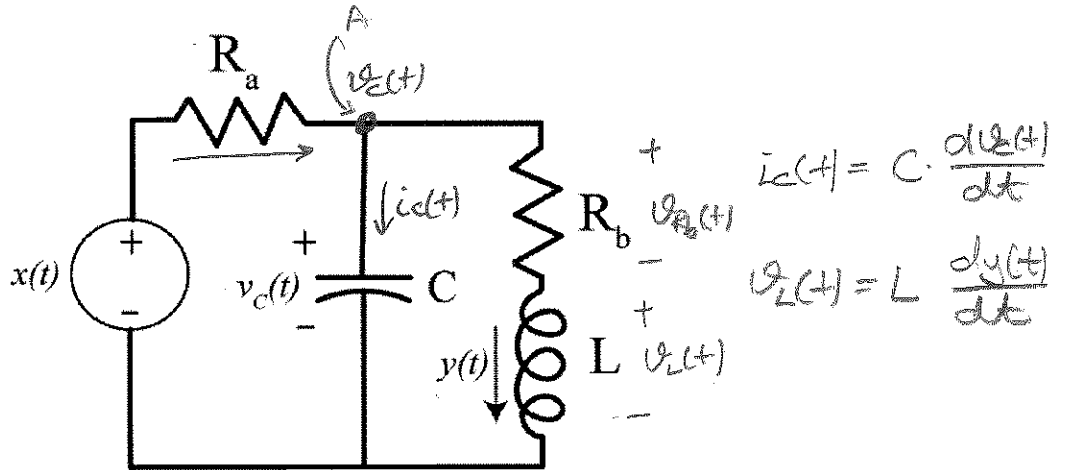③ $\dfrac{v_{out}(t)}{R_f} + \dfrac{v_B(t)}{R_e} = 0$

→ Plugging ① into ②, and rearranging:

④ $v_B(t) = \left(1 + \dfrac{R_d}{R_c}\right)\left(\dfrac{R_b}{R_a + R_b}\right) v_{in}(t)$

Plugging ④ into ③:

$v_{out}(t) = -\left(1 + \dfrac{R_d}{R_c}\right)\left(\dfrac{R_b}{R_a + R_b}\right)\left(\dfrac{R_f}{R_e}\right) v_{in}(t)$

$$\Rightarrow \boxed{G = -\left(\dfrac{R_b}{R_a + R_b}\right)\left(1 + \dfrac{R_d}{R_c}\right)\left(\dfrac{R_f}{R_e}\right)}$$

4

**4) (20 points)** Derive the governing differential equation for the following circuit. You do not need to put the differential equation into standard form.



- At node A: apply KCL

$$\frac{x(t) - v_c(t)}{R_a} = i_c(t) + y(t)$$

$$\Rightarrow \frac{x(t) - v_c(t)}{R_a} = C \cdot \frac{dv_c(t)}{dt} + y(t) \quad\text{------- ①}$$

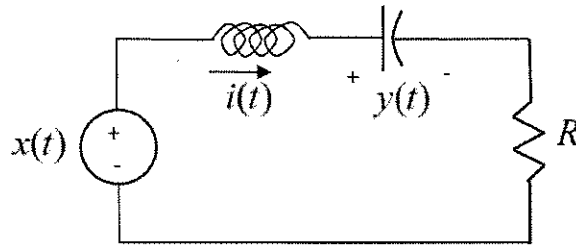- Apply KVL around the closed loop of $R_b, L, C$

$$v_L(t) + v_{R_b}(t) = v_c(t)$$

$$\Rightarrow L \cdot \frac{dy(t)}{dt} + y(t) R_b = v_c(t) \quad\text{------- ②}$$

- Plugging ② into ①, and rearranging

$$\Rightarrow \frac{x(t)}{R_a} - \frac{1}{R_a}\left(L\frac{dy(t)}{dt} + y(t)R_b\right) = LC\frac{d^2y(t)}{dt^2} + R_bC\frac{dy(t)}{dt} + y(t)$$

$$\Rightarrow \boxed{R_a L C \frac{d^2y(t)}{dt^2} + \left(R_a R_b C + L\right)\frac{dy(t)}{dt} + \left(R_a + R_b\right)y(t) = x(t)}$$

**5) (20 points)** Consider the circuit shown in the figure below:



The input $x(t)$ is 1 V for $t > 0$ and the output, $y(t)$, is the voltage across the capacitor. For this circuit, the inductor has $L = 33$ mH, the capacitor has $C = 0.01$ µF, and the resistor has $R = 6.6$ kΩ . Assume that the circuit is initially at rest (i.e., $y(0) = \dot{y}(0) = 0$). In this problem use the approximations $\dfrac{1}{LC} \approx 3.0 \times 10^9$

and $\dfrac{R}{L} \approx 2 \times 10^5$

Recall that the governing second order differential equation of the above circuit is given by

$$\frac{d^2 y(t)}{dt^2} + \frac{R}{L}\frac{dy(t)}{dt} + \frac{1}{LC}y(t) = \frac{1}{LC}x(t)$$

Solve for $y(t)$ and put your final answer in a box. You must show all your work to receive credit.

— Forced response : $y_f(t)$

  $y_f(t) = KA = (1)(1) = 1$

— Natural response : $y_n(t)$

  $r^2 + \frac{R}{L}r + \frac{1}{LC} = 0$

  $r = \dfrac{-2 \times 10^5 \pm \sqrt{(2 \times 10^5)^2 - 4(3 \times 10^9)}}{2}$  ⟹ $r_1 \cong -16334, \ r_2 \cong -183666$

  (two real distinct roots)

  ⟹ $y_n(t) = C_1 e^{-16334t} + C_2 e^{-183666t}$

— Overall response : $y(t)$

  $y(t) = 1 + C_1 e^{-16334t} + C_2 e^{-183666t}$

— Initial Conditions

  $y(0) = 0 = 1 + C_1 + C_2$  ⟹ $C_1 = -1 - C_2$

  $\dot{y}(0) = -16334 C_1 - 183666 C_2 = 0$

  $= +16334 + 16334 C_2 - 183666 C_2$

  ⟹ $C_1 = -1.0976$

  $C_2 = 0.0976$

  ∴ $\boxed{y(t) = 1 - 1.0976 e^{-16334t} + 0.0976 e^{-183666t}}$  for $t > 0$

6

Name _____ Mailbox _____

**Problems 6-8, 3 points each, no partial credit (9 points)**

Problems 6-8 refer the following graph showing the response of a second order system to a step input.
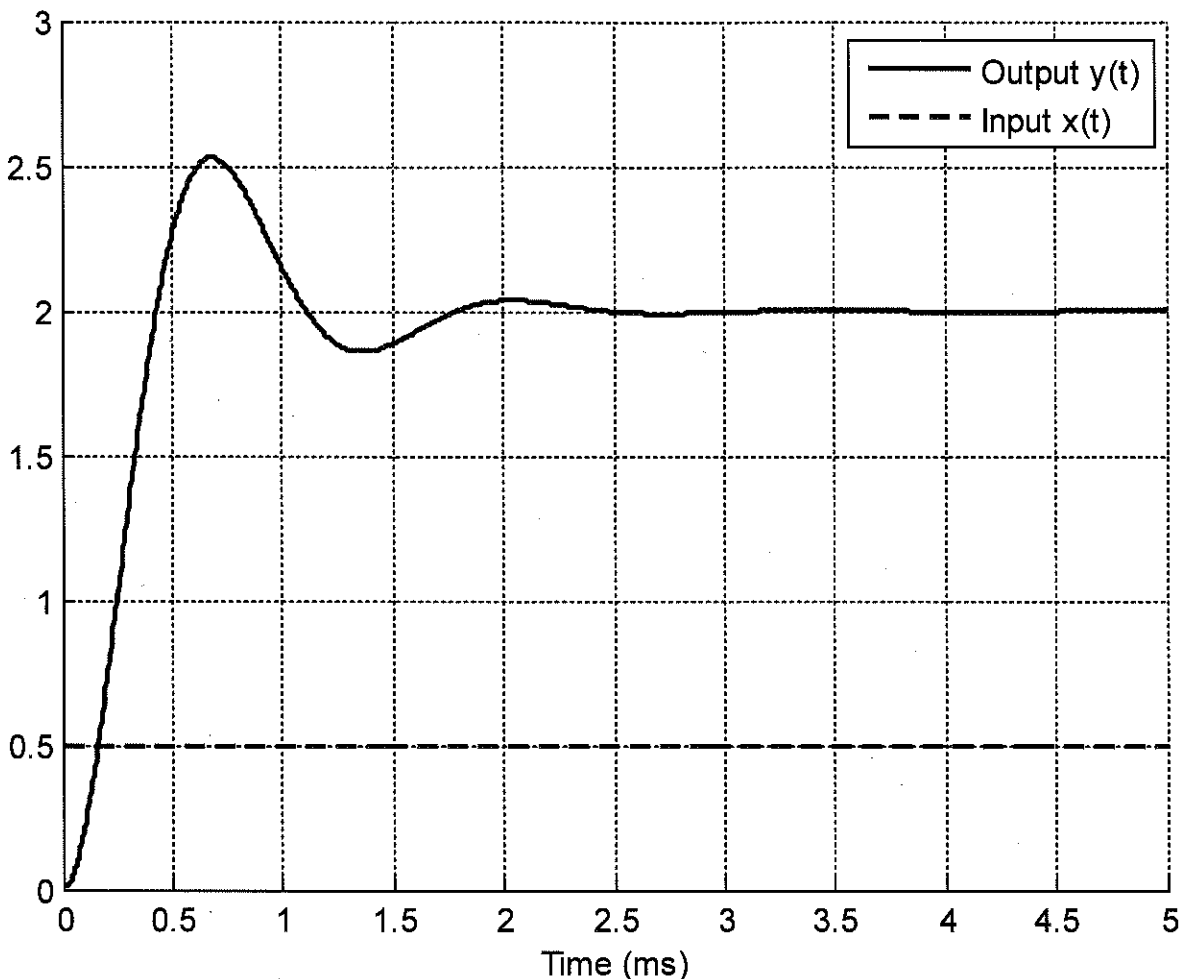


6) The percent overshoot for this system is best estimated as

a) 400%   b) 250 %   c) 200%   d) 150 %   e) 100 %   f) 25%

7) The (2%) settling time for this system is best estimated as

a) 1.5 ms   b) 2.5 ms   c) 4 ms   d) 5 ms

8) The static gain for this system is best estimated as

a) 1       b) 2       c) 3       d) 4