ECE-205 Dynamical Systems

**Course Notes
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1.0 Electrical Systems

The types of dynamical systems we will be studying can be modeled in terms of algebraic equations, differential equations, or integral equations. We will begin by looking at familiar mathematical models of ideal resistors, ideal capacitors, and ideal inductors. Then we will begin putting these models together to develop models for RL and RC circuits. Finally, we will review solution techniques for the first order differential equation we derive to model the systems.

1.1 Ideal Resistors

The governing equation for a resistor with resistance R is given by Ohm's law,

$$v(t) = Ri(t)$$

where $v(t)$ is the voltage across the resistor and $i(t)$ is the current through the resistor. Here R is measured in Ohms, $v(t)$ is measured in volts, and $i(t)$ is measured in amps. The entire expression must be in volts, so we get the unit expression

$$[volts] = [Ohms][amps]$$

1.2 Ideal Capacitors

The governing equation for a capacitor with capacitance C is given by

$$i(t) = C\frac{dv(t)}{dt}$$

Here C is measured in farads, and again $v(t)$ is measured in volts and $i(t)$ is measured in amps. This expression also helps us with the units. The entire expression must be in terms of current , so looking at the differential relationship we can determine the unit expression

$$[amps] = [farads][volts]/[seconds]$$

We can integrate this equation from an initial time t_0 up to the current time t as follows:

$$i(t) = C\frac{dv(t)}{dt}$$

$$\frac{1}{C}i(t)dt = dv(t)$$

Next, since we want to integrate up to a final time t, we need to use a dummy variable in the integral that is not t. This is an important habit to get into—do not use t as the dummy variable of integration if we expect a function of time as the output! Here we have chosen to use the dummy variable λ. Also we incorporate the fact that at time t_0 the voltage is $v(t_0)$, while at time t the voltage is $v(t)$

$$\frac{1}{C}\int_{t_0}^{t} i(\lambda)d\lambda = \int_{v(t_o)}^{v(t)} dv(\lambda)$$

Carrying out the integration we get

$$\frac{1}{C}\int_{t_0}^{t} i(\lambda)d\lambda = v(t) - v(t_0)$$

which we can rearrange as

$$v(t) = v(t_0) + \frac{1}{C}\int_{t_0}^{t} i(\lambda)d\lambda$$

This expression tells us there are two components to the voltage across a capacitor, the initial voltage $v(t_0)$ and the part due to any current flowing through the capacitor after that time, $\frac{1}{C}\int_{t_0}^{t} i(\lambda)d\lambda$

Finally, these expressions help us determine some important characteristics of our ideal capacitor:

- If the voltage across the capacitor is constant, then the current through the capacitor must be zero since the current is proportional to the rate of change of the voltage. Hence, *a capacitor is an open circuit to dc.*

- *It is not possible to change the voltage across a capacitor in zero time* .The voltage across a capacitor must be a continuous function of time, otherwise an infinite amount of current would be required.

1.3 Ideal Inductors

The governing equation for an inductor with inductance L is given by

$$v(t) = L\frac{di(t)}{dt}$$

Here L is measured in henrys, and again $v(t)$ is measured in volts and $i(t)$ is measured in amps. This expression also helps us with the units. The entire expression must be in terms of voltage , so looking at the differential relationship we can determine the unit expression

[volts] = [henrys][amps]/[seconds]

We can integrate this equation from an initial time t_0 up to the current time t as follows:

$$v(t) = L\frac{di(t)}{dt}$$

$$\frac{1}{L}v(t)dt = di(t)$$

Next, since we want to integrate up to a final time t, so we again have chosen to use the dummy variable λ. Also we incorporate the fact that at time t_0 the current is, $i(t_0)$ while at time t the current is $i(t)$.

$$\frac{1}{L}\int_{t_0}^{t} v(\lambda)d\lambda = \int_{i(t_o)}^{i(t)} di(\lambda)$$

Carrying out the integration we get

$$\frac{1}{L}\int_{t_0}^{t} v(\lambda)d\lambda = i(t) - i(t_0)$$

which we can rearrange as

$$i(t) = i(t_0) + \frac{1}{L}\int_{t_0}^{t} v(\lambda)d\lambda$$

This expression tells us there are two components to the current through an inductor, the initial current $i(t_0)$ and the part due to any voltage across the inductor after that time, $\frac{1}{L}\int_{t_0}^{t} v(\lambda)d\lambda$.

Finally, these expressions help us determine some important characteristics of our ideal inductor:

- If the current thought an inductor is constant, then the voltage across the inductor must be zero since the voltage is proportional to the rate of change of the current. Hence, *an inductor is a short circuit to dc.*

- *It is not possible to change the current through an inductor in zero time .*The current through an inductor must be a continuous function of time, otherwise an infinite amount of voltage would be required.

2.0 First Order Circuits

A first order circuit is a circuit with one effective energy storage element, either an inductor or a capacitor. (In some circuits it may be possible to combine multiple capacitors or inductors into one equivalent capacitor or inductor.) We begin this section with the derivation of the governing differential equation for various first order circuits. We will then put the first order equation into a standard form that allows us to easily determine physical characteristics of the circuit. Next we show an alternative method for checking some parts of the governing differential equations. We then solve the differential equations for the case of piecewise constant inputs, and finish the section with an alternative method of solving the differential equations using integrating factors.

2.1 Governing Differential Equations for First Order Circuits

In this section we derive the governing differential equations that model various RL and RC circuits. We then put the governing first order differential equations into a standard form, which allows us to read off descriptive information about the system very easily. The standard form we will use is

$$\tau \frac{dy(t)}{dt} + y(t) = Kx(t)$$

Here we assume the *system input* is $x(t)$ and the *system output* is $y(t)$. τ is the system *time constant*, which indicates how long it will take the system to reach steady state for a step (constant) input. K is the *static gain* of the system. For a constant input of amplitude A ($x(t) = Au(t)$, where $u(t)$ is the unit step function), in steady state we have $\frac{dy(t)}{dt} = 0$ and $y(t) = Kx(t) = KA$. Hence the static gain lets us easily compute the steady state value of the output. For circuits with capacitors the differential equation will in general be in terms of a voltage (the output $y(t)$ will be a voltage), while for circuits with inductors the differential equation will in general be in terms of current (the output $y(t)$ will be a current) .

Example 2.1.1. Consider the RC circuit shown in Figure 2.1. The voltage source is $v_s(t)$. We start to derive the governing differential equation by determining the single current in the loop

$$i_R(t) = \frac{v_s(t) - v_c(t)}{R} = i_C(t) = C\frac{dv_c(t)}{dt}$$

or

$$C\frac{dv_c(t)}{dt} = \frac{v_s(t) - v_c(t)}{R}$$

where $v_c(t)$ is the voltage across the capacitor and the current in the loop is equal to the current through the resistor $i_R(t)$ and the current through the capacitor $i_C(t)$. We can put this into a more standard form by rearranging the terms

$$RC\frac{dv_c(t)}{dt} + v_c(t) = v_s(t)$$

If we define the *time constant* $\tau = RC$, then we have

$$\tau \frac{dv_c(t)}{dt} + v_c(t) = v_s(t)$$
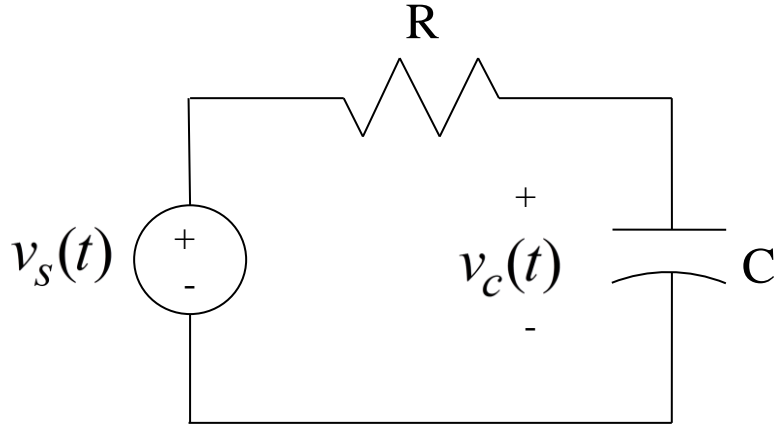
Here the static gain $K = 1$.

Figure 2.1. Circuit for Example 2.1.1.

Example 2.1.2. Consider the RC circuit shown in Figure 2.2. Again the voltage source is $v_s(t)$. We again start to derive the governing differential equation by determining the current through resistor R_a,

$$i(t) = \frac{v_s(t) - v_c(t)}{R_a}$$

This current must be equal to the sum of the currents through the capacitor and R_b,

$$i(t) = \frac{v_c(t)}{R_b} + C \frac{dv_c(t)}{dt}$$

Equating these we get the governing differential equation:

$$i(t) = \frac{v_s(t) - v_c(t)}{R_a} = \frac{v_c(t)}{R_b} + C \frac{dv_c(t)}{dt}$$

Rearranging terms we get

$$C \frac{dv_c(t)}{dt} + \left(\frac{1}{R_a} + \frac{1}{R_b} \right) v_c(t) = \frac{1}{R_a} v_s(t)$$

$$C \frac{dv_c(t)}{dt} + \frac{R_a + R_b}{R_a R_b} v_c(t) = \frac{1}{R_a} v_s(t)$$

or

$$\frac{R_a R_b C}{R_a + R_b} \frac{dv_c(t)}{dt} + v_c(t) = \frac{R_b}{R_a + R_b} v_s(t)$$

With time constant $\tau = \dfrac{R_a R_b C}{R_a + R_b}$ and static gain $K = \dfrac{R_b}{R_a + R_b}$ we get
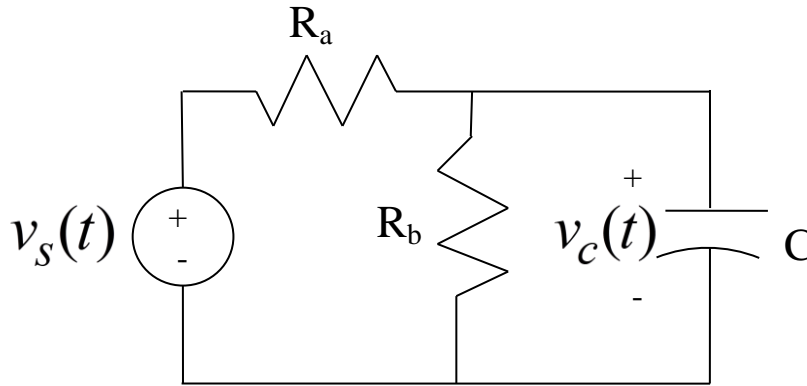
$$\tau \frac{dv_c(t)}{dt} + v_c(t) = K v_s(t)$$

Figure 2.2. Circuit used in Example 2.1.2.

Example 2.1.3. Consider the operational-amplifier circuit shown in Figure 2.3. The input voltage is again $v_s(t)$ and the output voltage (the voltage across the load resistor R_L) is the same as the voltage across the capacitor (since the + terminal of the op amp is assumed to be grounded). We will assume an ideal op amp, which implies the conditions

$$i^+(t) = i^-(t) = 0$$
$$v^+(t) = v^-(t)$$

Let's look at the currents flowing into the negative (feedback) terminal of the op-amp using the ideal op-amp model. Since for our example the non-inverting terminal is tied to ground we have $v^+(t) = 0$. With these assumptions our governing differential equation becomes

$$0 = \frac{v_s(t)}{R_a} + \frac{v_c(t)}{R_b} + C \frac{dv_c(t)}{dt}$$

Rearranging this gives

$$C \frac{dv_c(t)}{dt} + \frac{v_c(t)}{R_b} = -\frac{v_s(t)}{R_a}$$

or

$$R_b C \frac{dv_c(t)}{dt} + v_c(t) = -\frac{R_b}{R_a} v_s(t)$$

Setting the time constant $\tau = R_b C$ and static gain $K = -\dfrac{R_b}{R_a}$ we finally have

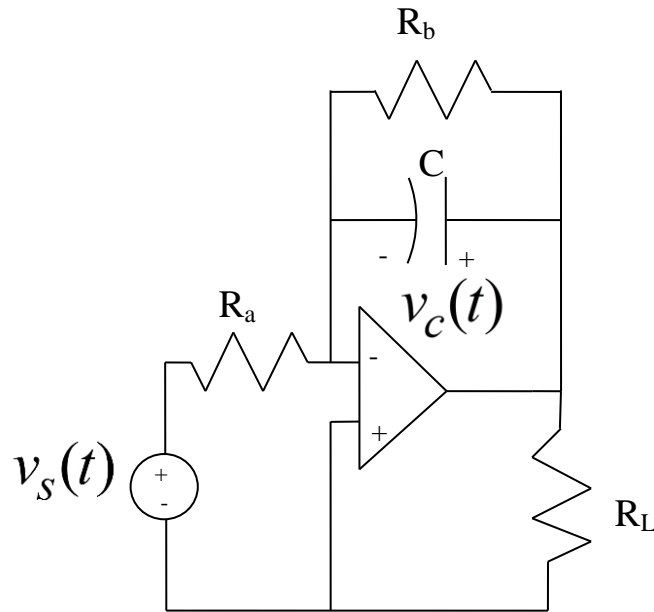$$\tau \frac{dv_c(t)}{dt} + v_c(t) = Kv_s(t)$$

Figure 2.3. Circuit for Example 2.1.3.

Example 2.1.4. Consider the RL circuit shown in Figure 2.4. The single current in the loop is given by

$$i(t) = \frac{v_s(t) - v_L(t)}{R}$$

where

$$v_L(t) = L \frac{di(t)}{dt}$$

Combining and rearranging we get

$$L \frac{di(t)}{dt} + Ri(t) = v_s(t)$$

or

$$\frac{L}{R} \frac{di(t)}{dt} + i(t) = \frac{1}{R} v_s(t)$$

With time constant $\tau = \dfrac{L}{R}$ and static gain $K = \dfrac{1}{R}$ the governing differential equation is
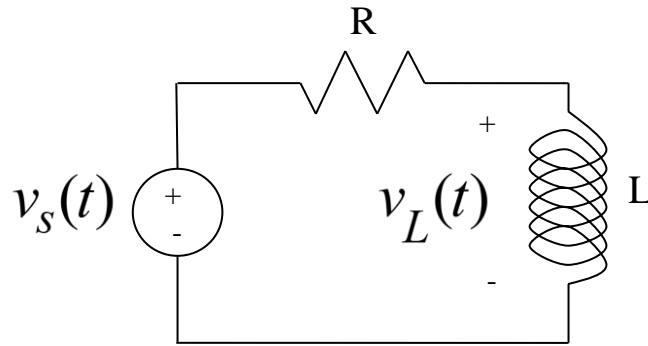
$$\tau \frac{di(t)}{dt} + i(t) = Kv_s(t)$$

Figure 2.4. Circuit for Example 2.1.4.

Example 2.1.5. Consider the RC circuit shown in Figure 2.5. The single current source must be divided between the current flowing through resistor R_b and the current flowing through the capacitor C,

$$i_s(t) = \frac{v_c(t)}{R_b} + C\frac{dv_c(t)}{dt}$$

Rearranging we get

$$R_b C\frac{dv_c(t)}{dt} + v_c(t) = R_b i_s(t)$$

With time constant $\tau = R_b C$ and static gain $K = R_b$ the governing differential equation is
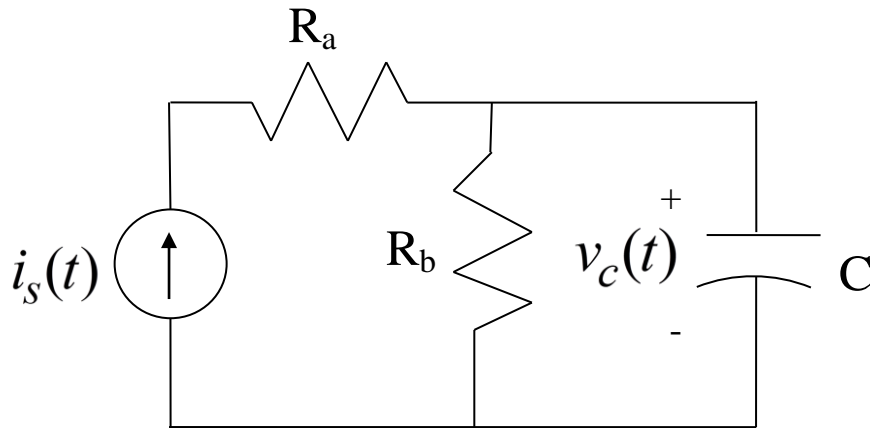
$$\tau\frac{dv_c(t)}{dt} + v_c(t) = Ki_s(t)$$

Figure 2.5. Circuit used in Example 2.1.5.

2.2 Thevenin Resistance, Time Constants, and Static Gain

Although we are focusing our attention on deriving the governing equations for first order circuits, it is useful and very convenient to be able to check our equations as much as possible.

First of all, for first order RC circuits the time constants will be of the form $\tau = R_{th} C_{eq}$ where R_{th} is the Thevenin resistance seen from the ports of the equivalent capacitor, C_{eq}. For first order RL circuits the time constants will be of the form $\tau = \dfrac{L_{eq}}{R_{th}}$ where R_{th} is the Thevenin resistance seen from the ports of the equivalent inductor, L_{eq}. Recall that *when determining the Thevenin resistance all independent voltage sources are treated as short circuits, and all independent current sources are treated as open circuits.*

Secondly, if we are looking at constant inputs, then we use the fact that *a capacitor is an open circuit to dc and an inductor is a short circuit to dc. In addition, for constant inputs in steady state all of the time derivatives are zero (in steady state nothing changes in time).*

Example 2.2.1. Consider the circuit shown in Figure 2.1 (Example 2.1.1). The Thevenin resistance seen from the capacitor is equal to R, so the time constant is $\tau = RC$. For a dc input, the capacitor looks like an open circuit, so in steady state the voltage across the capacitor is equal to v_s, the input voltage, so the static gain is $K = 1$. These results match our previous results.

Example 2.2.2. Consider the circuit shown in Figure 2.2 (Example 2.1.2). The Thevenin resistance seen from the capacitor is $R_{th} = R_a \parallel R_b = \dfrac{R_a R_b}{R_a + R_b}$, so the time constant is $\tau = R_{th} C = \dfrac{R_a R_b C}{R_a + R_b}$. For a dc input, the capacitor looks like an open circuit, so in steady state the voltage across the capacitor is given by the voltage divider relationship $v_c = \dfrac{R_b}{R_a + R_b} v_s$, so the static gain is $K = \dfrac{R_b}{R_a + R_b}$. These results match our previous results.

Example 2.2.3. Consider the circuit shown in Figure 2.3 (Example 2.1.3). The Thevenin resistance seen by the capacitor is a little more difficult to determine, and to do it correctly is beyond the scope of this course. For a dc input, the capacitor looks like an open circuit, so summing the currents into the negative terminal of the op amp we have $\dfrac{v_c}{R_b} + \dfrac{v_s}{R_a} = 0$, or in steady state $v_c = -\dfrac{R_b}{R_a} v_s$ Hence the static gain is

$$K = -\dfrac{R_b}{R_a}.$$

Example 2.2.4. Consider the circuit shown in Figure 2.4 (Example 2.1.4). The Thevenin resistance seen by the inductor is $R_{th} = R$. For a dc input, the inductor looks like a short circuit. Hence the steady state current flowing in the circuit for a dc input is $i = \dfrac{1}{R} v_s$, so the static gain is $K = \dfrac{1}{R}$.

Example 2.2.5. Consider the circuit shown in Figure 2.5 (Example 2.1.5). The Thevenin resistance seen by the capacitor is $R_{th} = R_b$ so the time constant is $\tau = R_b C$. For a dc input the capacitor looks like an open circuit, so in steady state $v_c = R_b i$, so the static gain is $K = R_b$.

2.3 Solving First Order Differential Equations

In this section we will go over two methods for solving first order differential equations. We will initially solve the equations by breaking the solution into the *natural response* (the response with no input) and then the *forced response* (the response when the input is turned on). We will apply this method to problems where the input is a constant value, or is switched between constant values. This method will also work with any input, and we will examine the results for a sinusoidal input later. In the last section we will go over a different method of solution using *integrating factors*, which will work for any type of input, and is an important method in helping us characterize how a system will respond to any type of input.

2.3.1 Solution using Natural and Forced Responses

Consider a system described by the first order differential equation

$$\tau \frac{dy(t)}{dt} + y(t) = Kx(t)$$

In this equation, τ is the time constant and K is the *static gain*. We will solve this equation in two parts. We will first determine the *natural response*, $y_n(t)$. The natural response is the when the input is zero. Then we will determine the *forced response*, $y_f(t)$. The forced response is the response due to the input only. The total response is then the sum of the natural and forced responses, $y(t) = y_n(t) + y_f(t)$.

Natural Response: To determine the natural response we assume there is no input in the system, so we have the equation

$$\tau \frac{dy_n(t)}{dt} + y_n(t) = 0$$

Let's assume a solution of the form $y_n(t) = c\, e^{rt}$, where c and r are parameters to be determined. Substituting this assumption into the differential equation we get

$$\tau r c e^{rt} + c e^{rt} = c e^{rt}[\tau r + 1] = 0$$

If $c = 0$ then we are done, and the natural response will be $y_n(t) = 0$. This solution certainly satisfies the differential equation. However, if $c \neq 0$, and since e^{rt} can never be zero, we must have $\tau r + 1 = 0$, or $r = -\frac{1}{\tau}$. In this case the natural response will be

$$y_n(t) = c e^{-t/\tau}.$$

Forced Response: To determine the forced response we must know the system input, $x(t)$. We will initially assume an input that is zero before $t = 0$ and then has constant amplitude A for $t \geq 0$,

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$$x(t) = \begin{cases} 0 & t < 0 \\ A & t \geq 0 \end{cases}$$

Then for $t \geq 0$ we have the equation

$$\tau \frac{dy_f(t)}{dt} + y_f(t) = KA$$

Since this is a linear ordinary differential equation we only need to find one solution. One obvious solution to this equation is the solution in steady state, when $\frac{dy_f(t)}{dt} = 0$. In steady state we have

$$y_f(t) = KA$$

Note that for a constant input, the steady state output is the product of the static gain and the amplitude of the input.

Total Solution: The total solution to the problem is the sum of these two solutions

$$y(t) = y_n(t) + y_f(t) = ce^{-t/\tau} + KA$$

Now assume the initial time is t_0 and at this time the output is denoted $y(t_0)$. Substituting this into our equation we have $y(t_0) = ce^{-t_0/\tau} + KA$, or $c = (y(t_0) - KA)e^{t_0/\tau}$, and our total solution is

$$y(t) = (y(t_0) - KA)e^{-(t-t_0)/\tau} + KA$$

For simplicity, let's write our steady state value explicitly, so $y(\infty) = KA$ and we have the solution

$$y(t) = (y(t_0) - y(\infty))e^{-(t-t_0)/\tau} + y(\infty)$$

Significance of the Time Constant

In much of what we do, we will be concerned with the time constants of a system in one way or another. Let's look at the response of our first order system assuming the system is initially at rest ($y(0) = 0$) and the final value is one ($y(\infty) = 1$). Let's look at the response of our system as the time t takes on the values of integer number of time constants:

Time (t)	t/τ	$y(t) = 1 - e^{-t/\tau}$
0	0	0
τ	1	0.632
2τ	2	0.865
3τ	3	0.950
4τ	4	0.982
5τ	5	0.993

Figure 2.6 show this result graphically, The way this information is usually interpreted is that a system is within 5% of its final value in 3 time constants, within 2% of its final value in 4 time constants, and within 1% of its final value in 5 time constants. Hence the use of time constants gives us a quick way to describe one aspect of the behavior of a system. As we will see, as the systems become more complex, the use of time constants indicates which part of the solution is the most important and how the system responds to periodic inputs (sines and cosines).
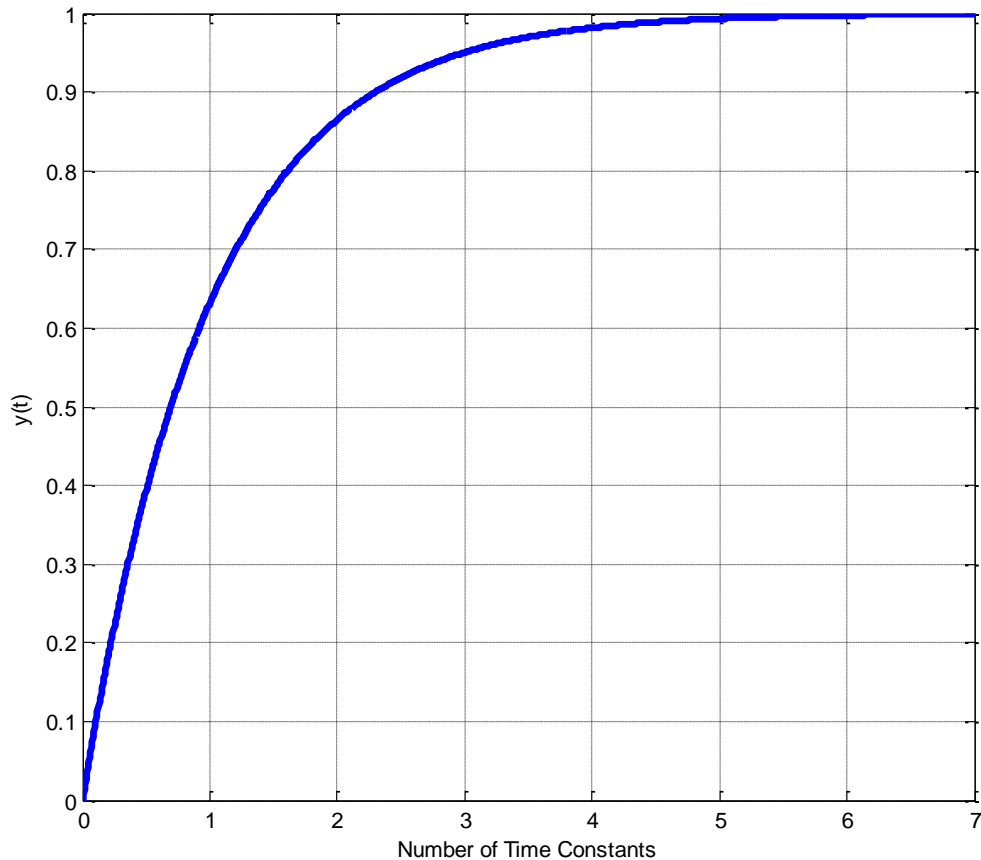
Figure 2.6. Graph of $y(t) = 1 - e^{-t/\tau}$ for $t = 0\tau$ up to $t = 7\tau$. $y(t)$ is within 5% of its final value in 3 time constant, within 2% of its final value in 4 time constants, and within 1% of its final value in 5 time constants.

Example 2.3.1. Consider the circuit in Figure 2.2 (Example 2.1.2). Let's first assume $R_a = R_b = 2k\Omega$ and $C = 1\mu F$. Then $R_{th} = 1k\Omega$, $\tau = 1ms$, and $K = 0.5$. Next we will assume the initial voltage on the capacitor is zero ($v_c(t_0) = v_c(0) = 0$) and the input is as follows:

$$v_s(t) = \begin{cases} 0 & t < 0 \\ 2 & 0 \le t \le 8 \\ -2 & 8 < t \le 16 \\ 1 & t > 16 \end{cases}$$

Here the input is in volts and the time is measured in milliseconds. We now want to determine the output. We will do this by looking at the initial and final values for each time interval, where the time intervals are determined by the times during which the input voltage is constant. The differential equation is

$$\tau \frac{dv_c(t)}{dt} + v_c(t) = Kv_s(t)$$

Clearly $y(t) = v_c(t)$ and $x(t) = v_s(t)$. The solution in each interval will be of the form

$$y(t) = \left[y(t_0) - y(\infty) \right] e^{-(t-t_0)/\tau} + y(\infty)$$

At this point we just need to be able to determine what $y(t_0)$ and $y(\infty)$ mean for each interval.

First interval ($0 \le t < 8$ ms) **:** We have the initial value in this interval $y(0) = v_c(0) = 0$ volts. To determine the final value, we use the static gain and the amplitude of the input for this interval.

$$v_c(\infty) = y(\infty) = K \cdot 2 = \frac{1}{2} \cdot 2 = 1$$

Hence for this interval, we have the solution

$$v_c(t) = y(t) = -e^{-t/\tau} + 1 = 1 - e^{-t/0.001}$$

Before we go on to the next interval we need to figure out the value of $y(t)$ at the end of this interval, this value will be the initial point during the next interval. At the end of the interval we will have

$$y(0.008) = 1 - e^{-0.008/0.001} = 1 - e^{-8} = 0.99966 \approx 1.0$$

Second interval ($8 < t \le 16$ ms) **:** The initial value for this interval will be the end point of the previous interval, so $y(t_0) = 1$. To determine the final value we again use the static gain

$$y(\infty) = K \cdot (-2) = \frac{1}{2} \cdot (-2) = -1$$

We now have almost everything we need, however, our solution assumed a time of zero was measured at the beginning of the interval. Hence to use our previous solution we need to subtract the time at the beginning of the interval from our actual time in our form of the solution, so our time will be measured from the beginning of the interval. Our solution for this interval is then

$$y(t) = [1 - (-1)]e^{-(t-0.008)/\tau} + (-1) = 2e^{-(t-0.008)/0.001} - 1$$

At the end of this interval we will have

$$y(0.016) = 2e^{-(0.016-0.008)/0.001} - 1 = 2e^{-8} - 1 = -0.99933 \approx -1.0$$

Third interval (t > 16 ms) **:** The initial value for this interval will be the end point of the previous interval, so $y(t_0) = -1$. To determine the final value we have

$$y(\infty) = K \cdot 1 = K = \frac{1}{2}$$

Again we must scale our solution so time is measured from the beginning of the interval, so we have

$$y(t) = [-1 - 0.5]e^{-(t-0.016)/\tau} + 0.5 = -1.5e^{-(t-0.016)/0.001} + 0.5$$

Total solution: To get the total solution, we list the solutions during each time interval:

$$v_c(t) = y(t) = \begin{cases} 0 & t < 0 \\ 1 - e^{-t/0.001} & 0 \le t < 8 \, ms \\ 2e^{-(t-0.008)/0.001} - 1 & 8 \le t < 16 \, ms \\ -1.5e^{-(t-0.016)/0.001} + 0.5 & t \ge 16ms \end{cases}$$

To get the current through the capacitor, we use the relationship $i_c(t) = C\dfrac{dv_c(t)}{dt}$ for each time interval above. Doing this we get

$$i_c(t) = C\frac{dv_c(t)}{dt} = \begin{cases} 0 & t < 0 \\ 0.001e^{-t/0.001} & 0 \le t < 8 \, ms \\ 0.002e^{-(t-0.008)/0.001} & 8 \le t < 16 \, ms \\ -0.015e^{-(t-0.016)/0.001} & t \ge 16ms \end{cases}$$

Here $i_c(t)$ is measured in amps.

Figure 2.7 shows the input voltage, the voltage across the capacitor, and the current through the capacitor as a function of time. Note that the voltage across the capacitor is continuous, as it must be. However for this input, which is discontinuous, the current through the capacitor is discontinuous. Let's also look at the answer to see if we can check our results and if the answer makes sense. When the source voltage is initially turned on, the voltage across the capacitor is zero and all of the voltage generated by the source is equal to the voltage across resistor R_a. If there were any voltage drop across R_b at the initial time, there would also be a voltage drop across the capacitor since they are in parallel. In steady state, the capacitor looks like an open circuit, so there is no current flowing through the capacitor and the maximum possible voltage at this time is half the voltage of the source, which agrees with our results. In this example the input was held constant for an equivalent of eight time constants, so the voltage across the capacitor had essentially reached steady state.

Finally is useful to point out that if the voltage across the capacitor is described by the relationship

$$v_c(t) = [v_c(0) - v_c(\infty)]e^{-t/\tau} + v_c(\infty)$$

Then the current through the capacitor is given by

$$i_c(t) = C\frac{dv_c(t)}{dt} = -\frac{C}{\tau}[v_c(0) - v_c(\infty)]e^{-t/\tau}$$

What this means is that if the voltage across a capacitor is growing exponentially, then the current through the capacitor is decreasing exponentially. Similarly, if the voltage across a capacitor is decreasing exponentially, the current through the capacitor will be growing exponentially. This is also behavior our results show. Similar results also hold for inductors.

Example 2.3.2. Consider the circuit in Figure 2.4 (Example 2.1.4). Let's first assume $R = R_{th} = 100\,\Omega$ and $L = 10\,mH$. Then $\tau = \frac{L}{R} = \frac{0.01}{100} = 0.0001 = 100\,\mu s$ and $K = 0.01$. Next we will assume the initial current through the inductor is $i(0) = 10\,mA$ and the input is as follows:

$$v_s(t) = \begin{cases} 0 & t < 0 \\ 2 & 0 \le t < 0.1 \\ -3 & 0.1 \le t < 0.25 \\ 4 & t \ge 0.25 \end{cases}$$

Here the input is in volts and the time is measured in milliseconds. We now want to determine the output. We will do this by looking at the initial and final values for each time interval, where the time intervals are determined by the times during which the input voltage is constant.

The differential equation for this system is again

$$\tau\frac{di(t)}{dt} + i(t) = Kv_s(t)$$

Clearly $y(t) = i(t)$ and $x(t) = v_s(t)$. The solution in each interval will be of the form

$$y(t) = [y(t_0) - y(\infty)]e^{-(t - t_0)/\tau} + y(\infty)$$

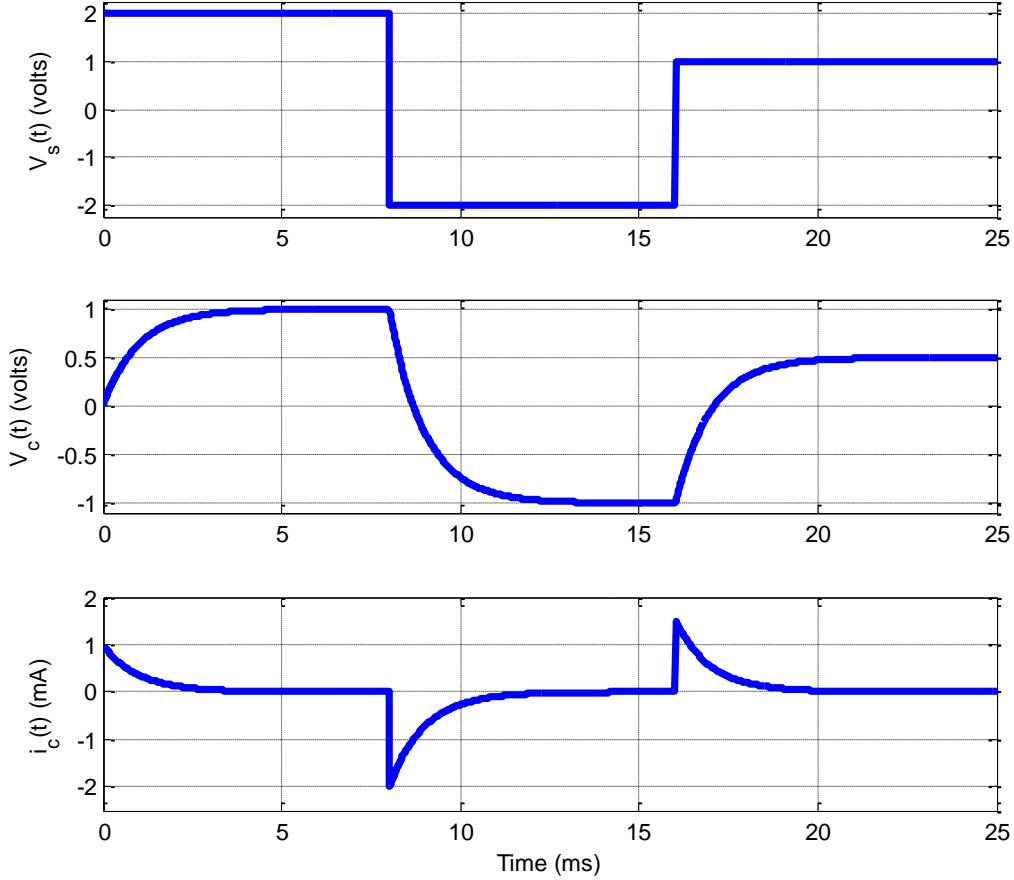At this point we just need to be able to determine what $y(0)$ and $y(\infty)$ mean for each interval.

Figure 2.7. Results for Example 2.3.1.

First *interval ($0 \leq t < 0.1\,ms$)* **:** We have the initial value $y(0) = i(0) = 0.01$ amps in this interval. To determine the final value, we use the static gain and the amplitude of the input for this interval

$$i(\infty) = y(\infty) = K \cdot 2 = \frac{1}{100} \cdot 2 = 0.02$$

Hence for this interval, we have the solution

$$y(t) = \left[y(0) - y(\infty) \right] e^{-t/\tau} + y(\infty) = [0.01 - 0.02]e^{-t/\tau} + 0.02 = -0.01e^{-t/0.0001} + 0.02$$

Before we go on to the next interval we need to figure out the value of $y(t)$ at the end of this interval, this value will be the initial point during the next interval. At the end of the interval we will have

$$y(0.0001) = -0.01e^{-0.0001/0.0001} + 0.02 = -0.01e^{-1} + 0.02 = 0.01632$$

Third interval $(0.1 \leq t < 0.25\ ms)$ **:** The initial value for this interval will be the end point of the previous interval, so $y(t_0) = 0.01632$. To determine the final value we again use the static gain

$$y(\infty) = K \cdot (-3) = 0.01 \cdot (-3) = -0.03$$

We again need to subtract the time at the beginning of the interval from our actual time in our form of the solution, so our time will be measured from the beginning of the interval. Our solution for this interval is then

$$y(t) = [0.01632 - (-0.03)]e^{-(t-0.0001)/\tau} + (-0.03) = 0.04632e^{-(t-0.0001)/0.0001} - 0.03$$

At the end of this interval we will have

$$y(0.00025) = 0.04632e^{-(0.00025-0.0001)/0.0001} - 0.03 = 0.04632e^{-1.5} - 0.03 = -0.01966$$

Fourth interval $(t \geq 0.25\ ms)$ **:** The initial value for this interval will be the end point of the previous interval, so $y(t_0) = -0.01966$. To determine the final value we have

$$y(\infty) = K \cdot 4 = 0.04$$

Again we must scale our solution so time is measured from the beginning of the interval, so we have

$$y(t) = [-0.01966 - 0.04]e^{-(t-0.0025)/\tau} + 0.04 = -0.05966e^{-(t-0.00025)/0.0001} + 0.04$$

Total solution: To get the total solution, we list the solutions during each time interval:

$$i_L(t) = y(t) = \begin{cases} 0 & t < 0 \\ -0.01e^{-t/0.0001} + 0.02 & 0 \leq t < 0.1\ ms \\ 0.04632e^{-(t-0.0001)/0.0001} - 0.03 & 0.1 \leq t < 0.25\ ms \\ -0.05966e^{-(t-0.00025)/0.0001} + 0.04 & t \geq 0.25\ ms \end{cases}$$

To get the voltage across the inductor we use the relationship $v_L(t) = L\dfrac{di_L(t)}{dt}$ and compute the voltage for each time interval. Doing this we get

$$v_L(t) = L\frac{di_L(t)}{dt} = \begin{cases} 0 & t < 0 \\ e^{-t/0.0001} & 0 \leq t < 0.1\ ms \\ -4.632e^{-(t-0.0001)/0.0001} & 0.1 \leq t < 0.25\ ms \\ 5.966e^{-(t-0.00025)/0.0001} & t \geq 0.25\ ms \end{cases}$$

Figure 2.8 shows the input voltage, the current through the inductor, and the voltage across the inductor as a function of time. Note that the current through the inductor is continuous, as it must be, while in this case the voltage across the inductor is not continuous. Again let's look at our solution to see if it makes

sense. First of all, the voltage/current relationships for the inductor are consistent with what we expect. The initial current in the inductor is 10 mA, as we require, and the initial voltage from the source is 2 volts. Applying Kirchhoff's laws around the loop, we expect the initial voltage drop across the inductor to be given by $v_s(0) - i(0)R = 2 - (0.01)(100) = 1$ volts, which is what we have. In steady state the inductor looks like a short circuit, so there should be no voltage drop across the inductor once the system reaches steady state, which again matches our results. Note that the system only reaches steady state near 0.7 or 0.8 ms. In addition, in steady state the voltage drop across the resistor must match the voltage supplied by the source, or $v_s(\infty) - i(\infty)R = 4 - (0.04)(100) = 0$ volts, which again matches our results. Let's look at the results at one other convenient point in time, say $t = 0.2\, ms$. Using the equations we derived above (and the known input) we have

$$v_s(0.0002) = -3\ volts$$
$$i_l(0.0002) = -12.96\ mA$$
$$v_L(0.0002) = -1.70\ volts$$

Applying Kirchhoff's laws around the loop we have

$$v_s(0.0002) - i_s(0.0002)R - v_L(0.0002) = -3 - (-0.01296)(100) - (-1.70) \approx 0.0$$

We can obviously check as many points in time as we want in this way. This type of checking does not guarantee our answer is correct, but it does help find obvious errors.

2.3.2 Solution Using Integrating Factors

An alternative method of solution of first order differential equations is by the use of *integrating factors*. This method of solution is important to understand because as we start to analyze different types of *systems*, we need to be able to understand how we would solve for the output when we don't actually know what the input is. This helps us characterize systems independent of the actual (specific) input.

The use of integrating factors for solving first order differential equations is based on the fact that when we differentiate an exponential, we get the same exponential back multiplied by some other term. For example, if $x(t) = e^{\phi(t)}$, then

$$\frac{d}{dt}x(t) = \frac{d}{dt}e^{\phi(t)} = e^{\phi(t)}\frac{d\phi(t)}{dt} = x(t)\frac{d\phi(t)}{dt}$$

In what follows, the method looks fairly lengthy, but with practice most of the steps can be done in your head. Let's apply this idea to our equation
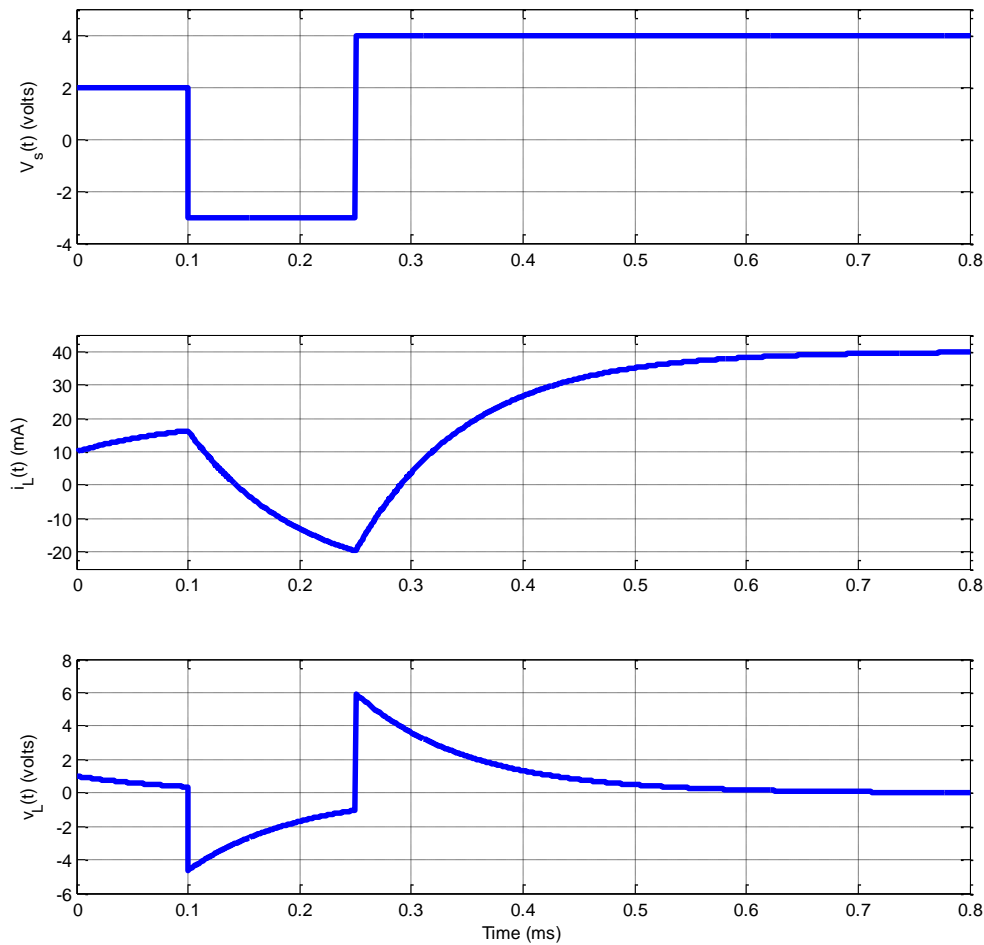
$$\tau\frac{dy(t)}{dt} + y(t) = Kx(t)$$

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Figure 2.8. Results for Example 2.3.2.

This method will work better if we rearrange our equation a bit to the form

$$\frac{dy(t)}{dt} + \frac{1}{\tau} y(t) = \frac{K}{\tau} x(t)$$

Next, we look at differentiating the product $y(t)e^{\phi(t)}$, where $\phi(t)$ will be determined by the differential equation we are trying to solve. This leads to the equation

$$\frac{d}{dt}\Big[y(t)e^{\phi(t)} \Big] = \frac{dy(t)}{dt} e^{\phi(t)} + y(t) \frac{d\phi(t)}{dt} e^{\phi(t)} = e^{\phi(t)} \bigg[\frac{dy(t)}{dt} + \frac{d\phi(t)}{dt} y(t) \bigg]$$

Next, we equate the term in brackets to the left hand side of our original differential equation,

$$\frac{dy(t)}{dt} + \frac{d\phi(t)}{dt} y(t) = \frac{dy(t)}{dt} + \frac{1}{\tau} y(t)$$

Clearly this means that

$$\frac{d\phi(t)}{dt} = \frac{1}{\tau}$$

Solving this simple equation we get

$$\phi(t) = \frac{t}{\tau}$$

Now we put this back into our equation above to get

$$\frac{d}{dt}\left[y(t)e^{t/\tau} \right] = \frac{dy(t)}{dt} e^{t/\tau} + y(t)\frac{1}{\tau}e^{t/\tau} = e^{t/\tau}\left[\frac{dy(t)}{dt} + \frac{1}{\tau} y(t) \right]$$

The term on the far right is the same as the left hand side of our differential equation multiplied by $e^{t/\tau}$, so this must equal the right hand side of our differential equation multiplied by the same thing,

$$\frac{d}{dt}\left[y(t)e^{t/\tau} \right] = e^{t/\tau}\left[\frac{dy(t)}{dt} + \frac{1}{\tau} y(t) \right] = e^{t/\tau}\left[\frac{K}{\tau} x(t) \right]$$

Next we eliminate the middle term to get the exact differential we want

$$\frac{d}{dt}\left[y(t)e^{t/\tau} \right] = e^{t/\tau}\left[\frac{K}{\tau} x(t) \right]$$

Finally we integrate from and initial time t_0 with initial value $y(t_0)$ to final time t with value $y(t)$,

$$\int_{t_0}^{t} \frac{d}{d\lambda}\left[y(\lambda)e^{\lambda/\tau} \right] d\lambda = \int_{t_0}^{t} e^{\lambda/\tau} \frac{K}{\tau} x(\lambda) d\lambda$$

The left hand side can be integrated as

$$\int_{t_0}^{t} \frac{d}{d\lambda}\left[y(\lambda)e^{\lambda/\tau} \right] d\lambda = y(t)e^{t/\tau} - y(t_0)e^{t_0/\tau} = \int_{t_0}^{t} e^{\lambda/\tau} \frac{K}{\tau} x(\lambda) d\lambda$$

or

$$y(t) = y(t_0)e^{-(t-t_0)/\tau} + \int_{t_0}^{t} e^{-(t-\lambda)/\tau} \frac{K}{\tau} x(\lambda) d\lambda$$

This is the general solution, for any input $x(t)$.

Example 2.3.1. Let's now look at the same input as before, $x(t) = A$ for $t \geq 0$ with initial condition $t_0 = 0$ and $y(t_0) = y(0)$. The solution to the differential equation becomes

$$y(t) = y(0)e^{-t/\tau} + \int_0^t e^{-(t-\lambda)/\tau} \frac{K}{\tau} A d\lambda$$

$$y(t) = y(0)e^{-t/\tau} + e^{-t/\tau} \int_0^t e^{\lambda/\tau} \frac{K}{\tau} A d\lambda$$

$$y(t) = y(0)e^{-t/\tau} + e^{-t/\tau} KA \left[e^{\lambda/\tau} \right]_{\lambda=0}^{\lambda=t}$$

$$y(t) = y(0)e^{-t/\tau} + e^{-t/\tau} KA \left[e^{t/\tau} - 1 \right]$$

$$y(t) = y(0)e^{-t/\tau} + KA \left[1 - e^{-t/\tau} \right]$$

With the substitution $y(\infty) = KA$, we get

$$y(t) = y(0)e^{-t/\tau} + y(\infty) \left[1 - e^{-t/\tau} \right]$$

or

$$y(t) = \left[y(0) - y(\infty) \right] e^{-t/\tau} + y(\infty)$$

the same solution as before.

Example 2.3.2. Let's use integration factors to determine the solution to the differential equation

$$\frac{dy(t)}{dt} = ay(t) + bx(t)$$

The first thing we need to do is put all of the y terms on the left hand side,

$$\frac{dy(t)}{dt} - ay(t) = bx(t)$$

Then we need

$$\frac{d\phi(t)}{dt} = -a$$

or

$$\phi(t) = -at$$

Then we have

$$\frac{d}{dt} \left[y(t)e^{-at} \right] = e^{-at} bx(t)$$

Integrating both sides we get

$$\int_{t_0}^{t} \frac{d}{d\lambda}\left[e^{-a\lambda}y(\lambda)\right]d\lambda = e^{-at}y(t) - e^{-at_0}y(t_0) = \int_{t_0}^{t} e^{-a\lambda}bx(\lambda)d\lambda$$

or

$$y(t) = e^{a(t-t_0)}y(t_0) + e^{at}\int_{t_0}^{t} e^{-a\lambda}bx(\lambda)d\lambda$$

Example 2.3.3. Let's use integration factors to determine the solution to the differential equation

$$\frac{dy(t)}{dt} - ty(t) = 2x(t)$$

Then we need

$$\frac{d\phi(t)}{dt} = -t$$

or

$$\phi(t) = -\frac{t^2}{2}$$

Then we have

$$\frac{d}{dt}\left[y(t)e^{-\frac{t^2}{2}}\right] = e^{-\frac{t^2}{2}}2x(t)$$

Integrating both sides we get

$$\int_{t_0}^{t} \frac{d}{d\lambda}\left[e^{-\frac{\lambda^2}{2}}y(\lambda)\right]d\lambda = e^{-\frac{t^2}{2}}y(t) - e^{-\frac{t_0^2}{2}}y(t_0) = \int_{t_0}^{t} e^{-\frac{\lambda^2}{2}}2x(\lambda)d\lambda$$

or

$$y(t) = e^{(\frac{t^2}{2}-\frac{t_0^2}{2})}y(t_0) + e^{\frac{t^2}{2}}\int_{t_0}^{t} e^{-\frac{\lambda^2}{2}}2x(\lambda)d\lambda$$

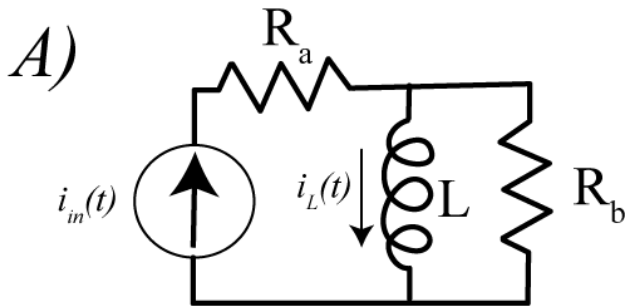# Chapter 2 Problems

2.1) For each of the circuits below:

i) Determine the governing differential equation using Kirchhoff's Laws and write it in standard form. For part *F* the output is the voltage across resistor R_C and this cannot be written in standard form.

ii) Determine the time constant and static gain from the differential equation you derive in (i)

iii) For all circuits except F, determine the Thevenin resistance from the ports of the capactor or inductor and verify the time constants.

iv) For all circuits except F, determine the static gain by determining the DC voltage across the capacitor or the current through and inductor.

Answers:

$$\left(\frac{L}{R_b}\right)\frac{di_L(t)}{dt}+i_L(t)=i_{in}(t), \quad C(R_a+R_b)\frac{dv_C(t)}{dt}+v_C(t)=R_b i_{in}(t)$$

$$\left(\frac{L}{R_a+R_b}\right)\frac{di_L(t)}{dt}+i_L(t)=\left(\frac{R_a}{R_a+R_b}\right)i_{in}(t), \quad C\left(\frac{R_a R_b}{R_a+R_b}\right)\frac{dv_C(t)}{dt}+v_C(t)=\left(\frac{R_b}{R_a+R_b}\right)v_{in}(t)$$

$$C\left(\frac{R_a R_b+R_c R_b+R_a R_c}{R_a+R_c}\right)\frac{dv_C(t)}{dt}+v_C(t)=\left(\frac{R_c}{R_a+R_c}\right)v_{in}(t), \quad \left(\frac{L}{R_b}\right)\frac{dv_{in}(t)}{dt}+v_{in}(t)=-\left(\frac{R_a}{R_b}\right)v_C(t)$$

**2.2)** For a simple series RC circuit the response of the system when the input is a unit step is
$$y(t)=1-e^{-t/RC}=1-e^{-t/\tau}$$
The 10-90% rise time, t_r, as shown below. The rise time is simply the amount of time necessary for the output to rise from 10% to 90% of its final value. Show that for this system the rise time is given by $t_r=\tau\ln(9)$
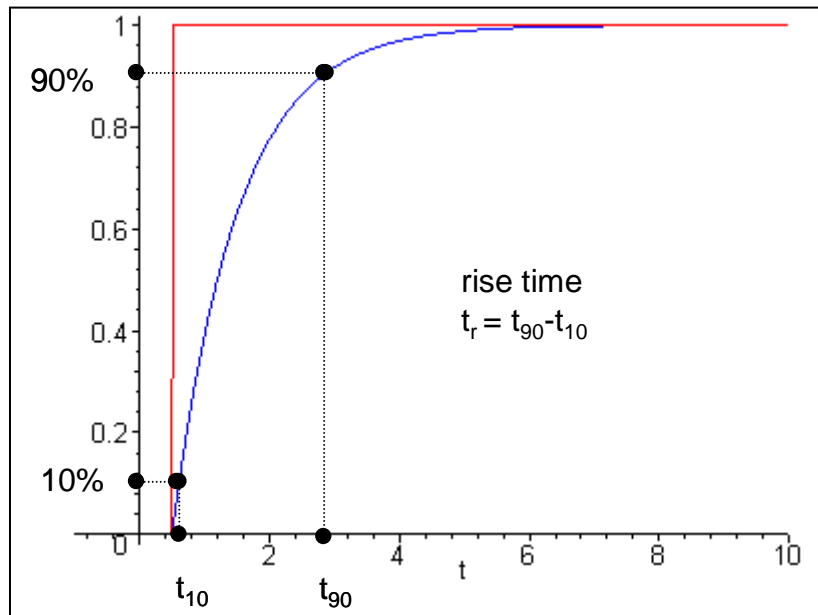
2.3) For each of the circuits below:

i) Determine the governing differential equation using Kirchhoff's Laws and write it in standard form.

ii) Determine the time constant and static gain from the differential equation you derive in (i)

iii) For all circuits except F, determine the Thevenin resistance from the ports of the capacitor or inductor and verify the time constants.

iv) For all circuits except F, determine the static gain by determining the DC voltage across the capacitor or the current through and inductor.

A)

B)

C)

D)

E)

F)
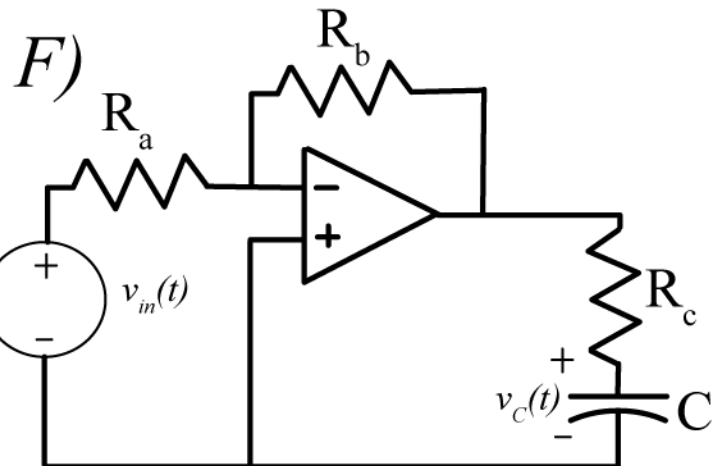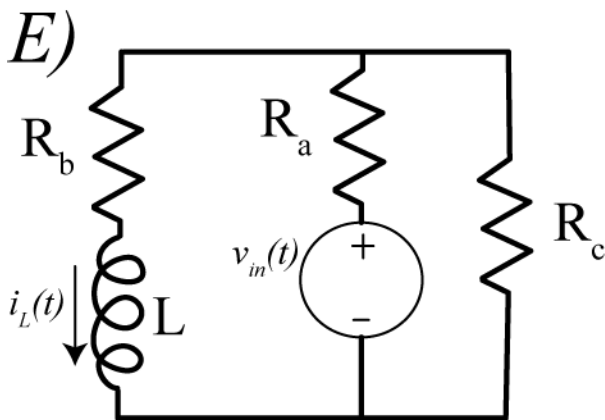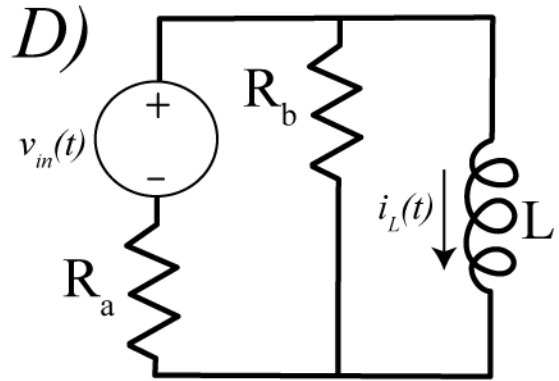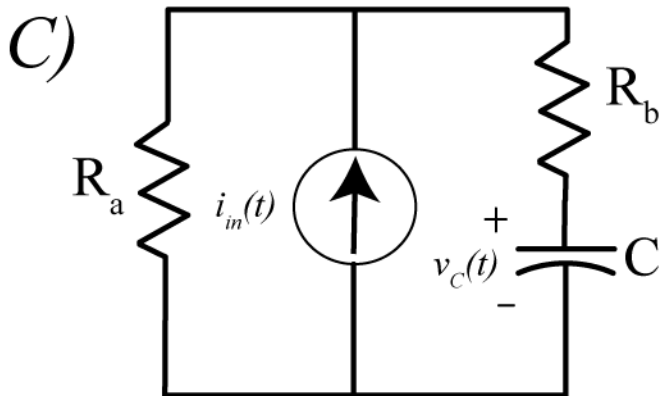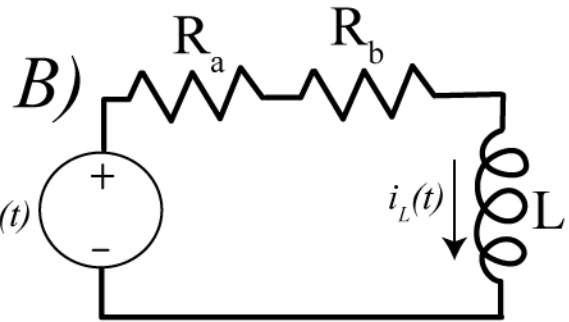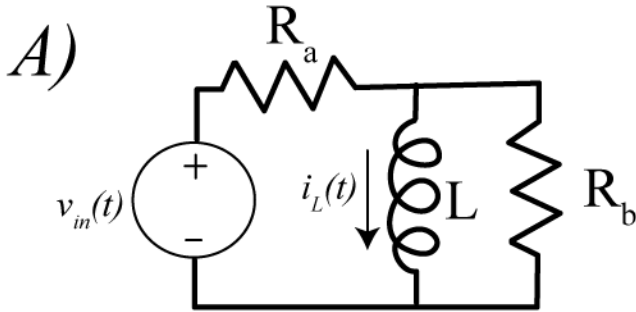
Answers:

$$\frac{(R_a + R_b)L}{R_a R_b}\frac{di_L(t)}{dt} + i_L(t) = \frac{1}{R_a}V_{in}(t), \qquad \frac{L}{R_a + R_b}\frac{di_L(t)}{dt} + i_L(t) = \frac{1}{R_a + R_b}V_{in}(t)$$

$$(R_a + R_b)C\frac{dv_C(t)}{dt} + v_C(t) = R_a i_{in}(t), \qquad \frac{(R_a + R_b)L}{R_a R_b}\frac{di_L(t)}{dt} + i_L(t) = \frac{1}{R_a}v_{in}(t)$$

$$\frac{(R_a + R_c)L}{R_a R_c + R_b R_c + R_a R_b}\frac{di_L(t)}{dt} + i_L(t) = \frac{R_c}{R_a R_c + R_b R_c + R_a R_b}v_{in}(t), \quad CR_c\frac{dv_C(t)}{dt} + v_C(t) = -\frac{R_b}{R_a}v_{in}(t)$$

2.4) Consider a first order sytem described by the differential equation $\tau \dot{y}(t) + y(t) = Kx(t)$.

a) If the initial value is $y(0) = 0$ and the final value is $y(\infty) = 10$, what is $y(4\tau)$?
b) If the initial value is $y(0) = -2$ and the final value is $y(\infty) = 8$, what is $y(4\tau)$?
c) If the initial value is $y(0) = 1$ and the final value is $y(\infty) = -4$, what is $y(4\tau)$?

Answers: 9.82, 7.82, -3.90

2.5) An RC circuit has paramters $R_{th} = R_a \| R_b$, $\tau = R_{th}C$, $K = R_{th}$ and is described by the first ordeer equation

$$\tau \frac{dv_c(t)}{dt} + v_c(t) = Ki_{in}(t)$$

For this circuit $R_a = 100\,\Omega$, $R_b = 200\,\Omega$, and $C = 2$ mf .The input current is

$$i_{in}(t) = \begin{cases} 0 & t \le 0 \\ 1\,mA & 0 \le t < 0.2\ sec \\ 2\,mA & 0.2 \le t < 0.4\ sec \\ -3\,mA & t \ge 0.4\ sec \end{cases}$$

a) Determine an analytical expression for the voltage across the capacitor in each of these regions
b) Enter your analytical expression as an anonymous function in Matlab, and simulate the system using Simulink. Show that all three of your answers are identical. Turn in your Matlab (driver file) code and your neatly labeled plots. Be sure to set the initial value of the integrator to zero.

2.6) An RL circuit has paramters $R_{th} = R_a + R_b + R_c$, $\tau = L/R_{th}$, $K = (R_a + R_b)/R_{th}$ and is described by the first ordeer equation

$$\tau \frac{di_L(t)}{dt} + i_L(t) = Ki_{in}(t)$$

For this circuit, $R_a = R_b = R_c = 10\,\Omega$ and $L = 8$ mh .The input current is

$$i_{in}(t) = \begin{cases} 0 & t \le 0 \\ 0.02A & 0 \le t < 0.4\ ms \\ -0.03\ A & 0.4 \le t < 1.0\ ms \\ 0.05\ A & t \ge 1.0\ ms \end{cases}$$

a) Determine an analytical expression for the current through the inductor in each of these regions

b) Enter your analytical expression as an anonymous function in Matlab, and simulate the system using Simulink. Show that all three of your answers are identical. Turn in your Matlab (driver file) code and your neatly labeled plots.

2.7) An RC circuit has paramters $R_{th} = R_a \| R_b$, $\tau = R_{th}C$, $K = R_{th}$ and is described by the first ordeer equation

$$\tau \frac{dv_c(t)}{dt} + v_c(t) = Ki_{in}(t)$$

For this circuit $R_a = 1\, k\Omega$, $R_b = 5\, k\Omega$, $C = 0.1$ mf and the *initial voltage* is $v_c(0) = 0.5V$

The input current is

$$i_{in}(t) = \begin{cases} 0 & t \leq 0 \\ -2\ mA & 0 \leq t < 0.2\ sec \\ 4\ mA & 0.2 \leq t < 0.3\ sec \\ -3\ mA & t \geq 0.3\ sec \end{cases}$$

a) Determine an analytical expression for the voltage across the capacitor in each of these regions

b) Enter your analytical expression as an anonymous function in Matlab, and simulate the system using Simulink. Run the simulation for 1 second (from 0 to 1 second). Show that all three of your answers are identical. Turn in your Matlab (driver file) code and your neatly labeled plots. *Be sure to set the initial value of the integrator to the correct value.*

2.8) An RL circuit has paramters $R_{th} = R_a \| R_b$, $\tau = L / R_{th}$, $K = \dfrac{1}{R_a}$ and is described by the first ordeer equation

$$\tau \frac{di_L(t)}{dt} + i_L(t) = Kv_{in}(t)$$

For this circuit, $R_a = 100\, \Omega$, $R_b = 400\Omega$, $L = 30$ mh and the initial current through the inductor is zero. The input current is

$$v_{in}(t) = \begin{cases} 0 & t \leq 0 \\ 1V & 0 \leq t < 0.6\ ms \\ -2V & 0.6\ ms \leq t < 2\ ms \\ 6V & t \geq 2\ ms \end{cases}$$

a) Determine an analytical expression for the current through the inductor in each of these regions

b) Enter your analytical expression as an anonymous function in Matlab, and simulate the system using Simulink. Run the simulation for 5 ms. Show that all three of your answers are identical. Turn in your Matlab (driver file) code and your neatly labeled plots. *Be sure to set the initial value of the integrator to the correct value.*

2.9) For each of the following first order differential equations, use an integrating factor to write $y(t)$ as a function of its initial value $y(t_0)$ and an integral of the input (plus some other functions)

$$a\dot{y}(t) + by(t) = cx(t) \qquad \frac{1}{2}\dot{y}(t) - ty(t) = x(t+2)$$

$$\dot{y}(t) + \frac{1}{t}y(t) = x(t) \qquad \dot{y}(t) + \cos(t)y(t) = e^t x(t)$$

Answers:

$$y(t) = e^{-\frac{b}{a}(t-t_0)}y(t_0) + \int_{t_0}^{t}\frac{c}{a}e^{-\frac{b}{a}(t-\lambda)}x(\lambda)d\lambda \qquad y(t) = e^{t^2-t_0^2}y(t_0) + \int_{t_0}^{t}2e^{(t^2-\lambda^2)}x(\lambda+2)d\lambda$$

$$y(t) = \left(\frac{t_0}{t}\right)y(t_0) + \int_{t_0}^{t}\left(\frac{\lambda}{t}\right)x(\lambda)d\lambda \qquad y(t) = e^{-\sin(t)+\sin(t_0)}y(t_0) + \int_{t_0}^{t}e^{-\sin(t)}e^{\lambda+\sin(\lambda)}x(\lambda)d\lambda$$

2.10) For each of the following first order differential equations, use an integrating factor to write $y(t)$ as a function of its initial value $y(t_0)$ and an integral of the input (plus some other functions)

$$\dot{y}(t) = ay(t) + bx(t) \qquad \dot{y}(t) = aty(t) + bx(t)$$

$$\dot{y}(t) + \frac{1}{t}y(t) = x(t) \qquad \dot{y}(t) + \frac{1}{t^2}y(t) = x(t-1)$$

Answers:

$$y(t) = e^{a(t-t_0)}y(t_0) + \int_{t_0}^{t}be^{a(t-\lambda)}x(\lambda)d\lambda \qquad y(t) = e^{\frac{a}{2}(t^2-t_0^2)}y(t_0) + \int_{t_0}^{t}be^{\frac{a}{2}(t^2-\lambda^2)}x(\lambda)d\lambda$$

$$y(t) = \left(\frac{t_0}{t}\right)y(t_0) + \int_{t_0}^{t}\left(\frac{\lambda}{t}\right)x(\lambda)d\lambda \qquad y(t) = e^{\left(\frac{1}{t}-\frac{1}{t_0}\right)}y(t_0) + \int_{t_0}^{t}e^{\left(\frac{1}{t}-\frac{1}{\lambda}\right)}x(\lambda-1)d\lambda$$

3.0 Second Order Circuits

A second order circuit is a circuit with two effective energy storage elements, either two capacitors, two inductors, or one of each. (In some circuits it may be possible to combine multiple capacitors or inductors into one equivalent capacitor or inductor.) We begin this section with the derivation of the governing differential equation for various second order circuits. At this point we will focus on circuits that we can put into a standard form. Once we have covered Laplace transforms we will analyze different types of second order circuits. This standard second order form will again allow us to easily determine physical characteristics of the circuit and predict the time response. We then solve the differential equations for the case of a constant input.

3.1 Governing Differential Equations for Second Order Circuits: Standard Form

In this section we derive the governing differential equations that model various RL, RC, and RLC circuits. We then put the governing second order differential equations into a standard form, which allows us to read off descriptive information about the system very easily. The standard form we will use is

$$\frac{d^2 y(t)}{dt} + 2\zeta\omega_n \frac{dy(t)}{dt} + \omega_n^2 y(t) = K\omega_n^2 x(t)$$

or

$$\frac{1}{\omega_n^2}\frac{d^2 y(t)}{dt} + \frac{2\zeta}{\omega_n}\frac{dy(t)}{dt} + y(t) = Kx(t)$$

Here we assume the *system input* is $x(t)$ and the *system output* is $y(t)$. ω_n is the system *natural frequency*, which indicates the frequency at which the system will oscillate if there is no dampling. The natural frequency ω_n has units of radians/second. ζ is the damping ratio, which indicates how much damping there is in the system. A damping ratio of zero indicates there is no damping at all. The damping ratio ζ is dimensionless. K is the *static gain* of the system. For a constant input of amplitude A ($x(t) = Au(t)$, where $u(t)$ is the unit step function), in steady state we have $\frac{dy(t)}{dt} = 0$, $\frac{d^2 y(t)}{dt^2} = 0$, and $y(t) = Kx(t) = KA$. Hence the static gain lets us easily compute the steady state value of the output. To determine the units of the static gain we use

[units of y] = [units of K][units of x]

or

[units of K] = [units of y]/[units of x]

Note that not all second order circuits can be modeled by a differential equation of this form. While we can always write the left hand side of the differential equation in this form, for some circuits the right hand side of the differential equation may contain derivatives of the inputs. In addition, this form may not always be the best way to write the differential equation.

Example 3.2.1. Consider the RLC circuit shown in Figure 3.1. The input, $x(t)$, is the applied voltage and the output, $y(t)$, is the voltage across the capacitor. If we denote the current flowing in the circuit as $i(t)$, then applying Kirchhoff's voltage law around the single loop gives us the equation

$$-x(t) + L\frac{di(t)}{dt} + y(t) + i(t)R = 0$$

We can also relate the voltage across the capacitor with the current flowing through the capacitor

$$i(t) = C\frac{dy(t)}{dt}$$

Substituting this equation into our first expression we get

$$-x(t) + LC\frac{d^2 y(t)}{dt^2} + y(t) + RC\frac{dy(t)}{dt} = 0$$

or

$$LC\frac{d^2 y(t)}{dt^2} + RC\frac{dy(t)}{dt} + y(t) = x(t)$$

Comparing this expression with our standard form we get

$$\text{natural frequency: } \frac{1}{\omega_n^2} = LC \text{ , or } \omega_n = \frac{1}{\sqrt{LC}}$$

$$\text{damping ratio: } \frac{2\zeta}{\omega_n} = RC \text{ , or } \zeta = \frac{R}{2}\sqrt{\frac{C}{L}}$$
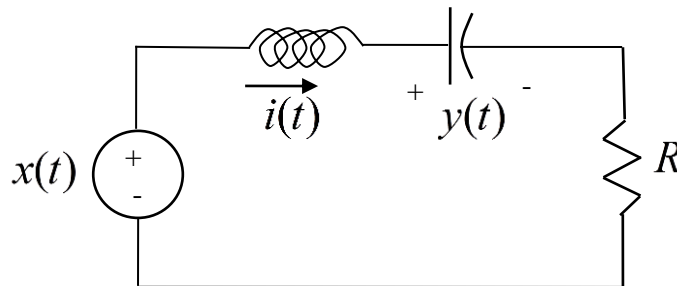
$$\text{static gain: } K = 1$$

Figure 3.1. Circuit for Example 3.2.1.

Example 3.2.2. Consider the RLC circuit shown in Figure 3.2. The input, $x(t)$, is the applied current and the output, $y(t)$, is the current through the inductor. If we denote the node voltage at the top of the circuit as $v^*(t)$, then applying Kirchhoff's current law give us

$$x(t) = \frac{v^*(t)}{R} + y(t) + C\frac{dv^*(t)}{dt}$$

We can also relate the voltage across the inductor with the current flowing through the inductor

$$v^*(t) = L\frac{dy(t)}{dt}$$

Substituting this equation into our first expression we get

$$x(t) = \frac{L}{R}\frac{dy(t)}{dt} + y(t) + LC\frac{d^2 y(t)}{dt^2}$$

or

$$LC\frac{d^2 y(t)}{dt^2} + \frac{L}{R}\frac{dy(t)}{dt} + y(t) = x(t)$$

Comparing this expression with our standard form we get

$$\text{natural frequency: } \frac{1}{\omega_n^2} = LC \text{, or } \omega_n = \frac{1}{\sqrt{LC}}$$

$$\text{damping ratio: } \frac{2\zeta}{\omega_n} = \frac{L}{R} \text{, or } \zeta = \frac{1}{2R}\sqrt{\frac{L}{C}}$$
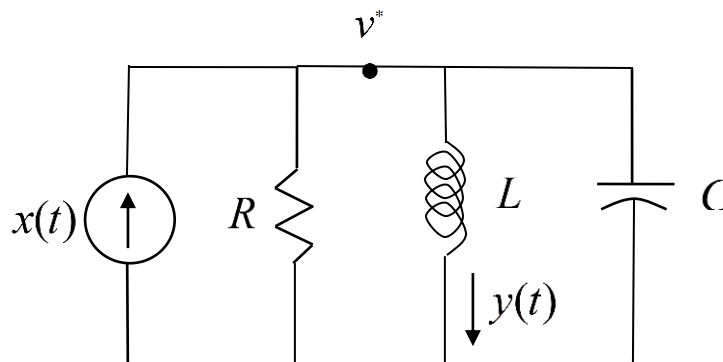
$$\text{static gain: } K = 1$$

Figure 3.2. Circuit used in Example 3.2.2.

Example 3.2.3. Consider the RLC circuit shown in Figure 3.3. The input, $x(t)$, is the applied current and the output, $y(t)$, is the current through the inductor. If we denote the node voltage at the top of the circuit as $v^*(t)$, then applyingKirchhoff's current law give us

$$x(t) = y(t) + C\frac{dv^*(t)}{dt}$$

We can then determine the node voltage $v^*(t)$ as

$$v^*(t) = Ry(t) + L\frac{dy(t)}{dt}$$

Substituting this equation into our first expression we get

$$x(t) = y(t) + C\frac{d}{dt}[Ry(t) + L\frac{dy(t)}{dt}] = y(t) + RC\frac{dy(t)}{dt} + LC\frac{d^2y(t)}{dt^2}$$

or

$$LC\frac{d^2y(t)}{dt^2} + RC\frac{dy(t)}{dt} + y(t) = x(t)$$

Comparing this expression with our standard form we get

$$\text{natural frequency: } \frac{1}{\omega_n^2} = LC \text{, or } \omega_n = \frac{1}{\sqrt{LC}}$$

$$\text{damping ratio: } \frac{2\zeta}{\omega_n} = RC \text{, or } \zeta = \frac{R}{2}\sqrt{\frac{C}{L}}$$
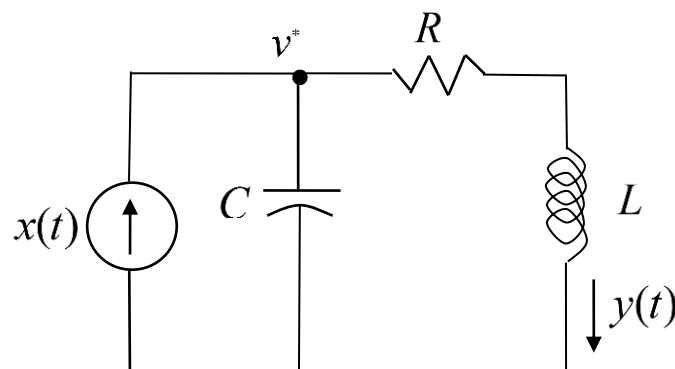
$$\text{static gain: } K = 1$$

Figure 3.3. Circuit used in Example 3.2.3.

Example 3.2.4. Consider the RLC circuit shown in Figure 3.4. The input, $x(t)$, is the applied voltage and the output, $y(t)$, is the voltage across resistor R_b. Node voltages $v_a(t)$ and $v_b(t)$ are as shown in the figure. Applying Kirchhoff's current law gives at node $v_a(t)$ gives

$$\frac{v_a(t) - x(t)}{R} + \frac{v_a(t) - v_b(t)}{R} + \frac{v_a(t)}{R} + C_a\frac{dv_a(t)}{dt} = 0$$

which we can simplify as

$$3v_a(t) - x(t) - v_b(t) = -RC_a \frac{dv_a(t)}{dt}$$

Summing the currents into the negative terminal of the op amp gives us

$$\frac{v_a(t)}{R} + C_b \frac{dv_b(t)}{dt} = 0$$

or

$$v_a(t) = -RC_b \frac{dv_b(t)}{dt}$$

Substituting this expression into our simplified equation above we get

$$3\left[-RC_b \frac{dv_b(t)}{dt} \right] - x(t) - v_b(t) = -RC_a \frac{d}{dt} \left[-RC_b \frac{dv_b(t)}{dt} \right]$$

or

$$R^2 C_a C_b \frac{d^2 v_b(t)}{dt^2} + 3RC_b \frac{dv_b(t)}{dt} + v_b(t) = -x(t)$$

Finally, we have

$$y(t) = v_b(t) \frac{R_b}{R_a + R_b}$$

or

$$v_b(t) = \frac{R_a + R_b}{R_b} y(t)$$

resulting in the differential equation

$$R^2 C_a C_b \frac{d^2 y(t)}{dt^2} + 3RC_b \frac{dy(t)}{dt} + y(t) = -\frac{R_b}{R_a + R_b} x(t)$$

Comparing this expression with our standard form we get

$$\text{natural frequency: } \frac{1}{\omega_n^2} = R^2 C_a C_b \text{, or } \omega_n = \frac{1}{R\sqrt{C_a C_b}}$$

$$\text{damping ratio: } \frac{2\zeta}{\omega_n} = 3RC_b \text{, or } \zeta = \frac{3}{2}\sqrt{\frac{C_b}{C_a}}$$
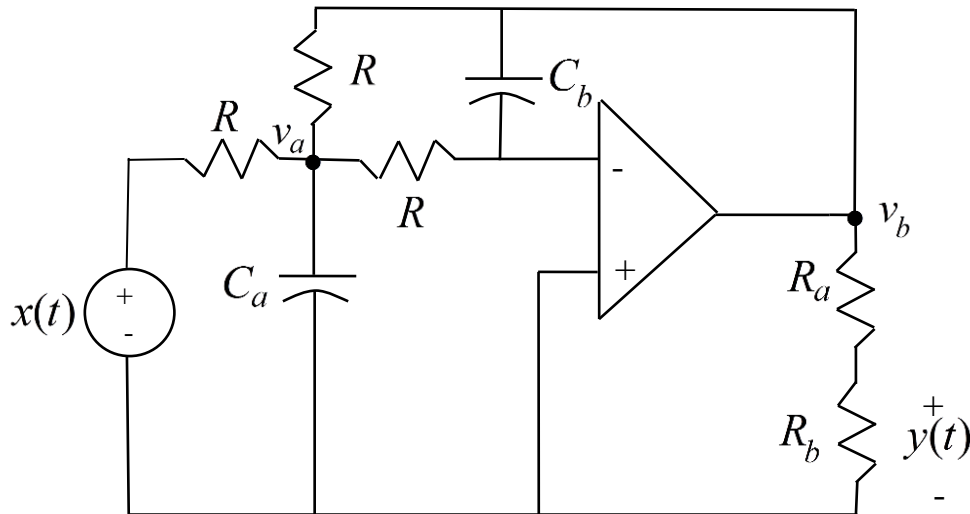
$$\text{static gain: } K = -\frac{R_b}{R_a + R_b}$$

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Figure 3.4. Circuit used in Example 3.2.4.

3.3 Solving Second Order Differential Equations in Standard Form

In this section we will solve second order differential equations the standard form

$$\frac{d^2 y(t)}{dt^2} + 2\zeta\omega_n \frac{dy(t)}{dt} + \omega_n^2 y(t) = K\omega_n^2 x(t)$$

for a constant (step) input. We will solve this equation in two parts. We will first determine the *natural response*, $y_n(t)$. The natural response is the response due only to initial conditions when no inputs are present. Then we will determine the *forced response*, $y_f(t)$. The forced response is the response due to the input only, assuming all initial conditions are zero. The total response is then the sum of the natural and forced responses,

$$y(t) = y_n(t) + y_f(t).$$

3.3.1 Natural Response. To determine the natural response we assume there is no input in the system, so we have the equation

$$\frac{d^2 y_n(t)}{dt^2} + 2\zeta\omega_n \frac{dy_n(t)}{dt} + \omega_n^2 y_n(t) = 0$$

Let's assume a solution of the form $y_n(t) = c\,e^{rt}$, where c and r are parameters to be determined. Substituting this assumption into the differential equation we get

$$r^2 c e^{rt} + 2\zeta\omega_n r c e^{rt} + \omega_n^2 c e^{rt} = 0$$

or

$$c e^{rt}[r^2 + 2\zeta\omega_n r + \omega_n^2] = 0$$

If $c = 0$ then we are done, and the natural response will be $y_n(t) = 0$. This solution certainly satisfies the differential equation. However, if $c \neq 0$, and since e^{rt} can never be zero, we must have

$$r^2 + 2\zeta\omega_n r + \omega_n^2 = 0$$

Using the quadratic formula, the roots of this equation are

$$r = \frac{-2\zeta\omega_n \pm \sqrt{(2\zeta\omega_n)^2 - 4\omega_n^2}}{2} = -\zeta\omega_n \pm \sqrt{\zeta^2\omega_n^2 - \omega_n^2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

We now have four cases to consider depending on the value of the damping ratio ζ. These four cases are: over damped ($\zeta > 1$), critically damped ($\zeta = 1$), undamped ($\zeta = 0$), and under damped ($0 < \zeta < 1$). We will consider each of these in turn.

Overdamped ($\zeta > 1$). In this case we have two real and distinct roots,

$$r_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$$
$$r_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$$

The natural response is then

$$y_n(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

where c_1 and c_2 are constants to be determined by the initial conditions. Note that both r_1 and r_2 are always negative, since $\zeta > \sqrt{\zeta^2 - 1}$.

Critically Damped ($\zeta = 1$). In this case we initially appear to have only one solution,

$$r = r_1 = r_2 = -\omega_n$$

From differential equations, we know that in this situation we should look for an additional solution of the form

$$y_n(t) = cte^{rt}$$

Let's check to see if this works. With $\zeta = 1$, the differential equation becomes

$$\frac{d^2 y_n(t)}{dt^2} + 2\omega_n \frac{dy_n(t)}{dt} + \omega_n^2 y_n(t) = 0$$

We have then

$$\frac{dy_n(t)}{dt} = \frac{d}{dt}(cte^{rt}) = ce^{rt} + ctre^{rt}$$

$$\frac{d^2 y_n(t)}{dt^2} = \frac{d^2}{dt^2}(cte^{rt}) = \frac{d}{dt}\left[ce^{rt} + ctre^{rt}\right] = cre^{rt} + cre^{rt} + ctr^2 e^{rt} = 2cre^{rt} + ctr^2 e^{rt}$$

Substituting these into the differential equation we get

$$[2cre^{rt} + ctr^2e^{rt}] + 2\omega_n[ce^{rt} + ctre^{rt}] + \omega_n^2[ctre^{rt}] = cte^{rt}[r^2 + 2\omega_n r + \omega_n^2] + 2ce^{rt}[r + \omega_n] = 0$$

Since we know $r = -\omega_n$ this equation is clearly satisfied. Hence our natural solution in this case will be of the form

$$y_n(t) = c_1 e^{rt} + c_2 t e^{rt}$$

where c_1 and c_2 are constants to be determined by the initial conditions.

Undamped ($\zeta = 0$). In this case there is no damping, and the system oscillates at frequency ω_n. The natural response is of the form

$$y_n(t) = c \sin(\omega_n t + \phi)$$

where c and ϕ are constants to be determined by the initial conditions.

Under Damped ($0 < \zeta < 1$). In this case will have two complex conjugate roots, which we can write as

$$r = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -\zeta\omega_n \pm j\omega_d$$

$\omega_d = \omega_n\sqrt{1-\zeta^2}$ is the *damped frequency*. This is the frequency this system will oscillate with. As we go on, it will be usually easier to remember the roots of this equation as

$$r = -\zeta\omega_n \pm j\omega_d = -\sigma \pm j\omega_d$$

It is also useful at this point to examine the roots of this equation in the complex plane and see what we can determine. Figure 3.5 illustrates the relationships we will be discussing.

First of all, since our equation for the roots is real we must have complex conjugate roots to the equation, which the figure shows. If we look at the magnitude of the roots, we get

$$|r|^2 = r \times r^* = (-\zeta\omega_n)^2 + (\omega_n\sqrt{1-\zeta^2})^2 = \zeta^2\omega_n^2 + \omega_n^2 - \omega_n^2\zeta^2 = \omega_n^2$$

So the roots will all lie on a circle with magnitude $|r| = \omega_n$. Secondly, if we look at the angle made with the negative real axis, we can see that

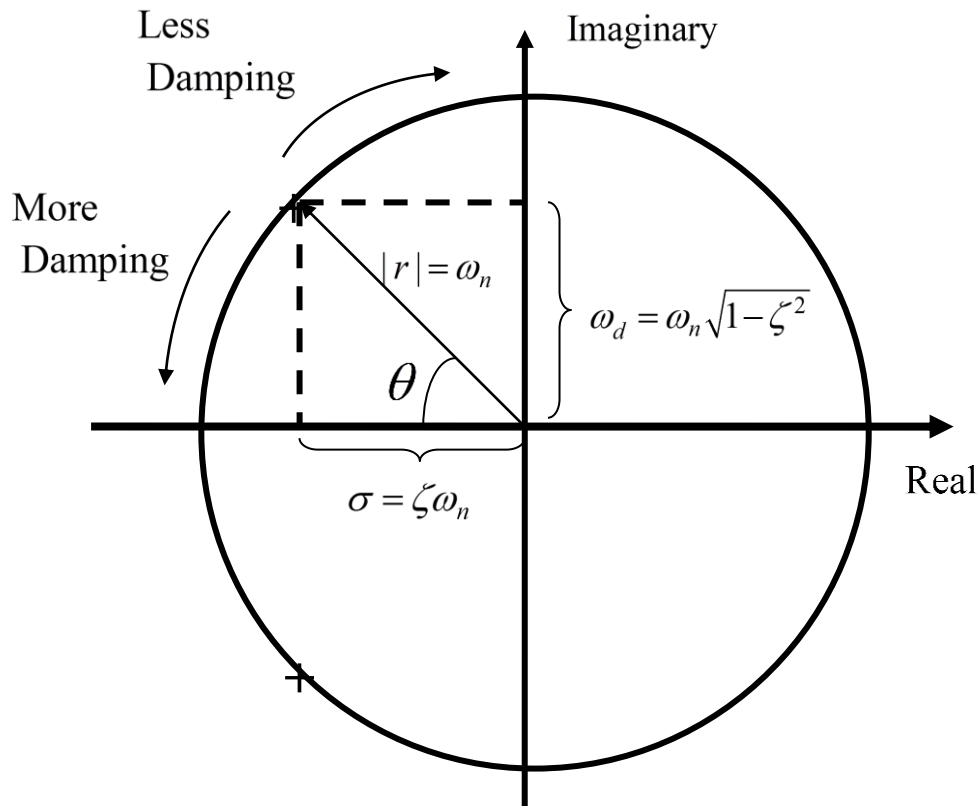$$\cos(\theta) = \frac{\zeta\omega_n}{\omega_n} = \zeta$$

Figure 3.5. Relationship between the location of the complex roots(+) and the natural frequency (ω_n, the magnitude of the roots) , the damped frequency ($\omega_d = \omega_n\sqrt{1-\zeta^2}$, the imaginary part of the root and the frequency of oscillation), and the damping ratio (ζ , $\cos(\theta) = \zeta$). When $\zeta = 0$ (*undamped*) the angle $\theta = 90^o$ and the roots are purely imaginary. In this case $\omega_d = \omega_n$ and the system just oscillates at the natural frequency. When $\zeta = 1$ (critically damped) the angle $\theta = 0^o$ and the roots are purely real and are repeated. If $\zeta \geq 1$ both roots are real, not repeated, and are on the real axis. In this case $\omega_d = 0$ and there is no oscillation. Between these extremes we have an *under damped* system ($0 < \zeta < 1$).

When $\zeta = 0$ (*undamped*) the angle $\theta = 90^o$ and the poles are purely imaginary. In this case $\omega_d = \omega_n$ and the system just oscillates at the natural frequency. When $\zeta = 1$ (critically damped) the angle $\theta = 0^o$ and the poles are purely real. In this case $\omega_d = 0$ and there is no oscillation. Between these two extremes we have an *under damped* system ($0 < \zeta < 1$).

Our solution at this point for the natural response to the under damped system is then

$$y_n(t) = c_1 e^{(-\zeta\omega_n + j\omega_d)t} + c_2 e^{(-\zeta\omega_n - j\omega_d)t} = e^{-\zeta\omega_n t}\left[c_1 e^{j\omega_d t} + c_2 e^{-j\omega_d t}\right]$$

Since we want a real valued solution, let's make an assumption about the relationship between the two unknown constants. Assume

$$c_1 = \frac{c}{2j} e^{j\phi}$$

$$c_2 = \frac{-c}{2j} e^{-j\phi}$$

Then we have

$$y_n(t) = \frac{c}{2j} e^{-\zeta\omega_n t} \left[e^{j(\omega_d t + \phi)} - e^{-j(\omega_d t + \phi)} \right]$$

Finally we expand this out using Euler's identity to get

$$y_n(t) = \frac{c}{2j} e^{-\zeta\omega_n t} \left[\{\cos(\omega_d t + \phi) + j\sin(\omega_d t + \phi)\} - \{\cos(\omega_d t + \phi) - j\sin(\omega_d t + \phi)\} \right]$$

or

$$y_n(t) = ce^{-\zeta\omega_n t} \sin(\omega_d t + \phi)$$

where c and ϕ are constants to be determined by the initial conditions.

3.3.2 Forced Response.

To determine the forced response we must know the system input, $x(t)$. For now we will assume an input that is zero before $t = 0$ and then has constant amplitude A for $t \geq 0$,

$$x(t) = \begin{cases} 0 & t < 0 \\ A & t \geq 0 \end{cases}$$

Then for $t \geq 0$ we have the equation

$$\frac{d^2 y_n(t)}{dt^2} + 2\omega_n \frac{dy_n(t)}{dt} + \omega_n^2 y_n(t) = KA$$

Since this is a linear ordinary differential equation we only need to find one solution. One obvious solution to this equation is the solution in steady state, when $\frac{d^2 y_f(t)}{dt^2} = \frac{dy_f(t)}{dt} = 0$. In steady state we have

$$y_f(t) = KA$$

Note that for a constant input, the steady state output is the product of the static gain and the amplitude of the input.

3.3.3 Total Solution. The total solution to our differential equation is the sum of the natural and forced responses, which is summarized below:

$\zeta = 0$	*undamped*	*imaginary roots*	$y(t) = KA + c\sin(\omega_n t + \phi)$
$0 < \zeta < 1$	*under damped*	*complex conjugate roots*	$y(t) = KA + ce^{-\zeta\omega_n t}\sin(\omega_d t + \phi)$
$\zeta = 1$	*crititally damped*	*real, repeated roots*	$y(t) = KA + c_1 e^{-\omega_n t} + c_2 t e^{-\omega_n t}$
$\zeta > 1$	*over damped*	*real, distinct roots*	$y(t) = KA + c_1 e^{-(\zeta\omega_n + \omega_n\sqrt{\zeta^2-1})t} + c_2 e^{-(\zeta\omega_n - \omega_n\sqrt{\zeta^2-1})t}$

3.4 Response of Under Damped Systems at Rest

For the under damped case, we have the solution

$$y(t) = KA + ce^{-\zeta\omega_n t}\sin(\omega_d t + \phi)$$

We will determine the solution assuming the system is initially at rest ($y(0) = 0$ and $\dot{y}(0) = 0$). Let's look first at the derivative term,

$$\frac{dy(t)}{dt} = ce^{-\zeta\omega_n t}(-\zeta\omega_n)\sin(\omega_d t + \phi) + ce^{-\zeta\omega_n t}(\omega_d)\cos(\omega_d t + \phi)$$

At the initial time ($t = 0$) we will have

$$\dot{y}(0) = -\zeta\omega_n \sin(\phi) + \omega_d \cos(\phi) = 0$$

or

$$\frac{\sin(\phi)}{\cos(\phi)} = \tan(\phi) = \frac{\omega_d}{\zeta\omega_n} = \frac{\omega_n\sqrt{1-\zeta^2}}{\zeta\omega_n} = \frac{\sqrt{1-\zeta^2}}{\zeta}$$

Hence

$$\phi = \tan^{-1}\left[\frac{\sqrt{1-\zeta^2}}{\zeta}\right]$$

From this we can determine that the hypotenuse of the triangle is $r = 1$, so that $\sin(\phi) = \sqrt{1-\zeta^2}$ and $\cos(\phi) = \zeta$.

Next we use the initial condition that the initial position is zero,

$$y(0) = 0 = KA + c\sin(\phi)$$

or

$$c = -\frac{KA}{\sin(\phi)} = -\frac{KA}{\sqrt{1-\zeta^2}}$$

Finally, our solution for $y(t)$ is

$$y(t) = KA\left[1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_d t + \phi)\right]$$

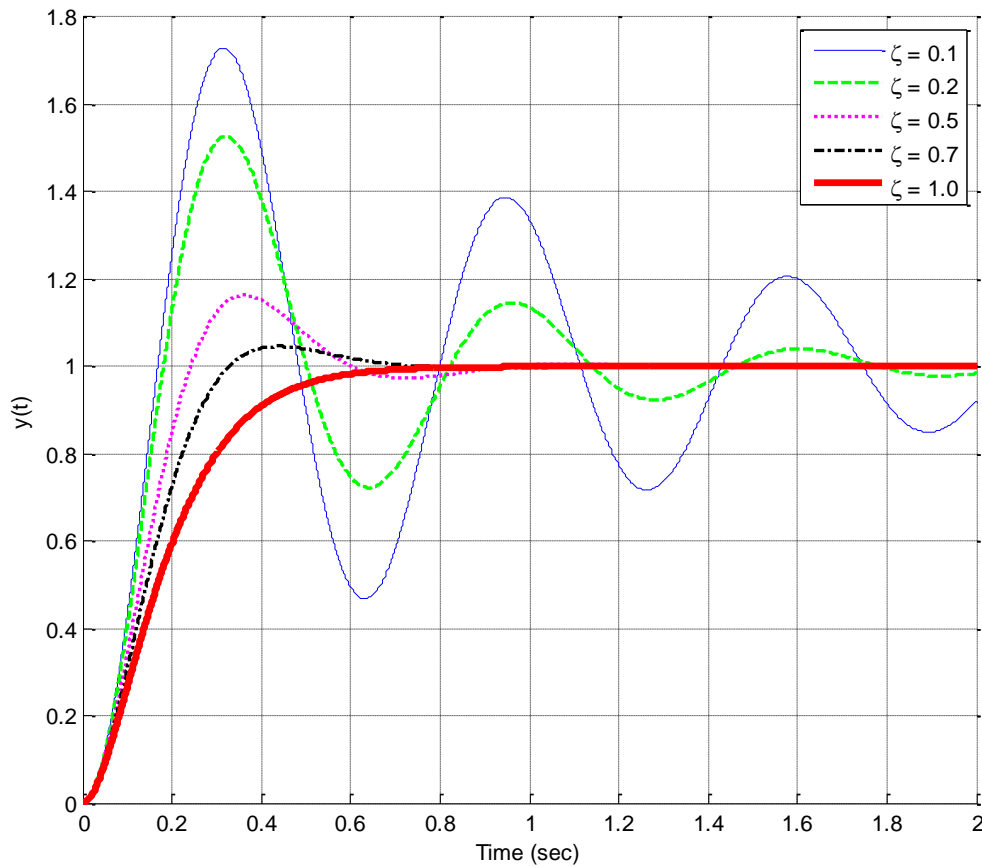$$\phi = \tan^{-1}\left[\frac{\sqrt{1-\zeta^2}}{\zeta}\right]$$

Figure 3.6 shows the response (output) of a system initially at rest. For this system $KA = 1$ and $\omega_n = 10$ rad/sec. There are four responses for under damped systems ($0 < \zeta < 1.0$) and one response for a critically damped system ($\zeta = 1.0$). The only thing changing in the responses shown in Figure 3.6 is the damping ratio, so it should be clear that the damping ratio can affect the response of the system quite a lot. For example, the peak amplitude and time at which the system reaches the peak amplitude is different for the different responses. Similarly, the frequency at which the system oscillates, the damped frequency, is different for the different responses. Finally, the time it takes the system to reach steady state is different for the different responses.

For systems which fit into our standard second order form, we can predict the response of the system and characterize the response in terms of our parameters (ω_n, ζ, K). The most common characterizations are depicted in Figure 3.7, which shows the response of a second order system ($\omega_n = 10 \, rad/sec$, $\zeta = 0.15$, $K = 2.0$) initially at rest to a unit step input (an input of constant amplitude 1.0 starting at time zero). Typical characterizations of second order systems shown in the figure are (1) the time to peak (T_p), the time it takes the output to reach its peak value; (2) the percent overshoot (PO), which indicates the amount the largest peak of the output overshoots the final (steady state) value of the output; and (3) the settling time (T_s), which indicates the time it takes for the transients in the output to settling out. After the settling time the system output remains within $\pm 2\%$ of the final (steady state) value. (This is a 2% definition of settling time, 1% definitions are also used, though the 1% definition is not as commonly used.)   In the following sections we indicate how we can determine these quantities in terms of the parameters that characterize our system.

3.4.1 Time to Peak. From our solution to the response of our under damped second order system to a step input, we can determine the time at which $y(t)$ reaches its peak value by taking the derivative of $y(t)$ and setting it equal to zero. This will give us the maximum value of $y(t)$ and the time this occurs at will be called the time to peak, T_p.

$$\frac{dy(t)}{dt} = -\frac{KA}{\sqrt{1-\zeta^2}}\left[-\zeta\omega_n \sin(\omega_d t + \phi) + \omega_d \cos(\omega_d t + \phi)\right] = 0$$

$$\zeta\omega_n \sin(\omega_d t + \phi) = \omega_d \cos(\omega_d t + \phi)$$

$$\tan(\omega_d t + \phi) = \frac{\omega_d}{\zeta\omega_n} = \frac{\omega_n\sqrt{1-\zeta^2}}{\zeta\omega_n} = \frac{\sqrt{1-\zeta^2}}{\zeta}$$

$$\omega_d t + \phi = \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta} = \phi$$

Hence $\omega_d t$ at the time to peak, $t = T_p$, must equal one period of the tangent, which is π, so

$$T_p = \frac{\pi}{\omega_d}$$

Remember that ω_d is equal to the imaginary part of the complex roots of

$$r^2 + 2\zeta\omega_n r + \omega_n^2 = 0$$

Figure 3.6. The response (output) of a system initially at rest. For this system $KA = 1$ and $\omega_n = 10$ rad/sec. There are four responses for under damped systems ($0 < \zeta < 1.0$) and one response for a critically damped system ($\zeta = 1.0$). The only thing changing in the responses is the damping ratio.

3.4.2 Percent Overshoot. Evaluating $y(t)$ at the peak time T_p we get the maximum value of $y(t)$,

$$y(T_p) = KA\left[1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n T_p} \sin(\omega_d T_p + \phi)\right]$$

$$y(T_p) = KA\left[1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n \frac{\pi}{\omega_d}} \sin(\omega_d \frac{\pi}{\omega_d} + \phi)\right]$$

$$y(T_p) = KA\left[1 + \frac{1}{\sqrt{1-\zeta^2}} e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \sin(\phi)\right]$$

We get the last equation by using the fact that $\sin(\phi+\pi) = -\sin(\phi)$. Finally, since we have previously determined that $\sin(\phi) = \sqrt{1-\zeta^2}$,

$$y(T_p) = KA\left[1 + e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}\right]$$

The percent overshoot is defined as

$$Percent\ Overshoot = PO = \frac{y(T_p) - y(\infty)}{y(\infty)} \times 100\%$$

Note that this is a standard definition for percent overshoot, independent of the system order or type of system we are analyzing. Note also that *the reference level is the value of the function in steady state.*

For our under damped second order system we have $y(\infty) = KA$, so we have

$$PO = \frac{KA\left[1 + e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}\right] - KA}{KA} \times 100\%$$

or

$$PO = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100\%$$

3.4.3 Settling Time. The settling time of a system is defined as the time it takes for the output of a system with a step input to stay within a given percentage of its final value. We will use the 2% settling time criteria, which is generally four time constants, $T_s = 4\tau$. For any exponential decay, the general form is written as $e^{-t/\tau}$, where τ is the time constant. Functions of the form $e^{-t/\tau} \cos(\omega_d t + \phi)$ or

$e^{-t/\tau}\sin(\omega_d t + \phi)$ such as we have in our solution will oscillate, but will still decay at the same rate as the exponential alone. For our system we have $\tau = \dfrac{1}{\zeta\omega_n}$ or $\sigma = \zeta\omega_n$, where σ is the absolute value of the real parts of the solutions to $r^2 + 2\zeta\omega_n r + \omega_n^2 = 0$. Hence for our system we estimate the settling time as

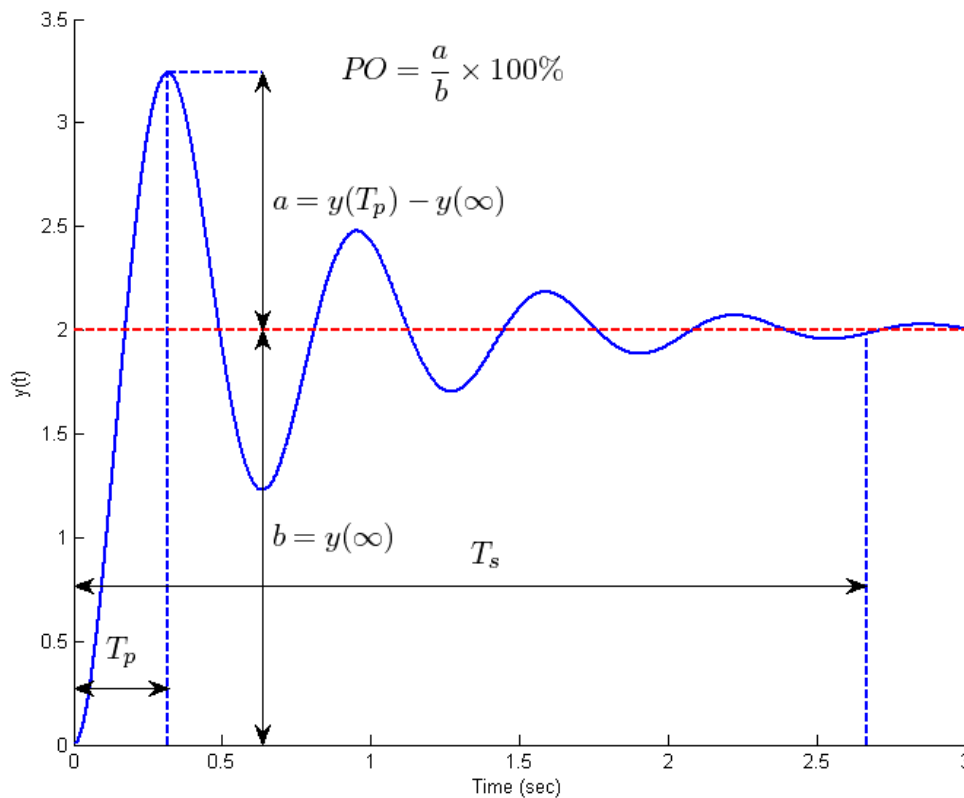$$T_s = 4\tau = \frac{4}{\zeta\omega_n} = \frac{4}{\sigma}$$

Figure 3.7. The response (output) of a system initially at rest. For this system $KA = 2$, $\omega_n = 10$ rad/sec, and $\zeta = 0.15$. Typical characterizations of second order systems (1) the time to peak (T_p), the time it takes the output to reach its peak value; (2) the percent overshoot (PO), which indicates the amount the largest peak of the output overshoots the final (steady state) value of the output ; and (3) the settling time (T_s), which indicates the time it takes for the transients in the output to settling out. After the settling time the system output remains within $\pm 2\%$ of the final (steady state) value.

3.5 Second Order System Examples

Example 3.5.1. Consider the circuit used in Example 3.2.2 with parameter values, $L = 10 \text{ mH}$, $C = 10 \mu\text{F}$, $R = 40 \Omega$. Assume the input is 0.5 amp step, $x(t) = 0.5u(t)$. We can then determine the parameters:

$$\omega_n = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(10 \times 10^{-3})(10 \times 10^{-6})}} = 3,162.2 \text{ rad / sec}$$

$$\zeta = \frac{1}{2R}\sqrt{\frac{L}{C}} = \frac{1}{(2)(40)}\sqrt{\frac{10 \times 10^{-3}}{10 \times 10^{-6}}} = 0.395$$

$$K = 1$$

$$T_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{3,162.2\sqrt{1-0.395^2}} = 0.011 \text{ sec}$$

$$T_s = \frac{4}{\zeta\omega_n} = \frac{4}{(0.395)(3162.2)} = 0.032 \text{ sec}$$

$$PO = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100\% = e^{\frac{-0.395\pi}{\sqrt{1-0.395^2}}} \times 100\% \approx 26\%$$

The time domain expression for the output is then given by

$$\phi = \tan^{-1}\left[\frac{\sqrt{1-\zeta^2}}{\zeta}\right] = \tan^{-1}\left[\frac{\sqrt{1-0.395^2}}{0.395}\right] = \tan^{-1}[2.3257] = 1.1647 \text{ rad}$$

$$y(t) = KA\left[1 - \frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin(\omega_d t + \phi)\right] = 0.5[1 - 1.09e^{-1249.07t}\sin(2905.05t + 1.1647)]$$

The response for this system is shown in Figure 3.8. The time to peak, settling time, and percent overshoot are displayed on the figure.
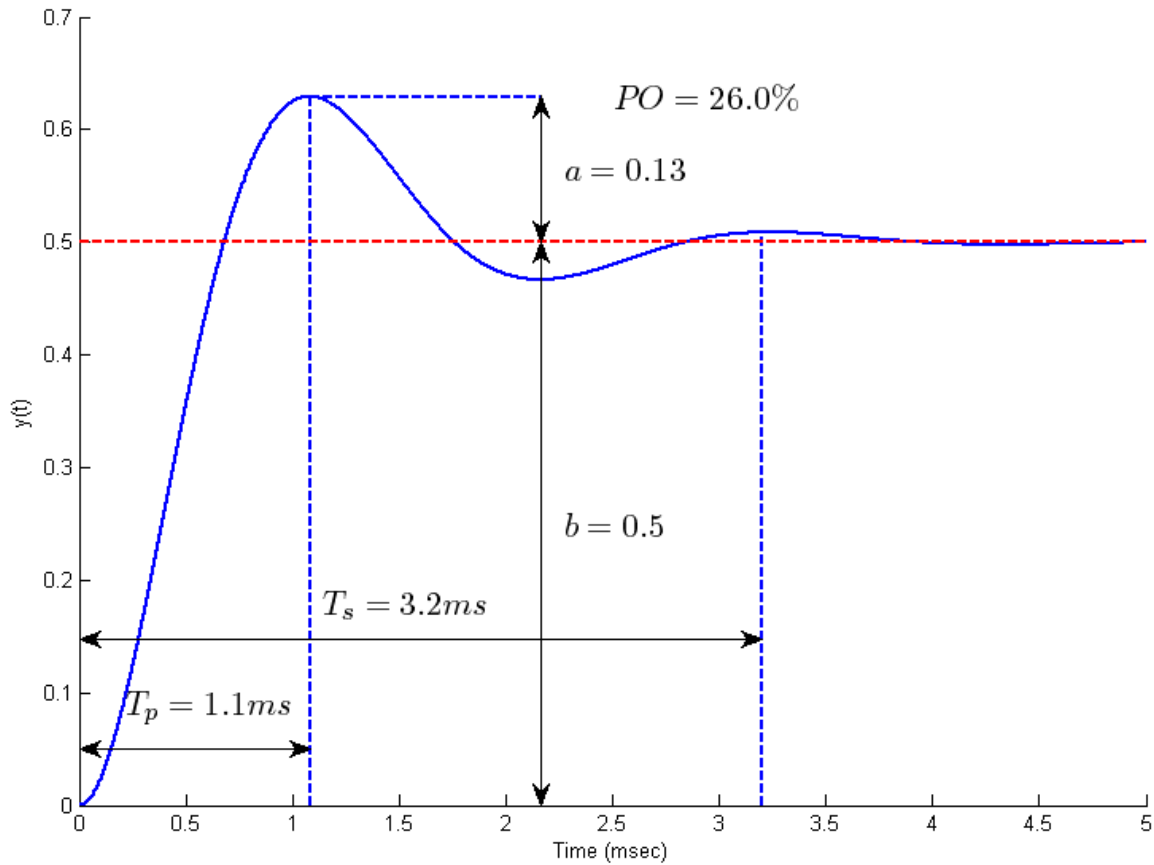
Figure 3.8. Response of the circuit used in Example 3.5.1 with parameter values $L = 10\,\text{mH}$, $C = 10\,\mu F$, $R = 40\,\Omega$. Assume the input is 0.5 amp step.

Example 3.5.2. Consider the circuit used in Example 3.2.4 with parameter values, $C_a = 10\,\mu F$, $C_b = 1\,\mu F$, $R = 1\,k\Omega$, $R_a = 3\,k\Omega$, $R_b = 2\,k\Omega$. Assume the input is a 1 volt step, $x(t) = u(t)$. We can then determine the parameters:

$$\omega_n = \frac{1}{R\sqrt{C_a C_b}} = \frac{1}{1\times10^3 \sqrt{(10\times10^{-6})(1\times10^{-6})}} = 316.2\,\text{rad/sec}$$

$$\zeta = \frac{3}{2}\sqrt{\frac{C_b}{C_a}} = \frac{3}{2}\sqrt{\frac{1\times10^{-6}}{10\times10^{-6}}} = 0.47$$

$$K = -\frac{R_b}{R_a + R_b} = -\frac{2\times10^3}{3\times10^3 + 2\times10^3} = -0.4$$

$$T_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{316.2\sqrt{1-0.47^2}} = 0.011\,\text{sec}$$

$$T_s = \frac{4}{\zeta\omega_n} = \frac{4}{(0.47)(316.2)} = 0.027 \text{ sec}$$

$$PO = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100\% = e^{\frac{-0.47\pi}{\sqrt{1-0.47^2}}} \times 100\% \approx 18.5\%$$

The time domain expression for the output is then given by

$$\phi = \tan^{-1}\left[\frac{\sqrt{1-\zeta^2}}{\zeta}\right] = \tan^{-1}\left[\frac{\sqrt{1-0.47^2}}{0.47}\right] = \tan^{-1}[1.657] = 1.0815 \text{ rad}$$

$$y(t) = KA\left[1 - \frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin(\omega_d t + \phi)\right] = -0.4[1 - 1.1329e^{-148.6t}\sin(279.1t + 1.0815)]$$

The response for this system is shown in Figure 3.9. Note that although the input is positive, the output has a negative steady state value. This is because the static gain is negative. However, all of the parameters we are interested in (time to peak, settling time, or percent overshoot) are still measured in the same way.

Example 3.5.3. Consider the response of an unknown second order system, with step input 4 volts, $x(t) = 4u(t)$. The measured output is also in volts. The response of this unknown system is shown in Figure 3.10. We want to try and determine the system parameters from the system output.

For this system, the steady state value is $y_{ss} = y(\infty) = b = 1.5$. Since we know the input was 4, we have

$KA = y_{ss} = K(4) = 1.5$, or $K = \frac{y_{ss}}{4} = \frac{1.5}{4} = 0.375$. The 2% settling time T_s occurs when

$|y(t) - y_{ss}| < 0.02$ for all $t \geq T_s$. Based on this graph, this occurs somewhere near 0.26 seconds, so $T_s \approx 0.26 \text{ sec}$. The time to peak T_p can be measured off the graph to be approximately 0.11 seconds, so $T_p \approx 0.11 \text{ sec}$. To determine the percent overshoot, we first need the steady state value of the output, which we have determined, and then how much beyond this the system has travelled. For this system we have $y(\infty) = y_{ss} = b = 1.5$ and $y(T_p) - y(\infty) = a \approx 1.75 - 1.5 \approx 0.25$.

Hence the percent overshoot is given by $PO = \frac{a}{b} \times 100\% = \frac{0.25}{1.5} \times 100\% \approx 17\%$. *It is important to remember that the percent overshoot is measured from the steady state value of the output, not from the value of the input.* We can now use these parameters, or various combinations of them, to determine the damping ratio and natural frequency.
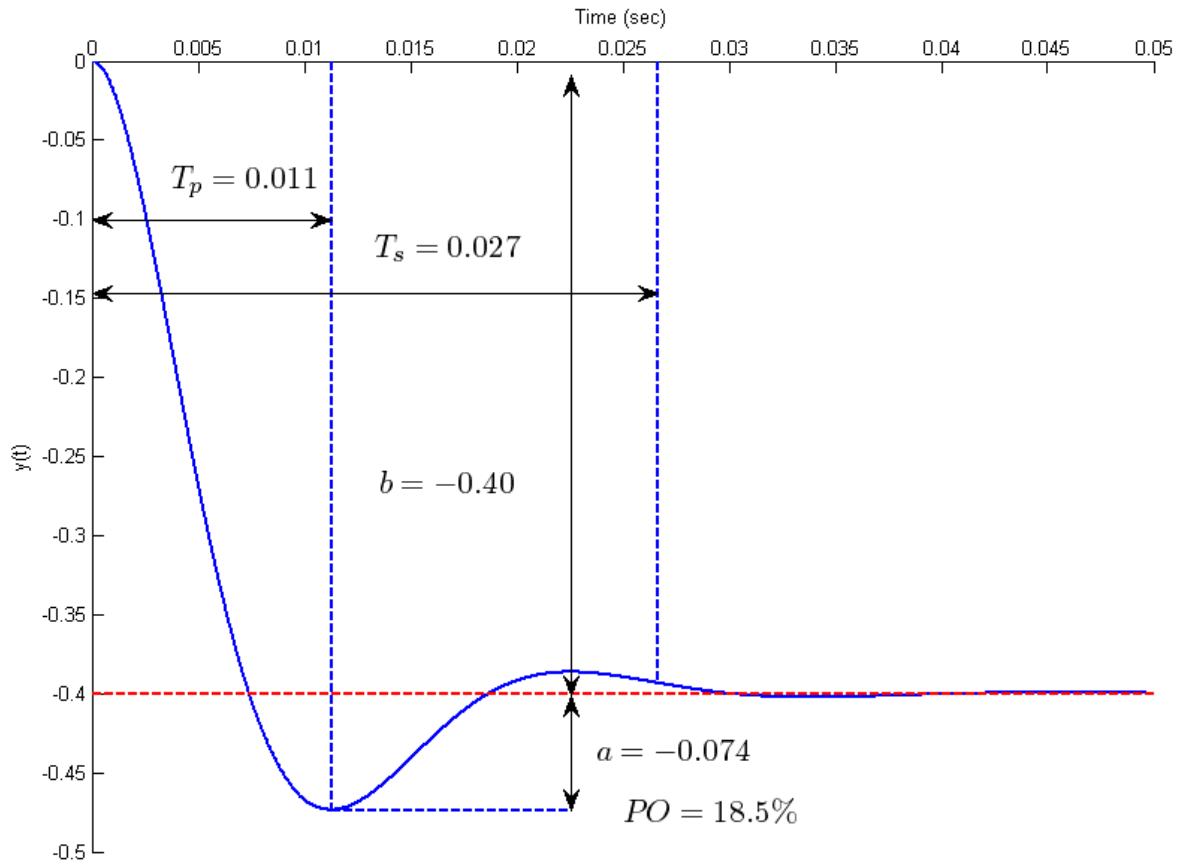
Figure 3.9. Response of the circuit used in Example 3.5.2 with parameter values $C_a = 10 \, \mu F$, $C_b = 1 \, \mu F$, $R = 1 \, k\Omega$, $R_a = 3 \, k\Omega$, $R_b = 2 \, k\Omega$. The input is a 1 volt step. Note that although the input to the circuit is a positive voltage, the output is negative because the static gain is negative.
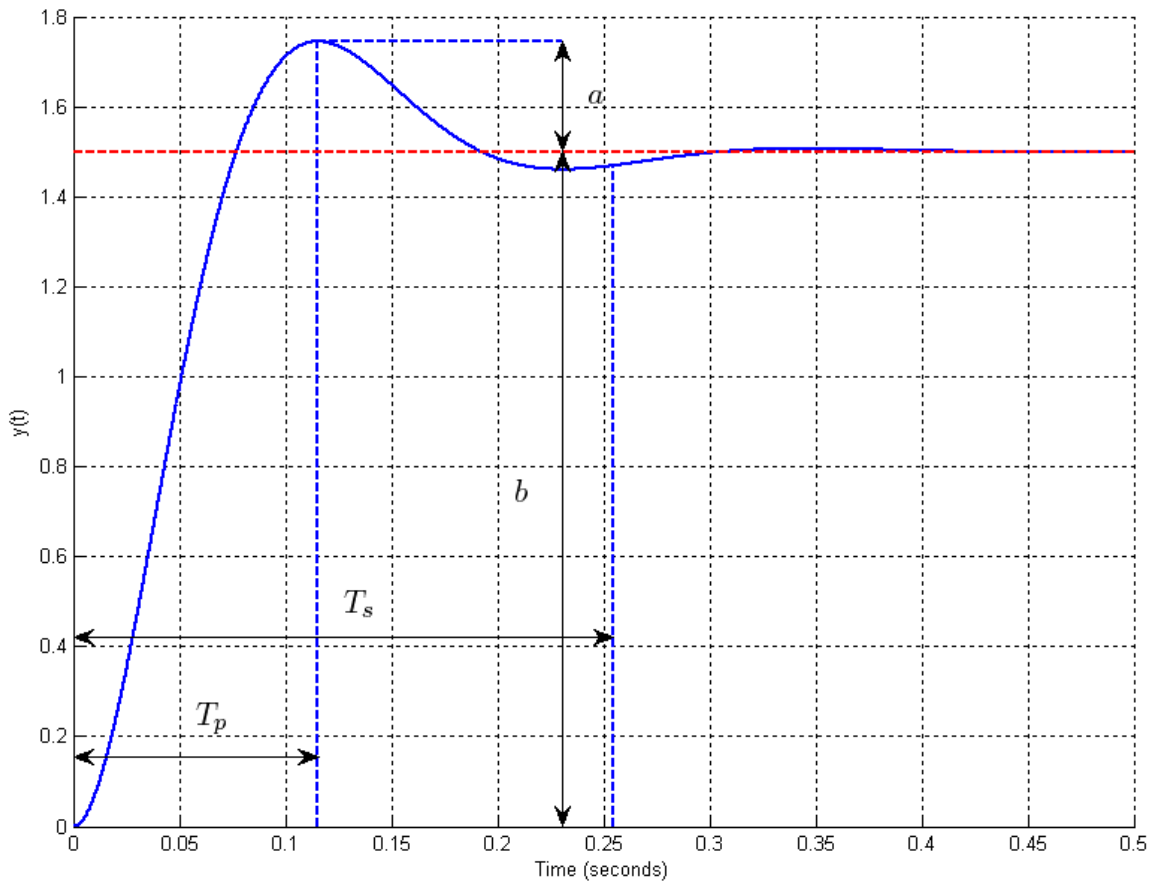
Figure 3.10. Output of an unknown second order system, analyzed in Example 3.5.3.

Example 3.5.4. Consider the circuit used in Example 3.2.3 with parameter values, $C = 1\,\text{mF}$, $L = 10\,\text{mH}$, $R = 10\,\Omega$. Assume the system is initially at rest and the input is a 1 amp step, $x(t) = u(t)$. We want to determine and characterize the output of the system. We start by determining the system parameters:

$$\omega_n = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(10\times10^{-3})(1\times10^{-3})}} = 316.2\,\text{rad/sec}$$

$$\zeta = \frac{R}{2}\sqrt{\frac{C}{L}} = \frac{10}{2}\sqrt{\frac{1\times10^{-3}}{10\times10^{-3}}} = 1.581$$

Since $\zeta > 1.0$ we have an over damped system. We know from our previous solution we will have a solution of the form

$$y(t) = KA + c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

where

$$r_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1} = -(1.581)(316.2) + (316.2)\sqrt{1.581^2 - 1} \approx -112.7$$

and

$$r_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1} = -(1.581)(316.2) - (316.2)\sqrt{1.581^2 - 1} \approx -887.1$$

In order to determine the complete response we will use the general form of the solution in what follows. Because the system starts at zero we have the condition

$$y(0) = 0 = KA + c_1 + c_2 \text{ or } c_1 + c_2 = -KA$$

Because the system starts at rest we need to look at the slope

$$\dot{y}(t) = r_1 c_1 e^{r_1 t} + r_2 c_2 e^{r_2 t}$$

At the initial time then we also have

$$\dot{y}(0) = r_1 c_1 + r_2 c_2 = 0 \text{ or } c_1 = -\frac{r_2}{r_1} c_2$$

Combining these conditions we have

$$c_1 + c_2 = -\frac{r_2}{r_1} c_2 + c_2 = \left[1 - \frac{r_2}{r_1}\right] c_2 = \frac{r_1 - r_2}{r_1} c_2 = -KA$$

or $c_2 = \dfrac{r_1}{r_2 - r_1} KA$ and $c_1 = \dfrac{-r_2}{r_2 - r_1} KA$. Our solution is then

$$y(t) = KA\left[1 - \frac{r_2}{r_2 - r_1} e^{r_1 t} + \frac{r_1}{r_2 - r_1} e^{r_2 t}\right]$$

For our system this becomes

$$y(t) = 1 - 1.145 e^{-112.7t} + 0.145 e^{-887.1t}$$

The response of this system is displayed in Figure 3.11. As this figure shows, there is no overshoot, so determining the time to peak or percent overshoot is meaningless. We can, however, determine the settling time. However, we cannot use our previous formula $T_s = \dfrac{4}{\zeta\omega_n}$, since this was derived for an under damped system. What we need is to use the more general form that the settling time is equal to four time constants, $T_s = 4\tau$. Recall that the general form of a decaying exponential is $e^{-t/\tau}$, where τ is the system time constant. For this system we are going to have two time constants since we have two exponential terms,

$$e^{-112.7t} = e^{-t/\tau}, \tau = \frac{1}{112.7} = 0.00887 \text{ sec}$$

$$e^{-887.1t} = e^{-t/\tau}, \tau = \frac{1}{887.1} = 0.00113 \text{ sec}$$

The system response and a plot of these exponentials is shown in Figure 3.12. As this figure shows, the response of the exponential with the smaller time constant is much more rapid than the response of the exponential with the larger time constant. The response of the system is more nearly like the response of the exponential with the larger time constant. Hence, to determine the settling time, we use the largest time constant

$$T_s = 4\tau \approx (4)(0.00887) = 0.0355 \text{ sec}$$

This is a general result that we will use later, ***the response of the system is dominated by the response of the part with the largest time constant.***
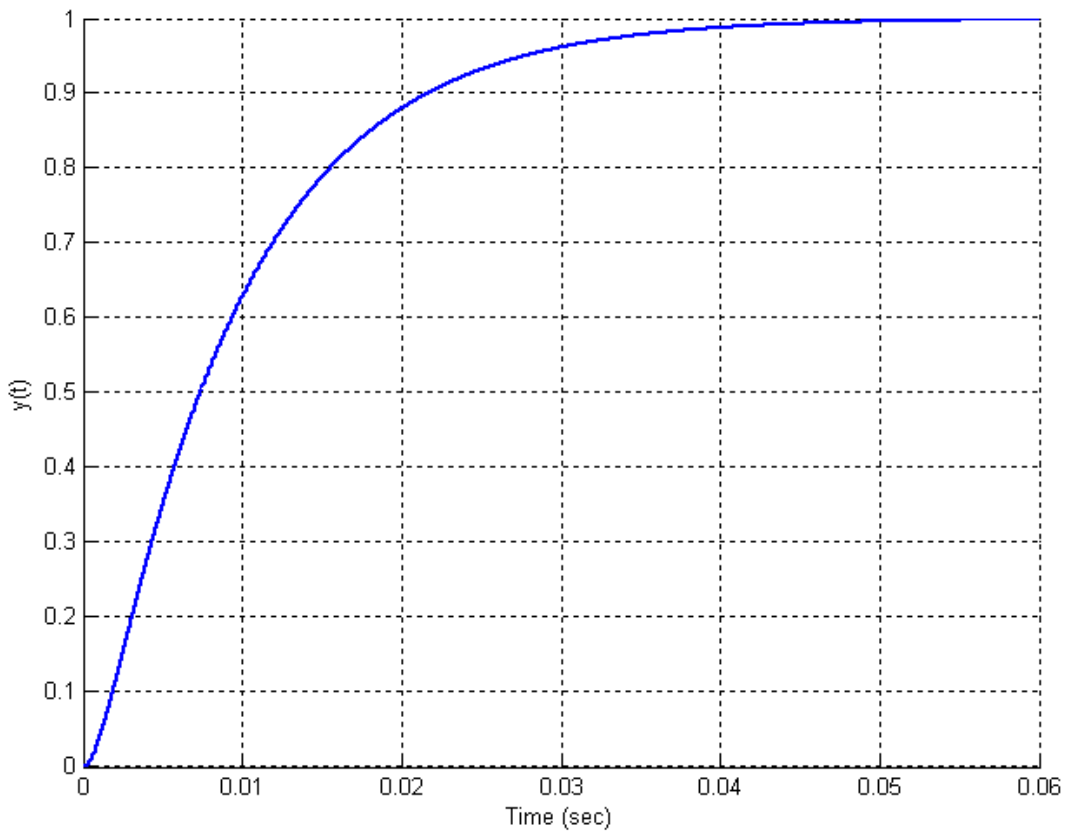
Figure 3.11. Response of system analyzed in Example 3.5.4. This is an over damped system. The settling time of the system is estimated to be $T_s = 0.0355 \text{ sec}$.
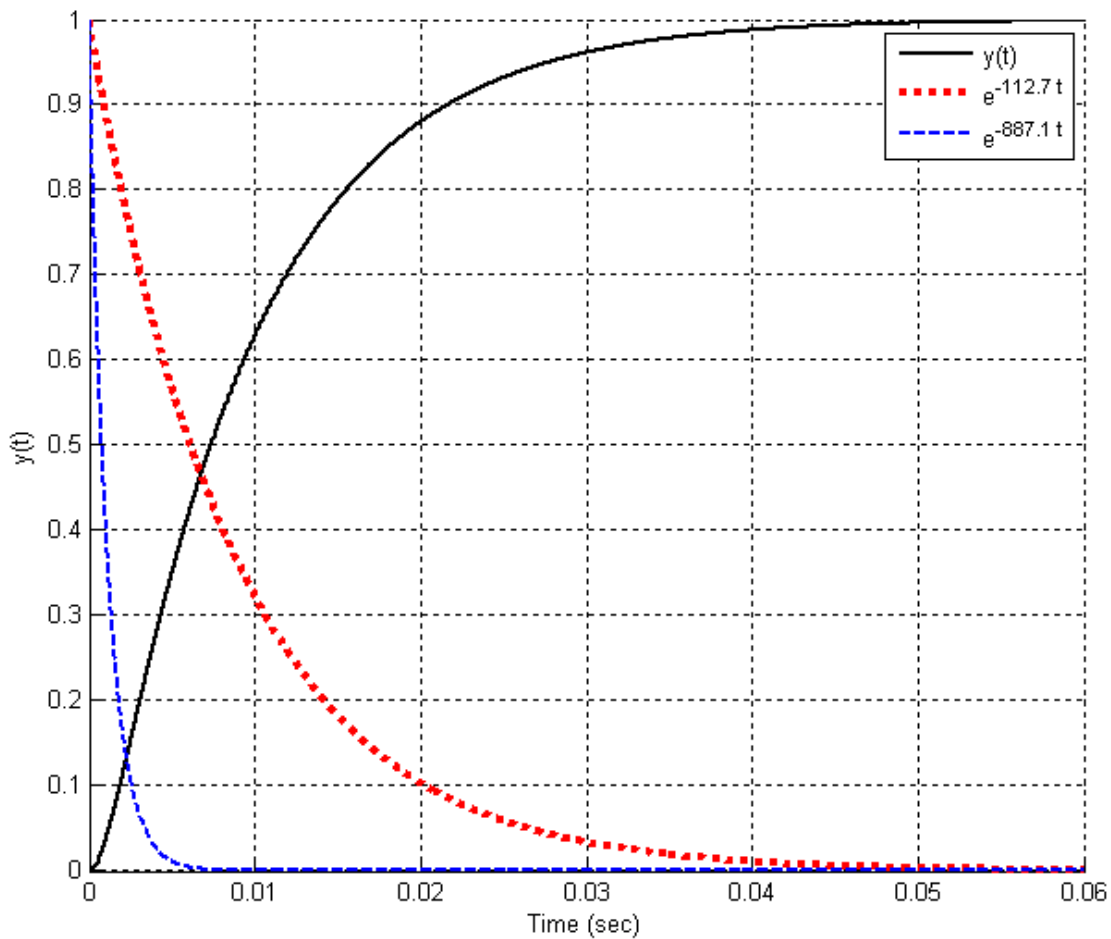
Figure 3.12. Response of system analyzed in Example 3.5.4 and plots of the exponentials that make up the response. For this system, $y(t) = 1 - 1.145e^{-112.7t} + 0.145e^{-887.1t}$. This system has components with time constants $\tau = \dfrac{1}{112.7} = 0.00887\,\text{sec}$ and $\tau = \dfrac{1}{887.1} = 0.00113\,\text{sec}$. The response of the system is clearly dominated by the component of the response with the largest time constant.

Chapter 3 Problems

3.1) Show that the governing differential equation for the following circuit is given by

$$R_b^2 C_a C_b \ddot{y}(t) + R_b (C_a + C_b) \dot{y}(t) + y(t) = \frac{R_b}{R_a} x(t)$$
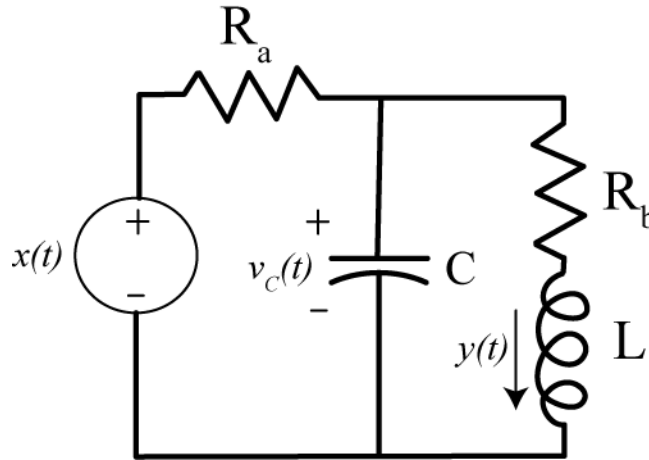
and that for this circuit $\omega_n = \dfrac{1}{R_b \sqrt{C_a C_b}}$, $\zeta = \dfrac{C_a + C_b}{2 \sqrt{C_a C_b}}$, and $K = \dfrac{R_a}{R_b}$
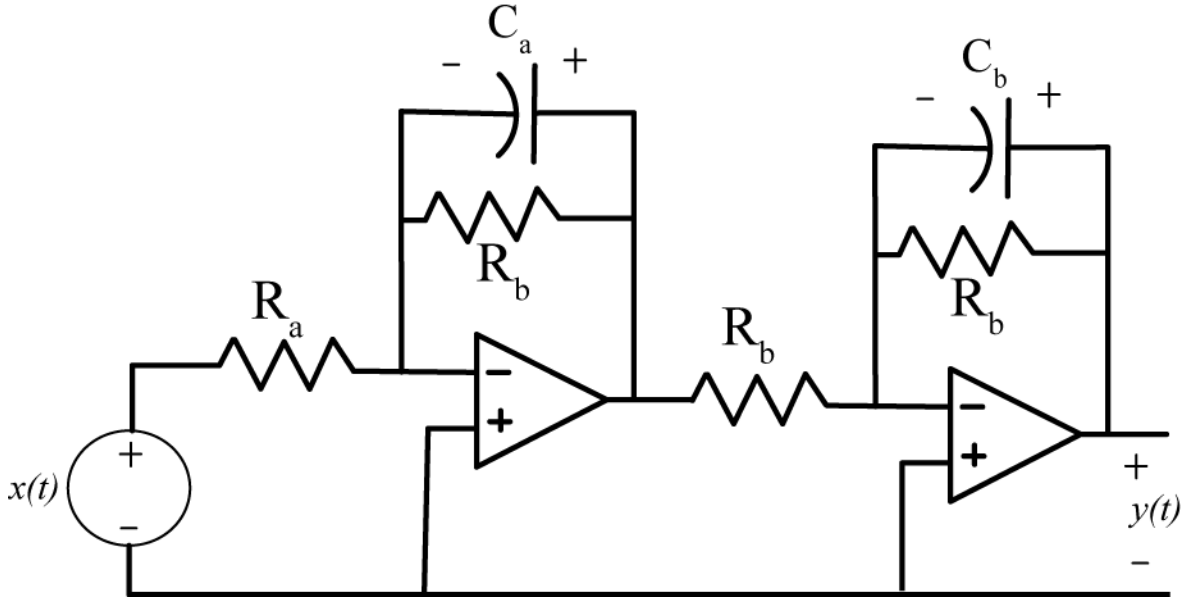
3.2) Show that the governing differential equation for the following circuit is given by

$$\left(\frac{LCR_a}{R_a+R_b}\right)\ddot{y}(t)+\left(\frac{L+R_aR_bC}{R_a+R_b}\right)\dot{y}(t)+y(t)=\left(\frac{1}{R_a+R_b}\right)x(t)$$

and that for this circuit $\omega_n=\sqrt{\dfrac{R_a+R_b}{LCR_a}}$, $\zeta=\dfrac{L+R_aR_bC}{2\sqrt{LCR_a}\sqrt{R_a+R_b}}$, and $K=\dfrac{1}{R_a+R_b}$.

3.3) For this problem, consider six second order systems described by the following differential equations:

$$\ddot{y}(t)+9\dot{y}(t)+20y(t)=20Kx(t)$$
$$\ddot{y}(t)+10\dot{y}(t)+25y(t)=25Kx(t)$$
$$\ddot{y}(t)+4\dot{y}(t)+13y(t)=13Kx(t)$$
$$\ddot{y}(t)+6\dot{y}(t)+8y(t)=8Kx(t)$$
$$\ddot{y}(t)+6\dot{y}(t)+9y(t)=9Kx(t)$$
$$\ddot{y}(t)+6\dot{y}(t)+13y(t)=13Kx(t)$$

a) Assume the systems are initially at rest and input is a step, $x(t)=Au(t)$, determine expressions for the system output by finding the forced and unforced responses and then solving for the unknown coefficients just as we did in class.

b) For the systems with real roots, show that your solution meets the two initial conditions ($y(0)=\dot{y}(0)=0$). For the systems with complex roots, determine ζ and ω_n from the governing differential equation, and show that your solution agrees with the form

$$y(t)=KA\left[1-\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t+\theta)\right] \qquad \theta=\cos^{-1}(\zeta) \qquad \omega_d=\omega_n\sqrt{1-\zeta^2}$$

Do not assume this is the form of the solution, but use it to check your answer.

Answers:

$$y(t) = KA - 5KAe^{-4t} + 4KAe^{-5t}, \quad y(t) = KA - KAe^{-5t} - 5KAte^{-5t}$$

$$y(t) = KA[1 - 1.202e^{-2t}\sin(3t + 56.3^\circ)], \quad y(t) = KA + KAe^{-4t} - 2KAe^{-2t}$$

$$y(t) = KA - KAe^{-3t} - 3KAte^{-3t}, \quad y(t) = KA[1 - 1.803e^{-3t}\sin(2t + 33.7^\circ)]$$

3.4) The response of a second order system is

$$y(t) = 1 - 1.0050e^{-t}\sin(9.9500t + 1.4706\,\text{rad})$$

a) Take the derivative of this function to determine the time at which the maximum occurs (the time to peak)

b) Determine the maximum value of this function (the value at the time to peak)

c) Determine the percent overshoot using your answer to (b)

d) For this response determine ζ, ω_d, and ω_n

e) Compute the percent overshoot using the formula

$$PO = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100\%$$

and verify your answer to (**c**).

3.5) The response of a second order system is

$$y(t) = 2[1 - 1.0328e^{-3.75t}\sin(14.5237t + 1.3181\,\text{rad})]$$

a) Take the derivative of this function to determine the time at which the maximum occurs (the time to peak)

b) Determine the maximum value of this function (the value at the time to peak)

c) Determine the percent overshoot using your answer to (b)

d) For this response determine ζ, ω_d, and ω_n

e) Compute the percent overshoot using the formula

$$PO = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100\%$$

and verify your answer to (**c**).

3.6) One of the methods that can be used to identify ζ and ω_n for mechanical systems the *log-decrement* method, which we will derive in this problem. If our system is at rest and we provide the mass with an initial displacement away from equilibrium, the response due to this displacement can be written

$$x_1(t) = Ae^{-\zeta\omega_n t}\cos(\omega_d t + \theta)$$

where

$x_1(t)$ = displacement of the mass as a function of time

ζ = damping ratio

ω_n = natural frequency

ω_d = damped frequency = $\omega_n\sqrt{1-\zeta^2}$

After the mass is released, the mass will oscillate back and forth with period given by $T_d = \dfrac{2\pi}{\omega_d}$, so if we measure the period of the oscillation (T_d) we can estimate ω_d. Let's assume t_0 is the time of one peak of the cosine. Since the cosine is periodic, subsequent peaks will occur at times given by $t_n = t_0 + nT_d$, where n is an integer.

a) Show that

$$\frac{x_1(t_0)}{x_1(t_n)} = e^{\zeta\omega_n T_d n}$$

b) If we define the log decrement as

$$\delta = \ln\left[\frac{x_1(t_0)}{x_1(t_n)}\right]$$

show that we can compute the damping ratio as

$$\zeta = \frac{\delta}{\sqrt{4n^2\pi^2 + \delta^2}}$$

c) Given the initial condition response shown in the Figures on the next page, estimate the damping ratio and natural frequency using the log-decrement method. (*You should get answers that include the numbers 15, 0.2, 0.1 and 15, approximately.*)
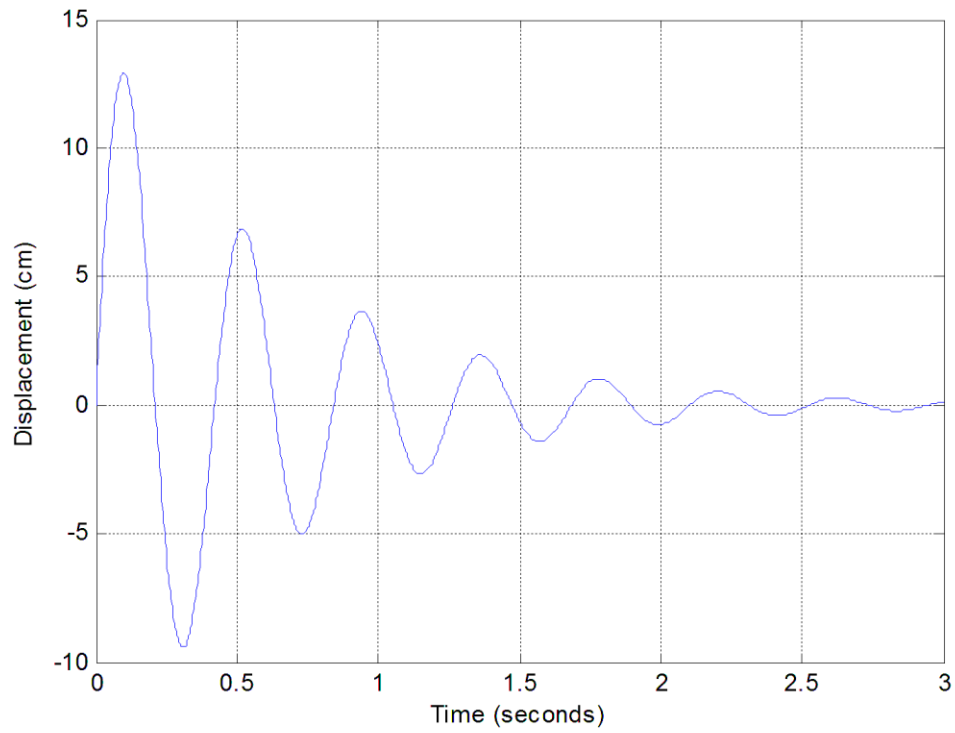
Figure 1. Initial condition response for second order system A.
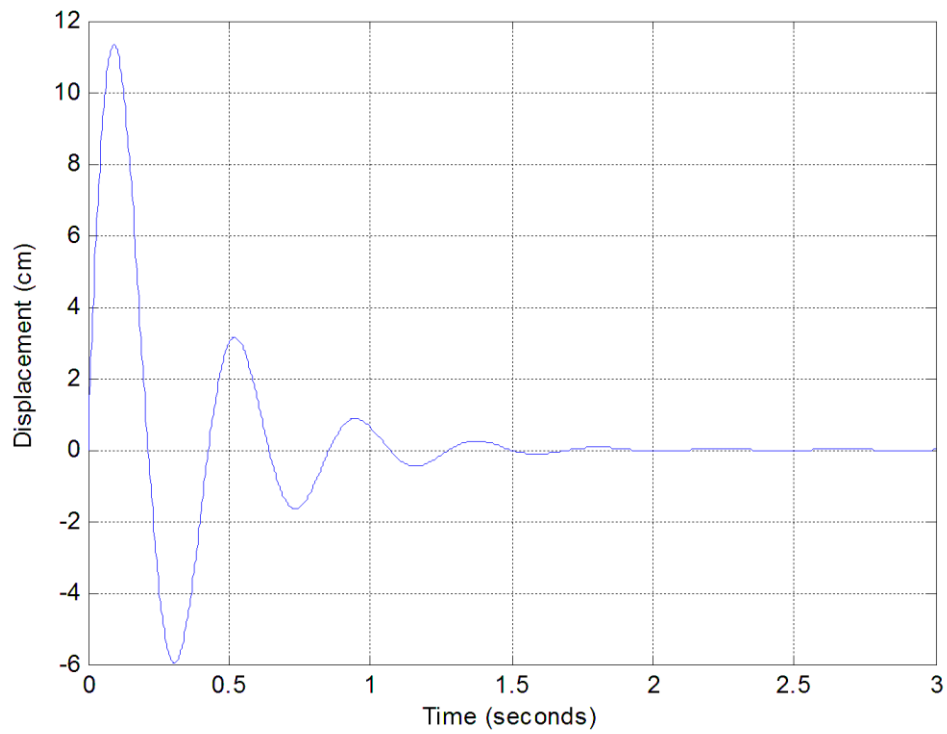

Figure 2. Initial condition response for second order system B.

4.0 System Properties

In this chapter we will start looking at various properties that can be used to characterize a system. We will initially illustrate these concepts as much as possible with examples from circuits and mechanical systems. However, since these are general concepts we will begin to explore abstract systems described only by algebraic, integral, or differential equations. Our goal is to be able to be able to determine whether or not a mathematical model of a system possesses these properties, and to develop the necessary vocabulary.

4.1 Linear (L) Systems

Let's assume we have a system with an input x(t) producing and output y(t). We can write this graphically as $x(t) \rightarrow y(t)$. A system possesses the *scaling* or *homogeneity* property if $\alpha x(t) \rightarrow \alpha y(t)$ for any constant α and any input $x(t)$. In particular, if a system possesses the homogeneity property, if the input is zero the output will be zero
($\alpha = 0$), if the input is doubled the output is doubled ($\alpha = 2$), and if we change the sign of the input we also change the sign of the output ($\alpha = -1$). These are very simple and common tests that can quickly be used to determine if a system does not possess the homogeneity property.

Next let's assume we name possible two inputs as $x_1(t)$ and $x_2(t)$, and we name the corresponding outputs $y_1(t)$ and $y_2(t)$. Hence we know $x_1(t) \rightarrow y_1(t)$ and $x_2(t) \rightarrow y_2(t)$.
A system possesses the *additivity* property if

$$\alpha_1 x_1(t) + \alpha_2 x_2(t) \rightarrow \alpha_1 y_1(t) + \alpha_2 y_2(t)$$

for all constants α_1 and α_2, and all inputs $x_1(t)$ and $x_2(t)$.

Definition: A *linear system* is any system that possesses both the homogeneity and the additivity properties.

Example 4.1.1. Consider the simple resistive circuit shown in Figure 4.1, with the system input defined as the input voltage $v^{in}(t)$ and the system output defined as the current flowing in the circuit, $i^{out}(t)$. For this simple system we have the mathematical model

$$i^{out}(t) = \frac{v^{in}(t)}{R}$$

Clearly this model satisfies the homogeneity requirement, since if we scale the input by a constant α we also scale the output by α,

$$\alpha\, i^{out}(t) = \frac{\alpha\, v^{in}(t)}{R}$$

Let's next assume we have two inputs $v_1^{in}(t)$ and $v_2^{in}(t)$, as shown in Figure 4.2. If we use superposition, we replace each voltage source with a short circuit and determine the resulting output current for each input voltage source acting alone. This gives us

$$i_1^{out}(t) = \frac{v_1^{in}(t)}{R} \text{ and } i_2^{out}(t) = \frac{v_2^{in}(t)}{R}$$

Adding these we clearly have

$$i^{out}(t) = i_1^{out}(t) + i_2^{out}(t) = \frac{v_1^{in}(t) + v_2^{in}(t)}{R} = \frac{v^{in}(t)}{R}$$

If should be clear for this example that if we scale both of the inputs, we would also scale both of the outputs, so

$$\alpha_1 i_1^{out}(t) + \alpha_2 i_2^{out}(t) = \frac{\alpha_1 v_1^{in}(t) + \alpha_2 v_2^{in}(t)}{R}$$

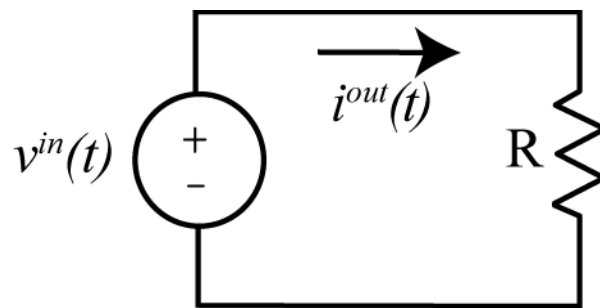Since the system has both the property of homogeneity and additivity, the system is linear.

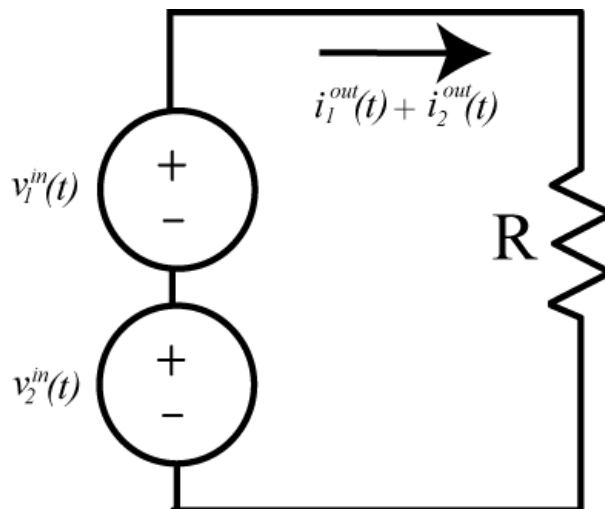**Figure 4.1.** Circuit used in Example 4.1.1.

Figure 4.2. Second circuit used in Example 4.1.1.

Example 4.1.2. Consider the simple resistive circuit shown in Figure 4.3, with system input equal to the input voltage $v^{in}(t)$ and the system output equal to the voltage $v^{out}(t)$ measured across the resistor R_b. We assume the voltage $v_0 \neq 0$. The current flowing through the circuit is

$$i(t) = \frac{v^{in}(t) - v_0}{R_a + R_b}$$

and the output voltage is

$$v^{out}(t) = i(t)R_b = \left(\frac{v^{in}(t) - v_0}{R_a + R_b} \right) R_b$$

Now let's check to see if the homogeneity condition is satisfied. If the system input (the input voltage) is zero, $v^{in}(t) = 0$, then we expect the system output (the output voltage) to be zero. However, it is clear that under these circumstances the system output will be

$$v^{out}(t) = \frac{-v_0}{R_a + R_b} R_b \neq 0$$

Hence the homogeneity condition is not satisfied, and thus the system is not linear. It is important to note in this example that we need to look carefully at the system input and the system output.
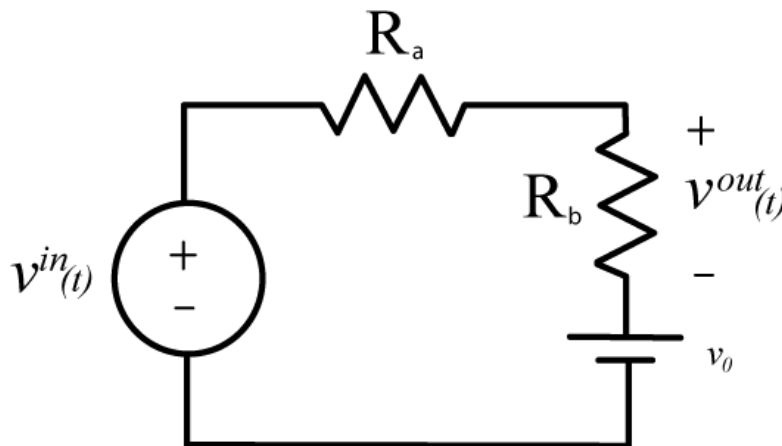

Figure 4.3. Circuit used in Example. 4.1.2.

Example 4.1.3. Consider the circuit shown in Figure 4.4. The system input is the input voltage $v^{in}(t)$ and the system output is the voltage across the capacitor, $v^{out}(t)$. The current flowing in the circuit is given by

$$i(t) = C \frac{dv^{out}(t)}{dt}$$

We then have

$$v^{in}(t) - \left(C \frac{dv^{out}(t)}{dt} \right) R = v^{out}(t)$$

or

$$\frac{dv^{out}(t)}{dt} + \frac{1}{RC} v^{out}(t) = \frac{1}{RC} v^{in}(t)$$

We can solve this using integrating factors, as before,

$$\frac{d}{dt} \left(v^{out}(t) e^{t/RC} \right) = v^{in}(t) e^{t/RC}$$

Next we integrate from some initial time t_0 up to the current time, t,

$$\int_{t_0}^{t} \frac{d}{d\lambda} \left(v^{out}(\lambda) e^{\lambda/RC} \right) d\lambda = \int_{t_o}^{t} v^{in}(\lambda) e^{\lambda/RC} d\lambda$$

$$v^{out}(t) e^{t/RC} - v^{out}(t_0) e^{t_0/RC} = \int_{t_0}^{t} v^{in}(\lambda) e^{\lambda/RC} d\lambda$$

Finally, we have the input-output relationship

$$v^{out}(t) = v^{out}(t_0) e^{-(t-t_0)/RC} + \int_{t_0}^{t} v^{in}(\lambda) e^{-(t-\lambda)} d\lambda$$
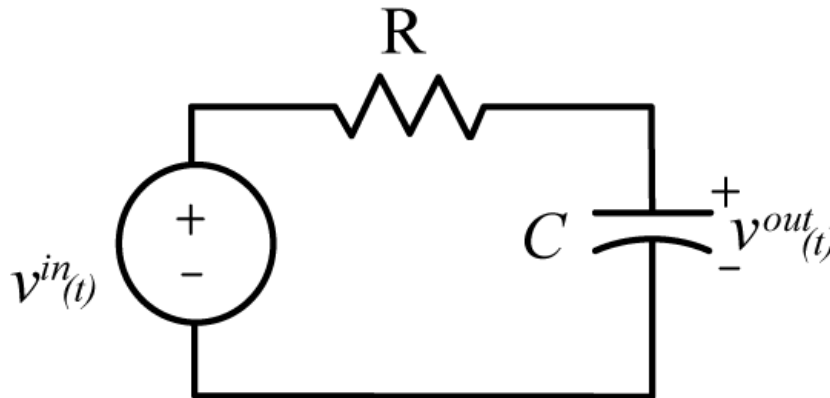

Figure 4.4. Circuit used in Example 4.1.3.

First let's check for homogeneity. If the input is zero, then we expect the output to be zero. However, in this case, if the input is zero, the output will be

$$v^{out}(t) = v^{out}(t_0) e^{-(t-t_0)/RC}$$

Hence, in order for the system to possess the homogeneity property, the initial conditions must be zero. ***This is a general requirement for all systems.*** Let's then assume the initial conditions are zero, then we have

$$v^{out}(t) = \int_{t_0}^{t} v^{in}(\lambda)e^{-(t-\lambda)}d\lambda$$

If we scale the input, we scale the output,

$$\left[\alpha v^{out}(t)\right] = \int_{t_0}^{t}\left[\alpha v^{in}(\lambda)\right]e^{-(t-\lambda)}d\lambda$$

Finally, if

$$v_1^{out}(t) = \int_{t_0}^{t} v_1^{in}(\lambda)e^{-(t-\lambda)}d\lambda$$

and

$$v_2^{out}(t) = \int_{t_0}^{t} v_2^{in}(\lambda)e^{-(t-\lambda)}d\lambda$$

then

$$\left[\alpha_1 v_1^{out}(t) + \alpha_2 v_2^{out}(t)\right] = \int_{t_0}^{t}\left[\alpha_1 v_1^{out}(t) + \alpha_2 v_2^{in}(\lambda)\right]e^{-(t-\lambda)}d\lambda$$

Hence the system also meets the additivity condition and is thus a linear system.

Example 4.1.4. The following models of systems, with system input $x(t)$, and system output $y(t)$, do not satisfy the homogeneity condition, and hence are not linear models:

$$y(t) = x(t) + 2$$
$$y(t) = |x(t)|$$
$$y(t) = e^{x(t)}$$
$$y(t) = \sin(x(t))$$
$$y(t) = \frac{1}{x(t)}$$

4.2 Testing for Linear Systems

We will present two different, though equivalent, methods for testing for linearity of a mathematical model of a system. A mathematical model of a system must pass one of these (or an equivalent) test for the model to be linear. It is much easier to show a mathematical model of a system is not linear than it is linear. The first method we will demonstrate assumes we have an algebraic or integral relationship between the system input and the system output. This test is straightforward, but we do not want to have to solve a differential equation to use it. Thus, we will present a second method to use with differential equations.

In order to simplify notation, we will assume we have a system operator, which we will denote as $\mathcal{H}$. The output of a system is the result of the system operator operating on the input. Hence if the system input is $x(t)$ and the system output is $y(t)$, then we would have $y(t) = \mathcal{H}\{x(t)\}$

Consider the two signal flow diagrams shown in Figure 4.5. In the top figure, we examine the output of the system, $z_1(t)$, when the input to the system is the input, $\alpha_1 x_1(t) + \alpha_2 x_2(t)$, $z_1(t) = \mathcal{H}\{\alpha_1 x_1(t) + \alpha_2 x_2(t)\}$.
In the bottom figure, we examine the output of the system to input $x_1(t)$, $y_1(t) = \mathcal{H}\{x_1(t)\}$, and input $x_2(t)$, $y_2(t) = \mathcal{H}\{x_2(t)\}$, then form the linear combination of these,

$$z_2(t) = \alpha_1 y_1(t) + \alpha_2 y_2(t) = \alpha_1 \mathcal{H}\{x_1(t)\} + \alpha_2 \mathcal{H}\{x_2(t)\}.$$

If the output is the same for both paths, i.e., if $z_1(t) = z_2(t)$, then the system is linear. If this is not true, then the system is not linear. Let's illustrate the method with a few examples. In the following examples, we assume the system input is $x(t)$ and the system output is $y(t)$.
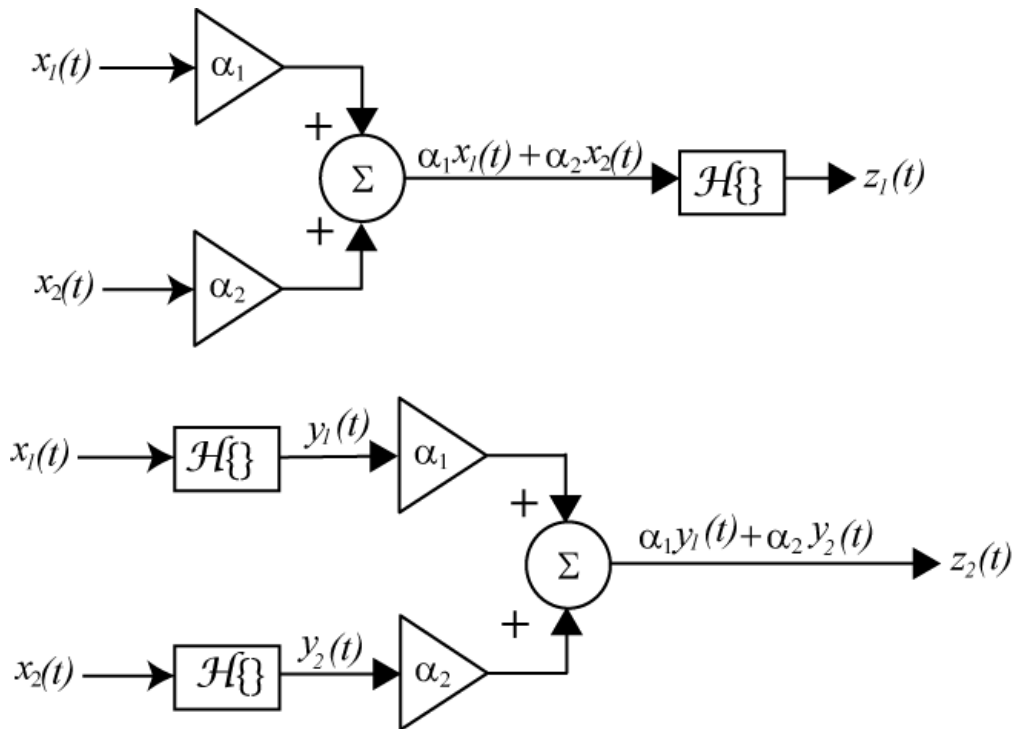

Figure 4.5. Signal flow graph of linearity test.

Example 4.2.1. Consider the mathematical model $y(t) = \sin^2(t)x(t)$. Does this represent a linear system? For this system, the linear operator is $\mathcal{H}\{x(t)\} = \sin^2(t)x(t)$. Along the top path we have
$$z_1(t) = \mathcal{H}\{\alpha_1 x_1(t) + \alpha_2 x_2(t)\} = \sin^2(t)\{\alpha_1 x_1(t) + \alpha_2 x_2(t)\}$$
Along the bottom path we have
$$z_2(t) = \alpha_1 \mathcal{H}\{x_1(t)\} + \alpha_2 \mathcal{H}\{x_2(t)\} = \alpha_1 \sin^2(t)x_1(t) + \alpha_2 \sin^2(t)x_2(t) = \sin^2(t)\{\alpha_1 x_1(t) + \alpha_2 x_2(t)\}$$

Since $z_1(t) = z_2(t)$ the mathematical model is linear.

Example 4.2.2. Consider the mathematical model $y(t) = x(t) + b$. Does this represent a linear system? Without even really trying, we know this equation does not satisfy the homogeneity condition, so it does not represent a linear system. However, let's see what happens with our new test. For this system, the linear operator is $\mathcal{H}\{x(t)\} = x(t) + b$. Along the top path we have

$$z_1(t) = \mathcal{H}\{\alpha_1 x_1(t) + \alpha_2 x_2(t)\} = \{\alpha_1 x_1(t) + \alpha_2 x_2(t)\} + b$$

Along the bottom path we have

$$z_2(t) = \alpha_1 \mathcal{H}\{x_1(t)\} + \alpha_2 \mathcal{H}\{x_2(t)\} = \alpha_1\{x_1(t) + b\} + \alpha_2\{x_2(t) + b\} = \{\alpha_1 x_1(t) + \alpha_2 x_2(t)\} + b\{\alpha_1 + \alpha_2\}$$

Now if we compare these, we do not have $z_1(t) = z_2(t)$ for all possible α_1 and α_2, hence the model is not linear.

Example 4.2.3. Consider the mathematical model $y(t) = \int_{-\infty}^{t} e^{-2(t-\lambda)} x(\lambda) d\lambda$. Does this represent a linear

system? Along the top path we have

$$z_1(t) = \mathcal{H}\{\alpha_1 x_1(t) + \alpha_2 x_2(t)\} = \int_{-\infty}^{t} \{\alpha_1 x_1(\lambda) + \alpha_2 x_2(\lambda)\} d\lambda$$

Along the bottom path we have

$$z_2(t) = \alpha_1 \mathcal{H}\{x_1(t)\} + \alpha_2 \mathcal{H}\{x_2(t)\}$$

$$z_2(t) = \alpha_1 \int_{-\infty}^{t} e^{-2(t-\lambda)} x_1(\lambda) d\lambda + \alpha_2 \int_{-\infty}^{t} e^{-2(t-\lambda)} x_2(\lambda) d\lambda = \int_{-\infty}^{t} e^{-2(t-\lambda)} \{\alpha_1 x_1(\lambda) + \alpha_2 x_2(\lambda)\} d\lambda$$

Since $z_1(t) = z_2(t)$, the mathematical model is linear.

The signal flow graph method we have presented works well for determining if a mathematical model of a system is linear, provided we have the output of the system written as an algebraic or integral function of the input. However, it is very common to model systems in terms of differential equations, and we would like to be able to determine if a system modeled by a differential equation is linear without having to solve the differential equation, as fun as that might be.

The second method we will present for determining if a mathematical description of a system is linear is easier to demonstrate then explain, but the general idea is as follows:

1. Write two differential equations, one with system input $x_1(t)$ and system output $y_1(t)$, the second with system input $x_2(t)$ and system output $y_2(t)$.

2. Multiply the $x_1(t) \rightarrow y_1(t)$ equation by α_1 and the $x_2(t) \rightarrow y_2(t)$ equation by α_2

3. Add the equations together and regroup, we want to write the resulting differential equation in terms of $\alpha_1 x_1(t) + \alpha_2 x_2(t)$ and $\alpha_1 y_1(t) + \alpha_2 y_2(t)$.

4. Make the substitutions $X(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t)$ and $Y(t) = \alpha_1 y_1(t) + \alpha_2 y_2(t)$ in the differential equation.

5. If the resulting differential equation is the same as the original differential equation, with $x(t)$ replaced by $X(t)$ and $y(t)$ replaced by $Y(t)$, then we have shown that

$$\alpha_1 x_1(t) + \alpha_2 x_2(t) \rightarrow \alpha_1 y_1(t) + \alpha_2 y_2(t)$$

and we can conclude that the system is linear. If this is not true, then the system is not linear.

Note that in order to satisfy the homogeneity conditions, we must assume the initial conditions for the system are all zero.

Example 4.2.4. Consider the mathematical model $\dot{y}(t) + \sin^2(t) y(t) = e^{-t} x(t)$. Does this represent a linear system? We have

$$\dot{y}_1(t) + \sin^2(t) y_1(t) = e^{-t} x_1(t) \quad \text{and} \quad \dot{y}_2(t) + \sin^2(t) y_2(t) = e^{-t} x_2(t)$$

Multiplying by α_1 and α_2 we have

$$\alpha_1 \dot{y}_1(t) + \alpha_1 \sin^2(t) y_1(t) = \alpha_1 e^{-t} x_1(t) \quad \text{and} \quad \alpha_2 \dot{y}_2(t) + \alpha_2 \sin^2(t) y_2(t) = \alpha_2 e^{-t} x_2(t)$$

Adding and regrouping we have

$$\left[\alpha_1 \dot{y}_1(t) + \alpha_2 \dot{y}_2(t)\right] + \sin^2(t)\left[\alpha_1 y_1(t) + \alpha_2 y_2(t)\right] = e^{-t}\left[\alpha_1 x_1(t) + \alpha_2 x_2(t)\right]$$

Substituting we have

$$\dot{Y}(t) + \sin^2(t) Y(t) = e^{-t} X(t)$$

Thus, this system represents a linear system.

Example 4.2.5. Consider the mathematical model $\dot{y}(t) + y(t)x(t) = x(t)$. Does this represent a linear system? We have

$$\dot{y}_1(t) + y_1(t)x_1(t) = x_1(t) \quad \text{and} \quad \dot{y}_2(t) + y_2(t)x_2(t) = x_2(t)$$

Multiplying by α_1 and α_2 we have

$$\alpha_1 \dot{y}_1(t) + \alpha_1 y_1(t)x_1(t) = \alpha_1 x_1(t) \text{ and } \alpha_2 \dot{y}_2(t) + \alpha_2 y_2(t)x_2(t) = \alpha_2 x_2(t)$$

Adding and regrouping we have

$$\left[\alpha_1 \dot{y}_1(t) + \alpha_2 \dot{y}_2(t)\right] + \alpha_1 y_1(t)x_1(t) + \alpha_2 y_2(t)x_2(t) = \left[\alpha_1 x_1(t) + \alpha_2 x_2(t)\right]$$

Substituting we have

$$\dot{Y}(t) + \alpha_1 y_1(t)x_1(t) + \alpha_2 y_2(t)x_2(t) = X(t)$$

At this points, it is clear that we cannot write this resulting differential equation just in terms of $X(t)$ and $Y(t)$, so the system is not linear.

Example 4.2.6. Consider the mathematical model $\dot{y}(t) + y(t) = x(t) + 2$. Does this represent a linear system? This one is easy if we use the homogeneity condition. If the input is zero the output should also be zero. However, in this model, if the input is zero, we still have the output being nonzero. Hence this model is nonlinear.

We should point out that this technique can be used for systems that are not differential equations, but it may sometimes be more difficult than the flow-graph techniques.

4.2 Time-Invariant (TI) Systems

A time-invariant (TI) system is one in which, if the input $x(t)$ is delayed by an amount T then the output $y(t)$ is delayed by the same amount, without changing shape. Symbolically, if for a system we have $x(t) \rightarrow y(t)$, then if the system is also time-invariant we will have $x(t-T) \rightarrow y(t-T)$. Figure 4.6 presents a signal flow graph test for time-invariance, assuming we can write the output as an algebraic function or integral of the input. Along the top path we delay the input and then determine the output. Along the bottom path, we put the usual input into the system and then delay the usual output. If the results of these two paths are identical, then the system is time-invariant. There are a few subtleties involved in this test, so read through the following examples carefully.
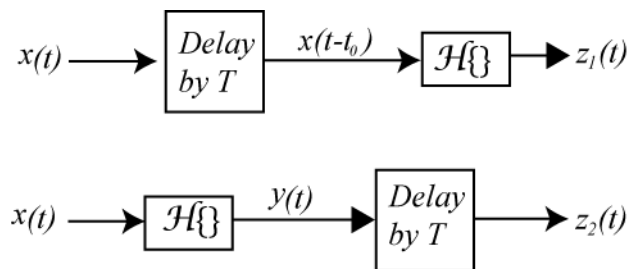
Figure 4.6. Signal flow graph of time invariance test.

Example 4.2.1. Consider a system with the mathematical model $y(t) = \alpha(t)x(t)$. Does this model represent a linear system? Along the top path, delaying the input we have

$$z_1(t) = \mathcal{H}\{x(t-T)\} = \alpha(t)x(t-T)$$

Along the second path (delaying the output) we have

$$z_2(t) = \left[\mathcal{H}\{x(t)\}\right]_{t=t-T} = \alpha(t-T)x(t-T)$$

Clearly $z_1(t) \neq z_2(t)$, so the model is not time-invariant.

Example 4.2.2. Consider the RC circuit shown in Figure 4.7, with an initial charge on the capacitor. The system input is the applied voltage $x(t)$ and the system output is the voltage across the capacitor $y(t)$. We need to write the output as a function of the input. The current in the loop is given by

$$\frac{x(t) - y(t)}{R} = C\frac{dy(t)}{dt}$$

or

$$\frac{dy(t)}{dt} + \frac{1}{RC}\,y(t) = \frac{1}{RC}\,x(t)$$

$$\frac{d}{dt}\left[y(t)e^{t/RC} \right] = e^{t/RC}x(t)$$

Integrating both sides and rearranging we get

$$\int_{t_0}^{t}\frac{d}{d\lambda}\left[y(\lambda)e^{\lambda/RC} \right]d\lambda = y(t)e^{t/RC} - y(t_0)e^{t_0/RC} = \int_{t_0}^{t}e^{\lambda/RC}x(\lambda)d\lambda$$

$$y(t) = y(t_0)e^{-(t-t_0)/RC} + \int_{t_0}^{t}e^{-(t-\lambda)/RC}x(\lambda)d\lambda$$

Along the top path of the signal flow graph, delaying the input, we have

$$z_1(t) = \mathcal{H}\{x(t-T)\} = y(t_0)e^{-(t-t_0)/RC} + \int_{t_0}^{t}e^{-(t-\lambda)/RC}x(\lambda-T)d\lambda$$

Note that we only delay the input, we do not change any other functions. Along the second path of the signal flow graph we only delay the output, which means we take the output and replace all instances of t with $t-T$. This leads to

$$z_2(t) = \left[\mathcal{H}\{x(t)\}\right]_{t=t-T} = y(t_0)e^{-(t-T-t_0)/RC} + \int_{t_0}^{t-T}e^{-(t-T-\lambda)/RC}x(\lambda)d\lambda$$

Now we want to see if $z_1(t) = z_2(t)$. Since we do not know what the input is, we will change variables in the integrals so both of them are simple functions of the dummy integral variable. $z_2(t)$ is already in the correct form, so we need to change variables in $z_1(t)$. Let's let $\sigma = \lambda - T$, or $\lambda = \sigma + T$. Then can rewrite $z_1(t)$ as

$$z_1(t) = y(t_0)e^{-(t-t_0)/RC} + \int_{t_0-T}^{t-T}e^{-(t-T-\sigma)/RC}x(\sigma)d\sigma$$

Now let's compare $z_1(t)$ and $z_2(t)$, and determine if they are equal or if we can make them equal. First of all, the terms associated with the initial conditions cannot be made equal, so if this system is going to be time invariant *the initial conditions must be zero*. Both of them have the same integrand, and both have the same upper limit on the integral. The only difference in the integral term is the lower limit. In order for the two integrals to be equal there are two choices. The most obvious choice is to assume that the input is zero for all times before the initial time t_0. Then the lower limit on the integral in $z_1(t)$ is effectively still effectively t_0,

$$z_1(t) = \int_{t_0-T}^{t-T} e^{-(t-T-\sigma)/RC} x(\sigma)d\sigma$$

$$= \int_{t_0-T}^{t_0} e^{-(t-T-\sigma)/RC} x(\sigma)d\sigma + \int_{t_0}^{t-T} e^{-(t-T-\sigma)/RC} x(\sigma)d\sigma$$

$$\underbrace{\qquad\qquad\qquad\qquad\qquad}_{x(\sigma)=0\,in\,this\,range}$$

$$= \int_{t_0}^{t-T} e^{-(t-T-\sigma)/RC} x(\sigma)d\sigma$$

The other option is for $t_0 = -\infty$. With this choice of initial value, the first terms are equal and we have

$$z_1(t) = z_2(t) = \int_{-\infty}^{t-T} e^{-(t-T-\sigma)/RC} x(\sigma)d\sigma$$

Note that in order for a system to be time-invariant, the initial conditions must be zero, just as they are for the linearity requirements.
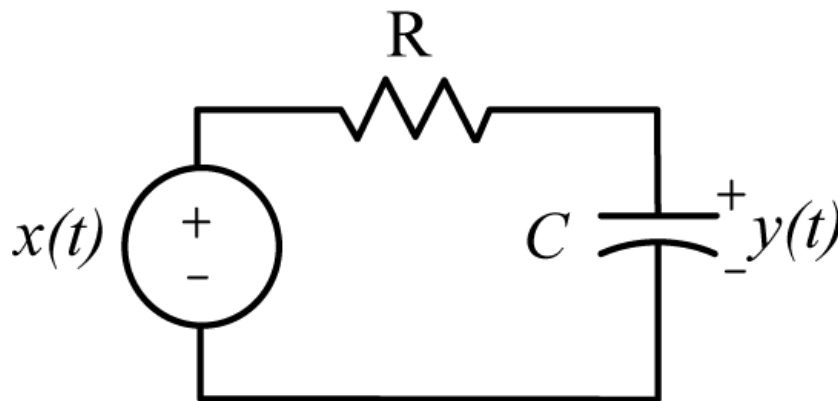
Figure 4.7. Circuit used in Example 4.2.2.

Example 4.2.3. Consider the model of a system $y(t) = x\left(\dfrac{t}{2}\right)$. Is this model time-invariant? This example is somewhat tricky, since we have to interpret the tests correctly. Along the top path, we are supposed to subtract T from the argument of $x(t)$, so we have

$$z_1(t) = \mathcal{H}\{x(t-T)\} = x\left(\frac{t}{2}-T\right)$$

This is not the result you expect, but it is the correct way to interpret the top path of our signal flow graph. Along the bottom path we delay the output by T so we have

$$z_2(t) = \left[\mathcal{H}\{x(t)\}\right]_{t=t-T} = x\left(\frac{t-T}{2}\right)$$

Clearly $z_1(t) \neq z_2(t)$, so the model is not time-invariant.

Example 4.2.4. Consider the model of a system $y(t) = x(1-t)$. Is this model time-invariant? We have

$$z_1(t) = \mathcal{H}\{x(t-T)\} = x(1-t-T)$$

and

$$z_2(t) = \left[\mathcal{H}\{x(t)\}\right]_{t=t-T} = x(1-t+T)$$

Clearly $z_1(t) \neq z_2(t)$, so the model is not time-invariant.

Testing differential equations for time-invariance in general is somewhat more difficult, so we will just state a result: If the differential equation is just a function of the input $x(t)$ and the output $y(t)$, and these are both just simple functions of t (e.g. there are no terms like $x(2t), y(t/2), x(1-t)$), then the differential equation is time-invariant *if there are no other functions of time other than the input and output functions.*

Example 4.2.5. The following models of systems are not time invariant (though they are linear!):

$$\dot{y}(t) = e^t x(t)$$
$$\ddot{y} + \sin(t)\dot{y}(t) = \cos(t)x(t)$$
$$\dot{y}(t) + y(t) = tx(t)$$

The following models of systems are time-invariant:

$$\dot{y}(t) + x(t)y(t) = x(t)$$
$$\ddot{y}(t) + \dot{y}(t) = x(t)$$
$$\dot{y}(t) + y(t) = x^2(t)$$

4.3 Causal Systems

If a system is *causal*, then the system output $y(t_0)$ at some arbitrary time t_0 can only depend on the system input $x(t)$ up until (and including) time t_0. Another way of describing a causal system is that it is nonanticipative, the output does not anticipate the input. While most of the systems we think of are causal, this is because they work in real-time. That is, we are collecting or storing data to be processed as the data comes in, not at a later time. However, many discrete-time systems, such as an ipod or MP3 player, have data or music stored in advance. When the music is played it is possible for the system to look at future values of the input (the discrete-time signal representing the music) and make adjustments based on the "future". This is possible because these systems do not need to work in real-time. In what follows, we again assume the system input is $x(t)$, the system output is $y(t)$, and $x(t) \rightarrow y(t)$.

Example 4.3.1. Consider the mathematical model of a system $y(t) = x(t+1)$. Does this model represent a causal system? Often the easiest way to analyze problems like this is to put in various values of t. Thus for $t = 0$ we have $y(0) = x(1)$, and clearly the output at time zero depends on the input at time one, and the system is not causal.

Example 4.3.2. Consider the mathematical model of a system $y(t) = e^{(t+1)}x(t)$. Does this model represent a causal system? This is a causal system, since the output $y(t)$ at any time t depends only on the system input at the same time. Remember that we are only concerned with input-output relationships. The exponential term does not affect the causality of the system.

Example 4.3.3. Consider the mathematical model of a system $y(t) = x(-t)$. Does this model represent a causal system? This system is not causal, since for $t = -1$ we have $y(-1) = x(1)$, and the output depends on a future value of the input.

Example 4.3.4. Consider the mathematical model of a system $y(t) = x\left(1 - \dfrac{t}{2}\right)$. Does this model represent a causal system? This system is a bit more difficult to analyze than the previous systems. If this system is causal, then we must have $t \geq 1 - \dfrac{t}{2}$ or $\dfrac{3}{2}t \geq 1$. This will not be true for all time, so the system is not causal.

Example 4.3.5. Consider the mathematical model of a system $\dot{y}(t) + 2ty(t) = x(t+1)$. Does this model represent a causal system? In order to answer this, we will solve for the output as a function of the input (and review integration factors as an added bonus!). We have

$$\frac{d}{dt}\left[y(t)e^{t^2}\right] = \dot{y}(t)e^{t^2} + 2te^{t^2}y(t) = e^{t^2}[\dot{y}(t) + 2ty(t)] = e^{t^2}\left[x(t+1)\right]$$

or

$$\frac{d}{dt}\left[y(t)e^{t^2}\right] = e^{t^2}\left[x(t+1)\right]$$

Integrating both sides from some initial time t_0 up to an arbitrary final time t we have

$$\int_{t_0}^{t} \frac{d}{d\lambda}[y(\lambda)e^{\lambda^2}]d\lambda = y(t)e^{t^2} - y(t_0)e^{t_0^2} = \int_{t_0}^{t} e^{\lambda^2}x(\lambda+1)d\lambda$$

Rearranging this we have

$$y(t) = y(t_0)e^{-t^2+t_0^2} + \int_{t_0}^{t} e^{-t^2+\lambda^2}x(\lambda+1)d\lambda$$

Now we can determine that the output at any time t, $y(t)$, depends on the input from time $t_0 + 1$ up until time $t + 1$. Thus the system is not causal.

4.4 Memoryless Systems

A system is *memoryless* or *instantaneous* if the output at any time t does not depend on past or future values of the input. If the output does directly depend on the input, then $y(t)$ must be an algebraic function of $x(t)$.

Example 4.4.1. Consider the mathematical model of a system $y(t) = x(t+1)$. Does this model represent a memoryless system? No, the system model is not memoryless, since the output at any time depends on a future input.

Example 4.4.2. Consider the mathematical model of a system $y(t) = x(t-1)$. Does this model represent a memoryless system? No, the system model is not memoryless, since the output at any time depends on a past input.

Example 4.4.3. Consider the mathematical model of a system $y(t) = x^2(t) + 2$. Does this model represent a memoryless system? Yes, the system model is memoryless, since the output at any time depends on the input only at that time.

Example 4.4..4. Consider the mathematical model of a system $\dot{y}(t) + 2ty(t) = x(t+1)$. Does this model represent a memoryless system? No, the system model is not memoryless, since the output at any time depends on past and future values of the input.

4.5 Invertible Systems

An *invertible* system a system in which each output is associated with a unique input. That is, there is a one-to-one relationship between the system input and the system output.

Example 4.5.1. The mathematical models of systems $y(t) = \cos(x(t))$ and $y(t) = x^2(t)$ are not invertible, since there is more than one input that produces the same output.

4.6 Bounded Input Bounded Output (BIBO) Stable Systems

A mathematical model of a system that produces a bounded output for every bounded input is a *bounded-input bounded-output stable system*. Note that we do not need to know what the output is for every input, only that it is bounded. In addition, since we are looking at the input output relationship *we assume all of the initial conditions are zero*. Mathematically, if $|x(t)| \le M$ for some finite constant M means $|y(t)| < N$ for some finite constant N, then the system is BIBO stable.

Example 4.6.1. Is the mathematical model of a system $y(t) = e^{x(t)}$ BIBO stable? If we assume $|x(t)| \le M$ then we have $y(t) \le e^M = N$ and the model is BIBO stable.

Example 4.6.2. Is the mathematical model of a system $y(t) = \cos\left(\dfrac{1}{x(t)}\right)$ BIBO stable ? The answer is

yes, since we know that the cosine is always bounded between -1 and 1, so the output is always bounded, even if we do not know what it is.

Example 4.6.3. Is the mathematical model of a system $y(t) = \displaystyle\int_0^t e^{-(t-\lambda)} x(\lambda) d\lambda$ BIBO stable? Since we

assume the input is bounded, $|x(t)| \le M$, we can write

$$y(t) = \int_0^t e^{-(t-\lambda)} x(\lambda) d\lambda \le M \int_0^t e^{-(t-\lambda)} d\lambda = Me^{-t} \int_0^t e^{\lambda} d\lambda = Me^{-t}(e^t - 1) = M(1 - e^{-t}) \le M$$

Hence $|y(t)| \le M$ and the output is bounded, so this is a BIBO stable model.

4.7 Linear Time-Invariant (LTI) Systems

In this course we will focus our attention on systems that are both linear and time-invariant, commonly referred to as LTI systems. If we have an LTI system and know the response of the system to specific input, then we can determine the response of the system to any linear and time shifted combination of those inputs. For example, assume we know the input/output relationships $x_i(t) \to y_i(t)$ for various inputs $x_i(t)$ and the corresponding outputs $y_i(t)$. Then, since the system is LTI, we know
$\alpha_i x_i(t) \to \alpha_i y_i(t)$,
$\displaystyle\sum_i \alpha_i x_i(t) \to \sum_i \alpha_i y_i(t)$, $x_i(t-t_i) \to y_i(t-t_i)$, and $\displaystyle\sum_i \alpha_i x_i(t-t_i) \to \sum_i \alpha_i y_i(t-t_i)$. Let's illustrate the
implications of this with a few examples.

Example 4.7.1. Assume that we know that if the input is $x(t) = u(t)$, a unit step (Heaviside) function,

then the output of an LTI system will be $y(t) = e^{-t} u(t)$. Now assume we want to use this information to determine the response of the system to a pulse of width (duration) T and amplitude A . The first thing we need to do is to write out new input $x_{new}(t)$ in terms of our known input. We can write the pulse input as

$$x_{new}(t) = Au(t) - Au(t-T) = Ax(t) - Ax(t-T)$$

Since the system is LTI we know

$$Ax(t) \to Ay(t)$$
$$Ax(t-T) \to Ay(t-T)$$

and the new output will be

$$y_{new}(t) = Ay(t) - Ay(t-T) = Ae^{-t} u(t) - Ae^{-(t-T)} u(t-T)$$

4.8 Linearizing Nonlinear System Models

While we will be focusing on linear systems, many systems we commonly use are actually nonlinear. However, if we operate them only over a limited range of inputs, the assumption of linearity is reasonably accurate. This is very common with electronic circuits, such as BJT and MOSFET transistors, where we use "small signal" models and assume these devices can be modeled as linear for small enough input signals.

In general, if we have an input-output relationship of the form, $y(t) = f(x(t))$, then we can use a Taylor series approximation of f that we can use for small $x(t)$. It may seem odd that we are allowing x to be a function of time, but the idea is the same. When the input is zero we have $y(0) = f(0)$, and this provides our nominal operating point. We can then approximate the output for small inputs as

$$y(t) \approx f(0) + \left[\frac{df(x)}{dx} |_{x=0} \right] x(t) = y(0) + \left[\frac{df(x)}{dx} |_{x=0} \right] x(t)$$

If we look at the deviations from the operating point, we have

$$y(t) - y(0) = \Delta y(t) \approx \left[\frac{df(x)}{dx} |_{x=0} \right] x(t) = mx(t)$$

where we have defined m as the slope near $x = 0$, $m = \dfrac{df(x)}{dx} |_{x=0}$. With this approximation we have the linear relationship between $\Delta y(t)$ and $x(t)$, $\Delta y(t) = mx(t)$.

Example 4.8.1. Consider the nonlinear system model $y(t) = \dfrac{1}{2 + x(t)}$. Determine a linear model for small signals. We can write this as

$$y(0) = f(0) = \frac{1}{2}$$
$$y(t) = f(x) = (2 + x)^{-1}$$

$$\frac{df(x)}{dx} |_{x=0} = -1(2 + x)^{-2} |_{x=0} = -\frac{1}{4}$$

so

$$y(t) \approx \frac{1}{2} - \frac{1}{4} x(t)$$

Looking at the deviation about the operating point we get the small signal linear model

$$\Delta y(t) = y(t) - \frac{1}{2} \approx -\frac{1}{4} x(t)$$

Example 4.8.2. Consider the nonlinear system model $y(t) = 1 + 3e^{x(t)}$. Determine a linear model for small signals. We can write this as

$$y(t) = f(0) = 4$$
$$y(t) = f(x) = 1 + 3e^x$$
$$\frac{df(x)}{dx}\bigg|_{x=0} = 3e^x \big|_{x=0} = 3$$

so

$$y(t) \approx 4 + 3x(t)$$

We then have the small signal linear model

$$\Delta y(t) = y(t) - 4 = 3x(t)$$

Chapter 4 Problems

1) For each of the following methematical description of a system, determine if the system is linear, time invariant, causal, and memoryless and fill in the following table. Unless the system does not meet the homogeneity condition, you must use a formal technique to show the system is linear. If it does not meet the homogeneity condition, you must give an example. For the memoryless and causal you can just say it's obvious (assuming it is).

System	Linea (Y/N)	Time Invariant (Y/N)	Memoryless (Y/N)	Causal (Y/N)
$\dot{y}(t) = x(t)$				
$y(t) = \dfrac{1}{1 + x(t)}$				
$\ddot{y}(t) + e^{-t} y(t) = \cos(t)x(t)$				
$y(t) = \displaystyle\int_{-\infty}^{t} e^{t+\lambda} x(\lambda)d\lambda$				
$y(t) = \displaystyle\int_{-\infty}^{t} e^{t-\lambda} x(\lambda+1)d\lambda$				
$y(t) = \displaystyle\int_{-\infty}^{t} e^{-\lambda} \lambda x(\lambda)d\lambda$				
$\dot{y}(t) + y(t)x(t) = x(t)$				

Answers: 5 are L, 4 are TI, 1 is memoryless, 6 are causal

2) For each of the following mathematical descriptions of a system, determine if the system is BIBO stable.

a) $y(t) = \dfrac{1}{1 + x(t)}$ **b)** $y(t) = e^{-x(t)}$ **c)** $y(t) = \cos(x(t))$

d) $y(t) = \displaystyle\int_{-\infty}^{t} e^{-2(t-\lambda)} x(\lambda)d\lambda$ **e)** $y(t) = \displaystyle\int_{-\infty}^{t} e^{2(t-\lambda)} x(\lambda)d\lambda$ **f)** $y(t) = tx(t)$

Answers: 3 are BIBO stable

3) Assume we have an LTI system with the input-output relationship $x(t) = u(t) \rightarrow y(t) = e^{-2(t-1)}u(t-1)$. This means that if the input is a unit step starting at zero, the output is a decaying exponential starting at $t = 1$. Determnine an expression for output the following inputs:

a) $x(t) = u(t-2)$ **b)** $x(t) = 6u(t+2)$ **c)** $x(t) = 2[u(t-1) - u(t-2)]$

The answer to c is $y(t) = 2e^{2(t-2)}u(t-2) - 2e^{2(t-3)}u(t-3)$

4) Matlab Problem

One property of linear systems that we have not discussed is that *if a system is linear, then if the input is a sinusoid, then the output must be a sinusoid.* The magnitude and phase can be different between the input and output signals, but they must both be sinusoids of the same frequency or the system is not linear.

a) Create a new Matlab m-file. Use **linspace** to create a time vector **t** from 0 to 20 seconds using 2000 points. Create the input sinusoid **x = cos(t)**; You can just type this command into your m-file, you should not use an ananymous function here. The variables **t** and **x** will be an array of the form

$$t = [t_1 \ t_2 ... t_{2000}] = [0...20] \qquad x = [\cos(t_1) \ \cos(t_2)... \cos(t_{2000})]$$

The systems we want to look at are represented by the following relationships

system A: $y = \tan(x)$ system B: $y = \exp(x)$ system C : $y = 1/(2+x)$ system D: $y = x^2$

Use Matlab to plot the system input and system output on four graphs (one a single page). Use the **subplot** command. Before your first plot, type **orient tall**, this will use more of the page. Your graph should look like that on the next page. You will also need the commands **xlabel, ylabel**, **grid**, and **legend** (among others). You may also need to use the commands **./** and **.*** for element by element division and multiplication. All of these systems are clearly non-linear. ***Turn in your code and your plot.***

b) Sometimes, although a system is nonlinear, we can approximate the system as linear if we only consider **small signals**. If the input signal is small enough, we know from calculus that we can approximate the output using the Taylor series (about $x = 0$, since we assume a small input),

$$y(x) \approx y(0) + \left(\frac{dy(x)}{dx} \bigg|_{x=0} \right) x$$

Even though x is a function of time, we can still use this approximation for functions. We can rewrite this as

$$\Delta y = y(x) - y(0) \approx mx = \left(\frac{dy(x)}{dx} \bigg|_{x=0} \right) x$$

where

$$m = \left(\frac{dy(x)}{dx} \bigg|_{x=0} \right)$$

If this first order approximation is valid, then we have the linear relationship $\Delta y \approx mx$. Now change your code so the input to the system is **x = cos(t)/10.** For each of your systems, determine $y(0)$ and plot time (t) versus y-y(0) using the same format as in part a. Do not plot the input signal, we only want to see the output signal. The first three system should now approximately be linear, while the fourth is still non-linear. *Note that if you have done this correctly, the output signal should be centered at zero.* ***Turn in your code and your plot.***
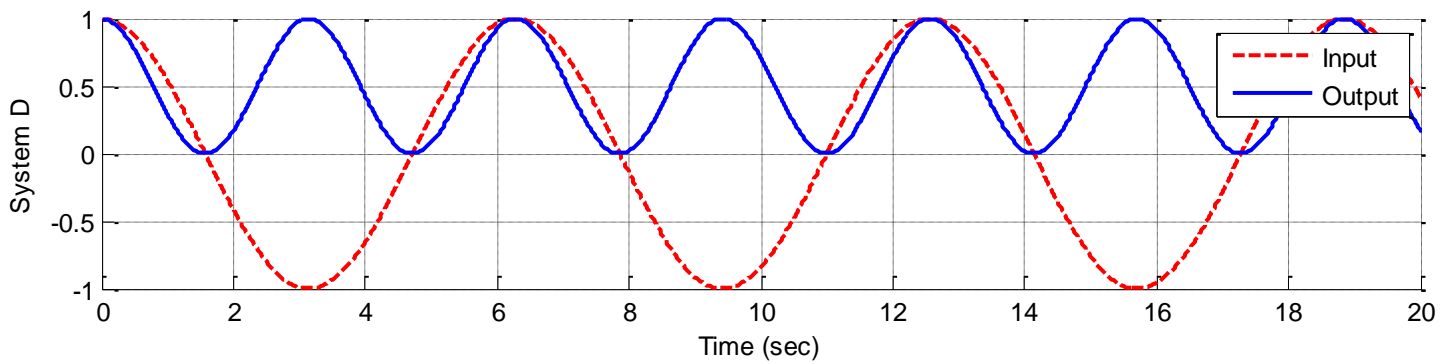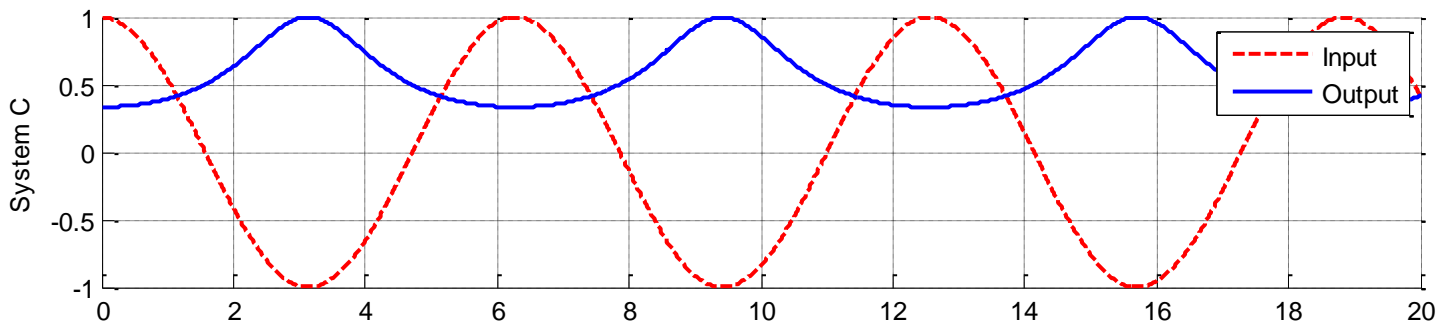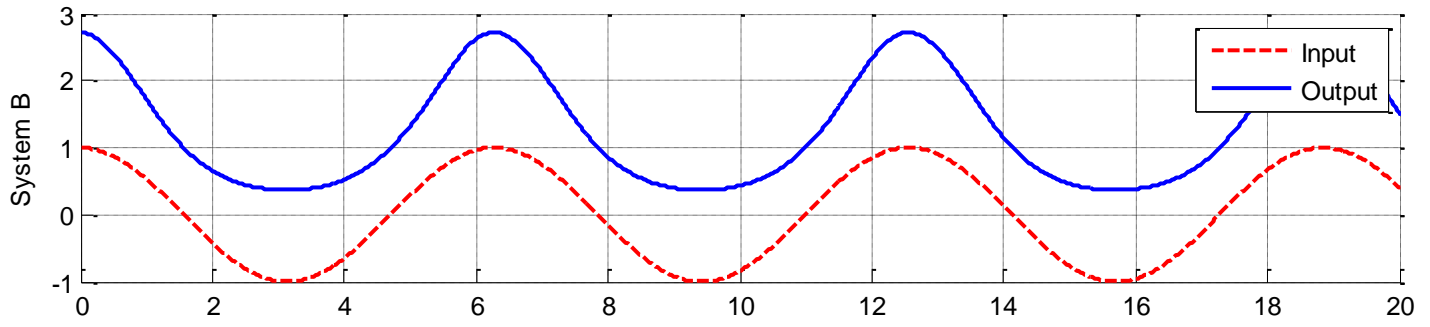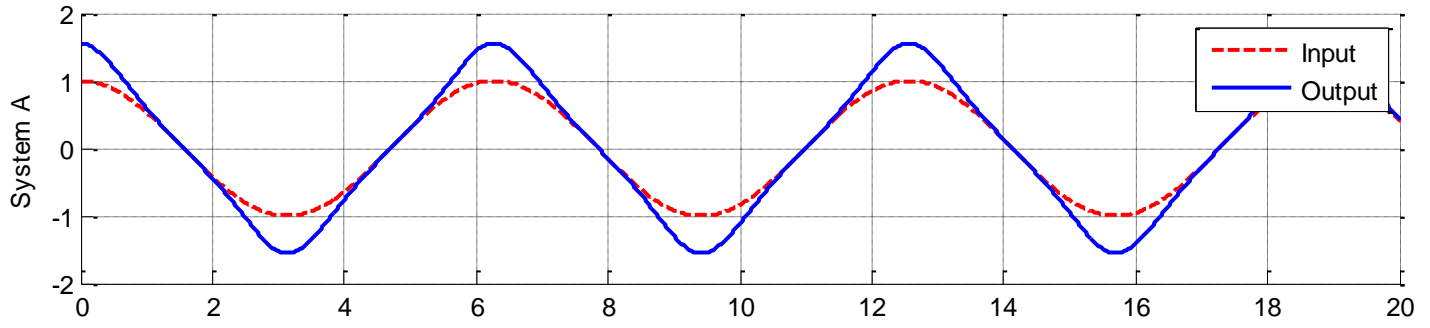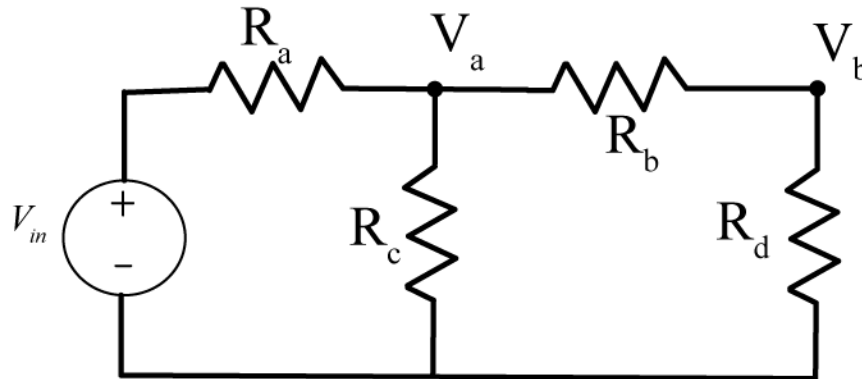
5) Matlab Problem

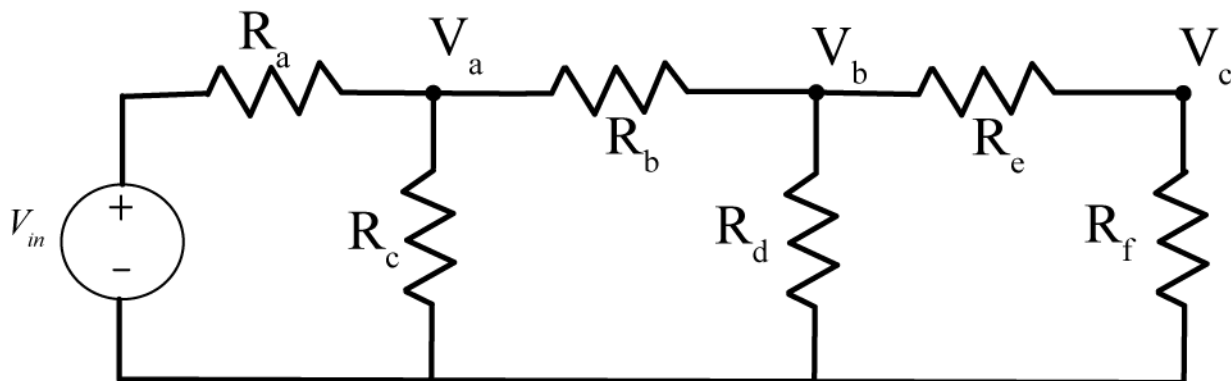Read the Appendix at the end of the problem section and then do the following:

a) Create an m-files to set up and solve matrix equations for the node values for the following circuits. Use **both** methods from the Appendix to set up the system of equations.
Assume $R_a = 1000\Omega, R_b = 5000\Omega, R_c = 2000\Omega, R_d = 1500\Omega$ and $V_{in} = 5V$.
Answer: $V_a = 3.023V, V_b = 0.698V$

b) Create m-files to set up and solve matrix equations for the node values for the following circuit. Use **both** methods from the Appendix to set up the system of equations.
Assume $R_a = 1000\Omega, R_b = 5000\Omega, R_c = 2000\Omega, R_d = 1500\Omega, R_e = 500\Omega, R_f = 5000\Omega$ and $V_{in} = 5V$.
Answer: $V_a = 3.0087V, V_b = 0.5739V, V_c = 0.5217V$
Turn in your m-files and write the node voltages obtained on your code.

c) Create m-files to set up and solve matrix equations for the node values for the following circuit. Use **_both_** methods from the Appendix to set up the system of equations.

Assume $R_a = 1000\,\Omega, R_b = 5000\,\Omega, R_c = 2000\,\Omega, R_d = 1500\,\Omega, R_e = 500\,\Omega, R_f = 5000\,\Omega$ and $V_{in} = 5V$.

Turn in your m-files and write the node voltages obtained on your code.
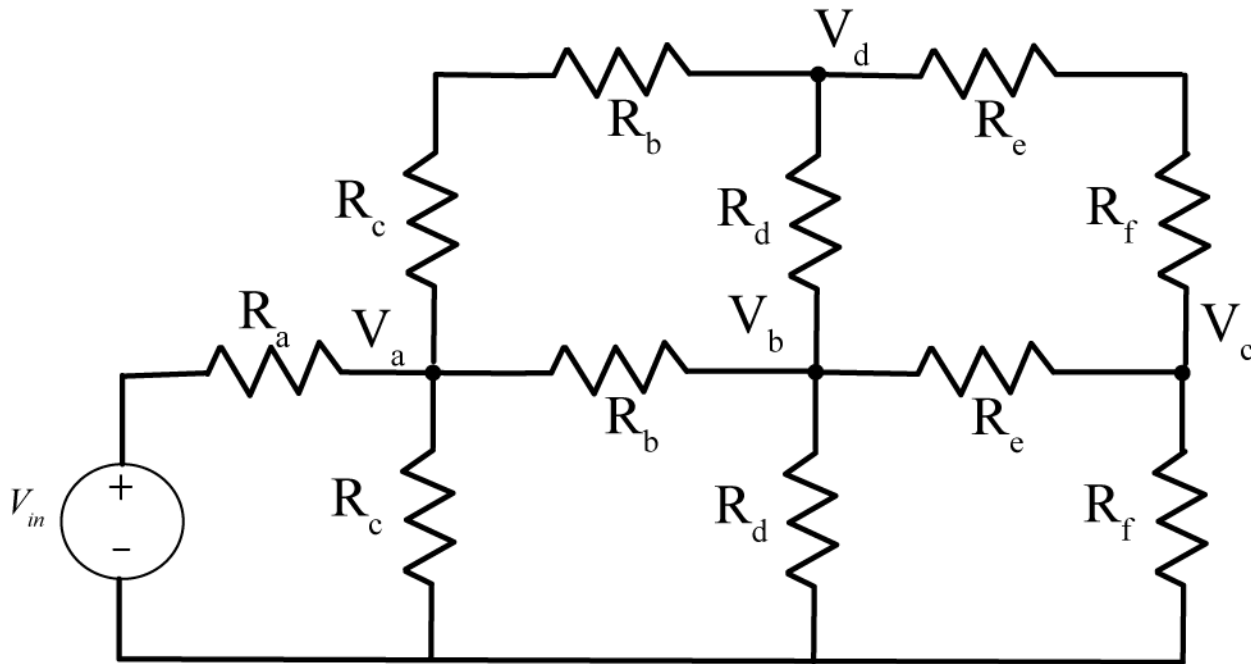
Appendix

In addition to working well with vectors, Matlab is also very good at matrices. In this Appendix we will show you how to set up and solve matrix equations in Matlab.

Consider the following circuit, with two nodes V_a and V_b.
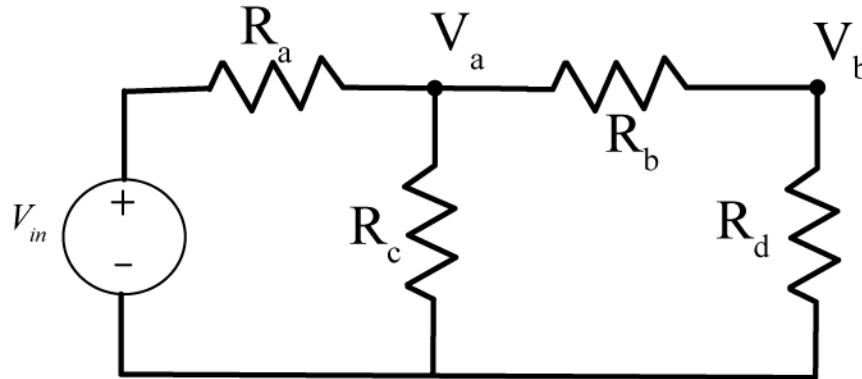
We can write node equations at these two nodes as

$$\frac{V_a - V_{in}}{R_a} + \frac{V_a}{R_c} + \frac{V_a - V_b}{R_b} = 0$$

$$\frac{V_b - V_a}{R_b} + \frac{V_b}{R_d} = 0$$

Rearranging these we get

$$\left[\frac{1}{R_a} + \frac{1}{R_b} + \frac{1}{R_c}\right]V_a + \left[-\frac{1}{R_b}\right]V_b = \left[\frac{1}{R_a}\right]V_{in}$$

$$\left[-\frac{1}{R_b}\right]V_a + \left[\frac{1}{R_b} + \frac{1}{R_d}\right]V_b = 0$$

We can then write these in standard matrix form, $Ax = b$, as

$$\begin{bmatrix} \dfrac{1}{R_a} + \dfrac{1}{R_b} + \dfrac{1}{R_c} & -\dfrac{1}{R_b} \\[3ex] -\dfrac{1}{R_b} & \dfrac{1}{R_b} + \dfrac{1}{R_d} \end{bmatrix} \begin{bmatrix} V_a \\[2ex] V_b \end{bmatrix} = \begin{bmatrix} \dfrac{V_{in}}{R_a} \\[2ex] 0 \end{bmatrix}$$

Before we put this into Matlab, we need to assign some values to the resistors. Let's assume we have the values $R_a = 1000\Omega, R_b = 5000\Omega, R_c = 2000\Omega, R_d = 1500\Omega$ and $V_{in} = 5V$. We can now construct our matrices in two different ways in Matlab, as shown by the following code segment

```matlab
%
%  circuit parameters
%
  Ra = 1000; Rb = 5000; Rc=2000;
  Rd = 1500; Re = 500; Rf = 5000;
  Vin = 5;
%
%  Method 1
%
%  create matrices and vectors that are all zeros, then fill them in
%
  A = zeros(2,2);   % construct a 2x2 matrix filled with zeros
  b = zeros(2,1);   % construct a 2x1 vector filled with zeros
%
%  fill in the values
%
  A(1,1) = 1/Ra + 1/Rb + 1/Rc;
  A(1,2) = -1/Rb;
  A(2,1) = -1/Rb;
  A(2,2) = 1/Rb + 1/Rd;
  b(1) = (1/Ra)*Vin;
%
%  solving the matrix equation
%
  x = A\b
%
%  Method 2
%
%  construct the matrices and vectors directly
%
%  note that the columns are separated by spaces, and each row
%  ends with a semicolon (;)
%
  A = [(1/Ra+1/Rb+1/Rc) -1/Rb; -1/Rb (1/Rb + 1/Rd)];
  b = [(1/Ra)*Vin; 0]
%
%  solving the matrix equation
%
  x = A\b
%
%  check your answer
%
  A*x
  b
%
% extract the node voltages
%
  Va = x(1)
  Vb = x(2)
```

5.0 Impulse Response, Step Response, and Convolution

In this chapter we confine ourselves to systems that can be modeled as linear and time-invariant, or LTI systems. For these types of systems, we can determine the output of the system to any input in a very systematic way. We can also determine a great deal about the system just by looking at how it responds to various types of inputs. The most fundamental of these inputs is the impulse response, or the response of a system at rest to an impulse. However, the response of a system to a step is much easier to determine and can be used to determine the impulse response of any LTI system.

5.1 Impulse or Delta Functions

An impulse, or delta function, $\delta(t)$, is defined as a function that is zero everywhere except at one point, and has an area of one. Mathematically, we can write this as

$$\delta(\lambda) = 0, \lambda \neq 0$$

$$\int_{-\mu}^{\mu} \delta(\lambda)d\lambda = 1, \ \mu > 0$$

Note that we do not know the value of the delta function at zero, it is undefined! We can think of, or model, delta functions as functions that exist in some type of limit. For example, the functions displayed in Figures 5.1, 5.2, and 5.3 can be thought of as different models for delta functions, since the meet our two (simplistic) requirements above.

$$\delta(t) \approx \lim_{\epsilon \to 0} \begin{cases} \frac{1}{2\epsilon} & |t| < \epsilon \\ 0 & |t| > \epsilon \end{cases}$$
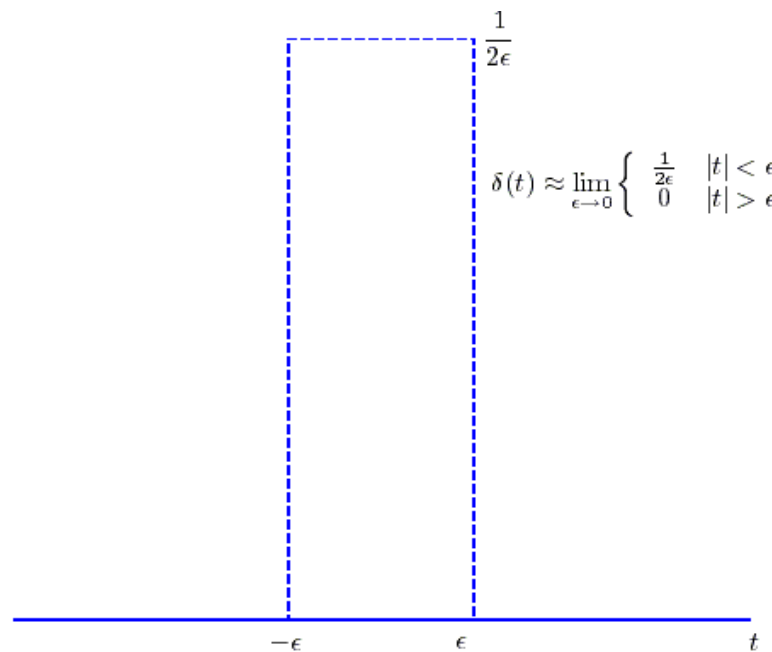
Figure 5.1. Rectangular model of an impulse (delta) function.

$$\delta(t) \approx \lim_{\epsilon \to 0} \begin{cases} \frac{1}{\epsilon^2}t + \frac{1}{\epsilon} & -\epsilon \le t \le 0 \\ -\frac{1}{\epsilon^2}t + \frac{1}{\epsilon} & 0 \le t \le \epsilon \\ 0 & t > |\epsilon| \end{cases}$$
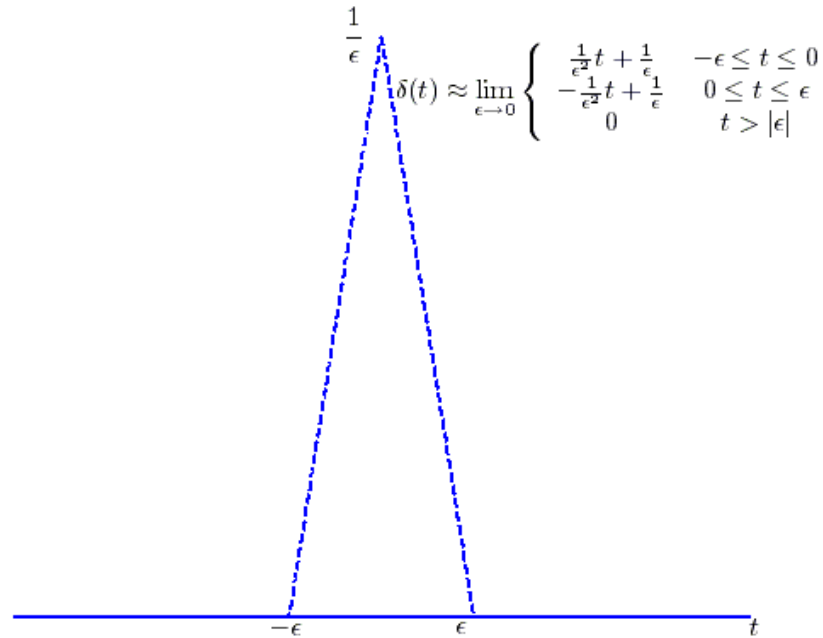
Figure 5.2. Triangular model of an impulse (delta) function.

$$\delta(t) \approx \lim_{\epsilon \to 0} \frac{\epsilon}{\sqrt{\pi}} e^{-\epsilon^2 t^2}$$
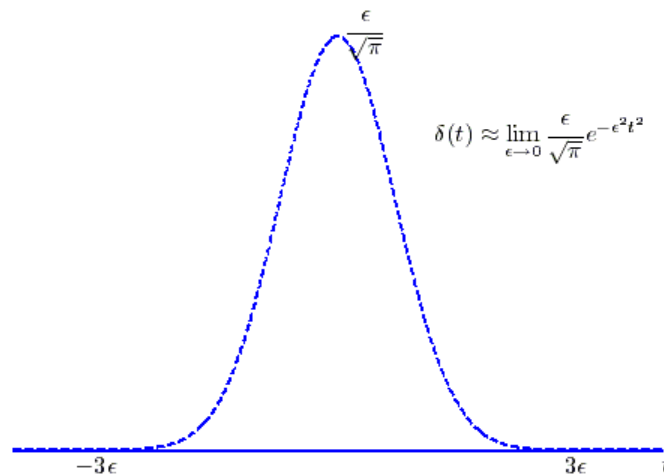
Figure 5.3. Gaussian model of an impulse (delta) function.

Although delta functions are really idealized functions, they form the basis for much or the study of systems. Knowing how a system will respond to an impulse (an impulse response) tells us a great deal about a system, and lets us determine how the system will response to any arbitrary input.

There following two very important properties of delta functions will be used extensively:

Property 1: $\phi(t)\delta(t-t_0) = \phi(t_0)\delta(t-t_0)$

Property 2 (Sifting Property): $\int_a^b \phi(t)\delta(t-t_0)dt = \phi(t_0) \quad a < t_0 < b$

The first property is pretty easy to understand if we think about the definition of a delta function. A delta function is zero everywhere except when its argument is zero, so both sides of the equation are zero everywhere except at t_0, and then at t_0 both sides have the same value.

The second property follows directly from the first property as follows:

$$\int_a^b \phi(t)\delta(t-t_0)dt = \int_a^b \phi(t_0)\delta(t-t_0)dt = \phi(t_0)\int_a^b \delta(t-t_0)dt = \phi(t_0) \quad a < t_0 < b$$

It is very important that the limits of the integral are such that the delta function is within the limits of the integral, or else the integral is zero.

Example 5.1.1. You should understand each of the following identities, and how to use the two properties to arrive at the correct solution.

$$e^t \delta(t-1) = e^1 \delta(t-1)$$

$$t^2 \delta(t-2) = 4\delta(t-2)$$

$$\int_0^\infty t^2 \delta(t-2)dt = 4$$

$$\int_0^{10} e^t \delta(t-1)dt = e^1$$

$$\int_{-10}^{10} e^t \delta(t-20)dt = 0$$

$$\int_{-\infty}^\infty \delta(t-1)\delta(t-2)dt = 0$$

5.2 Unit Step (Heaviside) Functions

We will define the unit step function as

$$u(\tau) = \begin{cases} 1 & \tau > 0 \\ 0 & \tau < 0 \end{cases}$$

We will not define $u(0)$, though some textbooks define $u(0) = \dfrac{1}{2}$. The argument of the unit step was deliberately not written as t, since this sometimes leads to some confusion when solving problems. It is

generally better to remember that the unit step is one whenever the argument (τ) is positive, and then try and figure out what this might mean in terms of t.

Example. 5.2.1. The following are some simple examples with unit step functions:

a) $u(t-1)=1$ for $t-1>0$ or $t>1$

b) $u(2-t)=1$ for $2-t>0$ or $2>t$

c) $u\left(4-\dfrac{t}{3}\right)=1$ for $4-\dfrac{t}{3}>0$ or $12>t$

Unit step functions also show up in integrals, and it is useful to be able to deal with them in that context. The usual procedure is to determine when the unit step function (or functions) are one, and then do the integrals. If the unit step functions are not one, then the integral will be zero. When you are done with the integral, you may need to preserve the information indicating that the integral is zero unless the unit step functions are "on", and this is usually done by including unit step functions. A few examples will hopefully clear this up.

Example 5.2.1. Simplify $\displaystyle\int_{-\infty}^{\infty} u(t-\lambda)u(\lambda-1)d\lambda$ as much as possible. We need both unit step functions to be one, or the integral is zero. We need then

$$u(t-\lambda)=1 \text{ for } t-\lambda>0 \text{ or } t>\lambda$$
$$u(\lambda-1)=1 \text{ for } \lambda-1>0 \text{ or } \lambda>1$$

The integral then becomes $\displaystyle\int_{1}^{t}(1)(1)d\lambda = t-1$. However, we are not done yet. We need to be sure both of the unit step functions are 1, which means we need $t>\lambda>1$, or $t>1$. So the answer is zero for $t<1$ and $t-1$ for $t>1$. The way we can write this compactly is $(t-1)u(t-1)$, which is the final answer.

Example 5.2.2. Simplify $\displaystyle\int_{-\infty}^{t+2} e^{-t}u(\lambda-2)d\lambda$ as much as possible. We need the step function to be one, or the integral is zero. We need then

$$u(\lambda-2)=1 \text{ for } \lambda-2>0 \text{ or } \lambda>2$$

The integral becomes

$$\int_{2}^{t+2} e^{-\lambda}(1)d\lambda = e^{-2}-e^{-(t+2)} = e^{-2}(1-e^{-t})$$

However, the integral will be zero unless $t+2>\lambda>2$ or $t>0$. The final answer is then $e^{-2}(1-e^{-t})u(t)$.

Example 5.2.3. Simplify $\int_{-\infty}^{\infty} u(t-\lambda)\delta(\lambda+2)d\lambda$ as much as possible. This integral has both an impulse and a unit step function. While we might be tempted to use the unit step function to set the limits of the integral, the best (and easiest) thing to do is to just use the sifting property of impulse functions. This gives the result $\int_{-\infty}^{\infty} u(t-\lambda)\delta(\lambda+2)d\lambda = u(t+2)$

Example 5.2.4. Simplify $\int_{-\infty}^{t} e^{-(t-\lambda)}\delta(\lambda-2)d\lambda$ as much as possible. We can again use the sifting property with this integral, but we must be careful. If we do not integrate past the impulse function, the integral will be zero. Hence we have

$$\int_{-\infty}^{t} e^{-(t-\lambda)}\delta(\lambda-2)d\lambda = \begin{cases} e^{-(t-2)} & t>2 \\ 0 & t<2 \end{cases}$$

which we can write in a more compressed form as $e^{-(t-2)}u(t-2)$.

Example 5.2.5. Simplify $\int_{t-1}^{\infty} e^{t+\lambda}\delta(\lambda+2)d\lambda$ as much as possible. Using the sifting property we have

$$\int_{t-1}^{\infty} e^{t+\lambda}\delta(\lambda+2)d\lambda = \begin{cases} e^{t-2} & t-1<-2 \\ 0 & t-1>-2 \end{cases}$$

which we can write more compactly as $e^{t-2}u(1-t)$.

Finally, if we consider integrating an impulse, $\int_{-\infty}^{t} \delta(\lambda)d\lambda$, we will either get a one (if we integrate past the impulse) or a zero (if we do not). Thus we have

$$\int_{-\infty}^{t} \delta(\lambda)d\lambda = \begin{cases} 1 & t>0 \\ 0 & t<0 \end{cases}$$

or

$$\int_{-\infty}^{t} \delta(\lambda)d\lambda = u(t)$$

If we differentiate both sides of this we get

$$\frac{du(t)}{dt} = \delta(t)$$

This relationship is important to remember, but when doing integrals it is generally a better idea to remember what conditions you may need to impose in order to determine if and what unit step functions will be required.

5.3 Impulse Response

The *impulse response* of an LTI system is the response of the system initially at rest (no initial energy, all initial conditions are zero) to an impulse at time $t = 0$. The most common way to denote the impulse response is by lower case letters h and g, though others are used.

Example 5.3.1. Consider the circuit shown in Figure 5.4. Determine the impulse response of the system. The circuit is a simple voltage divider, so we have

$$y(t) = \left(\frac{R_b}{R_a + R_b} \right) x(t)$$

and the impulse response is

$$h(t) = \left(\frac{R_b}{R_a + R_b} \right) \delta(t)$$
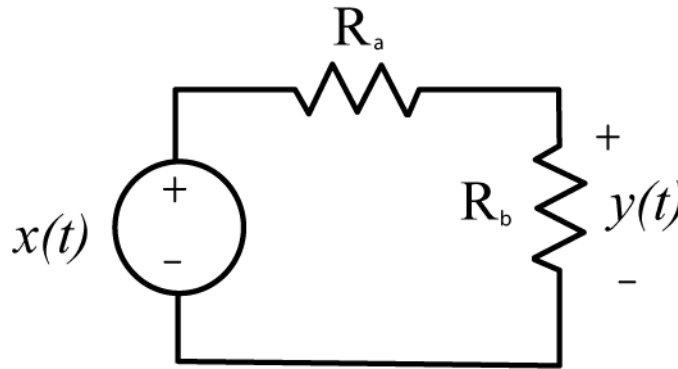
Figure 5.4. Circuit used for Example 5.3.1.

Example 5.3.2. Consider the circuit shown in Figure 5.5. Determine the impulse response of the system. We have

$$\frac{x(t) - y(t)}{R} = C \frac{dy(t)}{dt} + \frac{y(t)}{R}$$

or

$$\frac{dy(t)}{dt} + \frac{2}{RC} y(t) = \frac{1}{RC} x(t)$$

Then

$$\frac{d}{dt} \left[y(t) e^{2t/RC} \right] = e^{2t/RC} \frac{1}{RC} x(t)$$

Integrating from $-\infty$ up to t, and assuming the system is initially at rest, we have

$$y(t) = \frac{1}{RC} \int_{-\infty}^{t} e^{-2(t-\lambda)/RC} x(\lambda) d\lambda$$

The impulse response is then given by $h(t) = \frac{1}{RC} \int_{-\infty}^{t} e^{-2(t-\lambda)/RC} \delta(\lambda) d\lambda = \frac{1}{RC} e^{-2t/RC} u(t)$

89
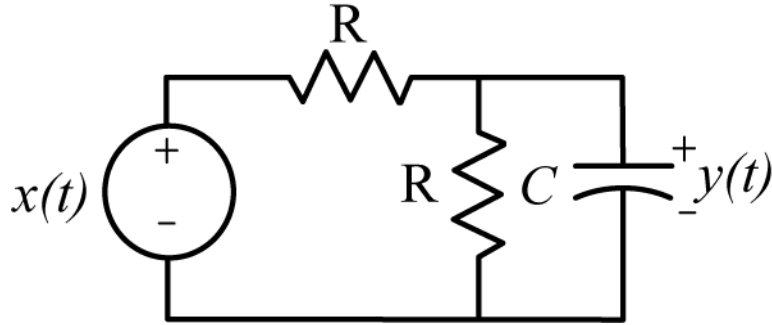
Figure 5.5. Circuit used for Example 5.3.2.

Example 5.3.3. Consider the system described by the mathematical model

$$y(t) = x(t-1) + \int_{-\infty}^{t} e^{-(t-\lambda)} x(\lambda+2) d\lambda$$

The impulse response will be given by

$$h(t) = \delta(t-1) + \int_{-\infty}^{t} e^{-(t-\lambda)} \delta(\lambda+2) d\lambda = \delta(t-1) + e^{-(t+2)} u(t+2)$$

5.4 Step Response

The *step response* of an LTI system is the response of the system initially at rest (no initial energy, all initial conditions are zero) to a step at time $t=0$. There is no common method for denoting the step response, but we will sometimes denote the step response as $s(t)$. If we know the step response of an LTI system, we can determine the impulse response of the system using the relationship

$$h(t) = \frac{d}{dt} [s(t)]$$

Example 5.4.1. Determine the step response and then use it to determine the impulse response for the system in Example 5.3.2. From the example we have

$$y(t) = \frac{1}{RC} \int_{-\infty}^{t} e^{-2(t-\lambda)/RC} x(\lambda) d\lambda$$

The step response is then given by

$$s(t) = \frac{1}{RC} \int_{-\infty}^{t} e^{-2(t-\lambda)/RC} u(\lambda) d\lambda$$

or

$$s(t) = \frac{1}{RC} e^{-2t/RC} \int_{0}^{t} e^{2\lambda/RC} d\lambda = e^{-2t/RC} \frac{1}{2} [e^{2t/RC} - 1] u(t) = \frac{1}{2} [1 - e^{-2t/RC}] u(t)$$

The impulse response is then given by

$$h(t) = \frac{d}{dt} s(t) = \frac{d}{dt} \left\{ [1 - e^{-2t/RC}] \frac{1}{2} u(t) \right\} = \frac{1}{RC} e^{-2t/RC} u(t) + [1 - e^{-2t/RC}] \frac{1}{2} \delta(t) = \frac{1}{RC} e^{-2t/RC} u(t)$$

which is the same answer we obtained before.

5.5. Convolution

Derivation of the Convolution Integral

We will derive the convolution integral using two different, though equivalent methods. Consider an LTI system with input $x(t)$. We can approximate $x(t)$ as a piecewise constant function over intervals of length ΔT, as shown in Figure 5.6. Thus we have the approximation

$$x(t) \approx \sum_{k=-\infty}^{k=\infty} x(k\Delta T) \left\{ u\left[t - \left(k - \frac{1}{2} \right) \Delta T \right] - u\left[t - \left(k + \frac{1}{2} \right) \Delta T \right] \right\}$$

Next, we can write the step response of the system as $s(t)$. Because the system is time-invariant, the response of the system to the input $u(t - (k - \frac{1}{2})\Delta T)$ is $s(t - (k - \frac{1}{2})\Delta T)$, and the response of the system to $u(t - (k + \frac{1}{2})\Delta T)$ is $s(t - (k + \frac{1}{2})\Delta T)$. Because the system is both linear and time-invariant, the response of the system to input $x(t)$ can then be approximated as

$$y(t) \approx \sum_{k=-\infty}^{k=\infty} x(k\Delta T) \left\{ s\left[t - \left(k - \frac{1}{2} \right) \Delta T \right] - s\left[t - \left(k + \frac{1}{2} \right) \Delta T \right] \right\}$$

Now we can approximate the derivative of the step response as

$$h(t - k\Delta t) \approx \frac{s\left[t - \left(k - \frac{1}{2} \right) \Delta T \right] - s\left[t - \left(k + \frac{1}{2} \right) \Delta T \right]}{\Delta T}$$

or

$$h(t - k\Delta t)\Delta T \approx s\left[t - \left(k - \frac{1}{2} \right) \Delta T \right] - s\left[t - \left(k + \frac{1}{2} \right) \Delta T \right]$$
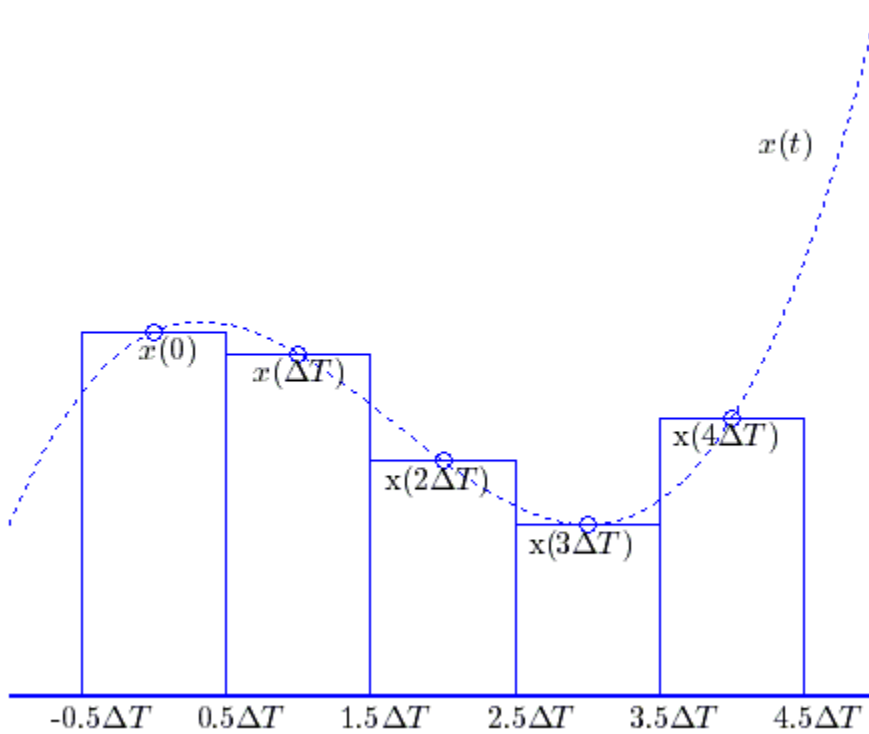
Thus the output can be approximated as

$$y(t) \approx \sum_{k=-\infty}^{k=\infty} x(k\Delta T) h(t - k\Delta T) \Delta T$$

If we define $\lambda = k\Delta T$, then as $\Delta T \to 0$ the sum becomes an integral and we have

$$y(t) = \int_{-\infty}^{\infty} x(\lambda) h(t - \lambda) d\lambda$$

This is one form of the *convolution integral*, which tells us how to determine the output of an LTI system if we know the impulse response of the system and the system input. We would write this as $y(t) = x(t) \star h(t)$, where $\star$ represents the *convolution operator*.

An alternative derivation for convolution would be to start with the same approximation

$$x(t) \approx \sum_{k=-\infty}^{k=\infty} x(k\Delta T) \left\{ u\left[t - \left(k - \frac{1}{2} \right)\Delta T \right] - u\left[t - \left(k + \frac{1}{2} \right)\Delta T \right] \right\}$$

We can then approximate the impulse response as

$$\delta(t - k\Delta T) \approx \frac{u\left[t - \left(k - \frac{1}{2} \right)\Delta T \right] - u\left[t - \left(k + \frac{1}{2} \right)\Delta T \right]}{\Delta T}$$

or

$$\delta(t - k\Delta T)\Delta T \approx u\left[t - \left(k - \frac{1}{2} \right)\Delta T \right] - u\left[t - \left(k + \frac{1}{2} \right)\Delta T \right]$$

Figure 5.6. Approximating the continuous function $x(t)$ as a piecewise constant function for the derivation of the convolution integral.

We then have

$$x(t) \approx \sum_{k=-\infty}^{k=\infty} x(k\Delta T)\delta(t - k\Delta T)\Delta T$$

If the input to the LTI system is $\delta(t - k\Delta T)$ the output will be $h(t - k\Delta T)$, so we can write the output as

$$y(t) \approx \sum_{k=-\infty}^{k=\infty} x(k\Delta T)h(t - k\Delta T)\Delta T$$

If we define $\lambda = k\Delta T$, then as $\Delta T \to 0$ the sum again becomes an integral and we have

$$y(t) = \int_{-\infty}^{\infty} x(\lambda)h(t - \lambda)d\lambda$$

By a change of variable we can show that convolution has *the commutative property*, or

$$y(t) = h(t) \star x(t) = x(t) \star h(t)$$

This means that we can compute the output in one of two equivalent ways:

$$y(t) = \int_{-\infty}^{\infty} x(\lambda)h(t - \lambda)d\lambda = \int_{-\infty}^{\infty} x(t - \lambda)h(\lambda)d\lambda$$

Now that we know how to determine the output of a system given the impulse response and the input, we need to determine the best way to compute this. There are two general ways for computing the convolution, analytical and graphical. Both methods give the same results, but usually the answers initially look different. Analytical evaluation of the convolution integral is generally used for very simple problems, and becomes unwieldy for very complicated problems. Graphical evaluation of the convolution integral is usually used for more complicated problems and is also useful for visualizing what is happening to the signals during convolution.

Analytical Evaluation of the Convolution Integral

In analytical evaluation, we basically just evaluate the integral. It is necessary to utilize any step function in the impulse response or system input to change the limits of the integral. In addition, it is important to remember to include any necessary unit step functions on the output, since the output of one system may be the input to another system.

Example 5.5.1. Determine the output of an LTI system with impulse response $h(t) = Ae^{-t/\tau}u(t)$ to input $x(t) = Bu(t-1) - Bu(t-2)$. To solve this problem we must first choose the way we are going to perform the convolution. We will use the form

$$y(t) = \int_{-\infty}^{\infty} h(t - \lambda)x(\lambda)d\lambda$$

Substituting in our functions we have

$$y(t) = \int_{-\infty}^{\infty} Ae^{-(t-\lambda)/\tau}u(t - \lambda)[Bu(\lambda-1) - Bu(\lambda-2)]d\lambda$$

or

$$y(t) = \int_{-\infty}^{\infty} ABe^{-(t-\lambda)/\tau}u(t-\lambda)u(\lambda-1)d\lambda - \int_{-\infty}^{\infty} ABe^{-(t-\lambda)/\tau}u(t-\lambda)u(\lambda-2)d\lambda$$

Using the step functions to change the limits on the integrals we have

$$y(t) = \int_{1}^{t} ABe^{-(t-\lambda)/\tau}d\lambda - \int_{2}^{t} ABe^{-(t-\lambda)/\tau}d\lambda = ABe^{-t/\tau}\int_{1}^{t} e^{\lambda/\tau}d\lambda - ABe^{-t/\tau}\int_{2}^{t} e^{\lambda/\tau}d\lambda$$

Finally we have

$$y(t) = ABe^{-t/\tau}\tau[e^{t/\tau}-e^{1/\tau}]u(t-1) - ABe^{-t/\tau}\tau[e^{t/\tau}-e^{2/\tau}]u(t-2)$$
$$= \tau AB[1-e^{-(t-1)/\tau}]u(t-1) - \tau AB[1-e^{-(t-2)/\tau}]u(t-2)$$

Graphical Evaluation of the Convolution Integral

As with analytical evaluation, the first thing to do is decide which of the two forms of the convolution integral to use. Let's assume that we are going to use the form

$$y(t) = \int_{-\infty}^{\infty} h(t-\lambda)x(\lambda)d\lambda$$

We need to keep in mind that we want the area under the product of two functions, $h(t-\lambda)$ and $x(\lambda)$. In addition, we need to remember that we are integrating with respect to the dummy variable λ, not t. This is important to understand, since the function $h(t-\lambda)$ will be at different places along the λ axis as the variable t varies. In fact, the whole point of doing graphical convolution is to sketch the function $h(t-\lambda)$ as a function of t and $x(\lambda)$, determine the overlap, and then perform the integration.

One simple method for being able to locate $h(t-\lambda)$ as a function of t and λ is to look at $h(t)$ and find suitable "marker" points. Let's call two such points t_1 and t_2. Then we can find where these marker points are on the λ axis as follows

$$h(t_1) = h(t-\lambda) \rightarrow \lambda = t - t_1$$
$$h(t_2) = h(t-\lambda) \rightarrow \lambda = t - t_2$$

This will all make more sense with a few examples.

Example 5.5.2. Determine the output of an LTI system with impulse response $h(t) = Ae^{-t/\tau}u(t)$ to input $x(t) = Bu(t-1) - Bu(t-2)$. (This is the same problem as Example 5.5.1.) Let's use the integral form

$$y(t) = \int_{-\infty}^{\infty} h(\lambda)x(t-\lambda)d\lambda$$

The top panel of Figure 5.7 displays the input signal. For this signal the most convenient markers are $x(1)$ and $x(2)$. If we can figure out how these points move we can determine the location of $x(t-\lambda)$ on the λ axis as a function of t. For these points we have

$$x(1) = x(t - \lambda) \rightarrow \lambda = t - 1$$
$$x(2) = x(t - \lambda) \rightarrow \lambda = t - 2$$

The bottom panel of Figure 5.7 displays $x(t - \lambda)$ as a function of λ. From this figure we can determine where this function is as t varies.

We next need to graph pictures of $h(\lambda)$ and $x(t - \lambda)$ and look for times when the product of the functions is not zero. For these functions there are three different pictures, corresponding to $t < 1$, $1 \le t \le 2$, and $t \ge 2$, as shown in Figure 5.8.

For $t < 1$ there is no place the product of the functions is not zero, so the output $y(t) = 0$.

For $1 \le t \le 2$ we have

$$y(t) = \int_0^{t-1} A e^{-\lambda/\tau} B d\lambda = AB\tau \left[1 - e^{-(t-1)/\tau} \right]$$

which agrees with our previous answer.
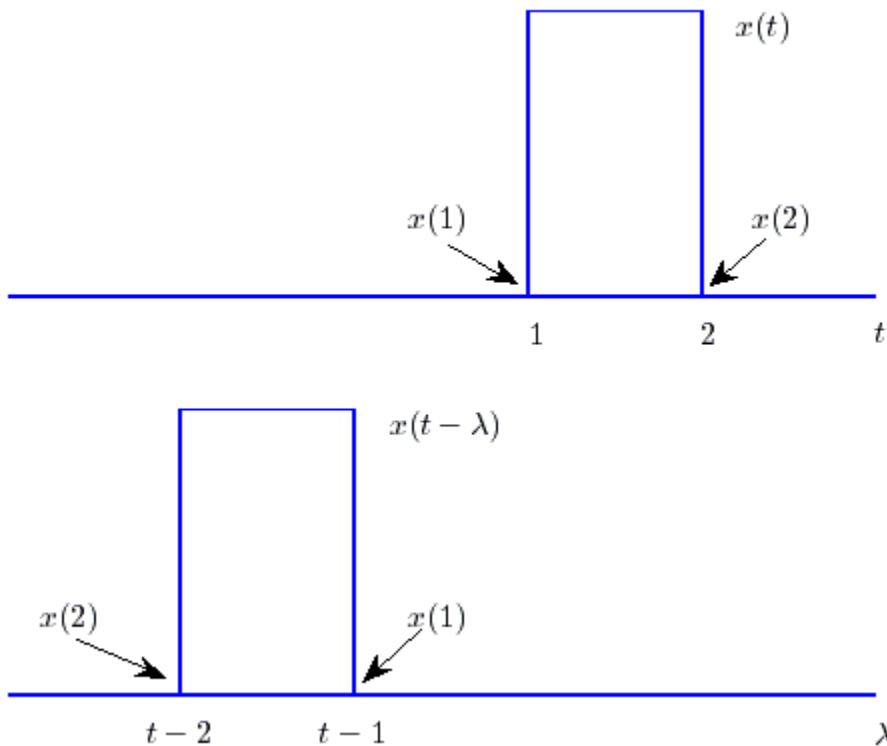
Figure 5.7. Input signal for Example 5.5.2. The original signal $x(t)$ is shown in the top panel, with the two convenient "marker" points. The lower panel shows $x(t - \lambda)$ as a function of λ, and shows how these marker points move as the parameter t is varied. Note that the function $x(t)$ has been flipped (reversed) from its original orientation.
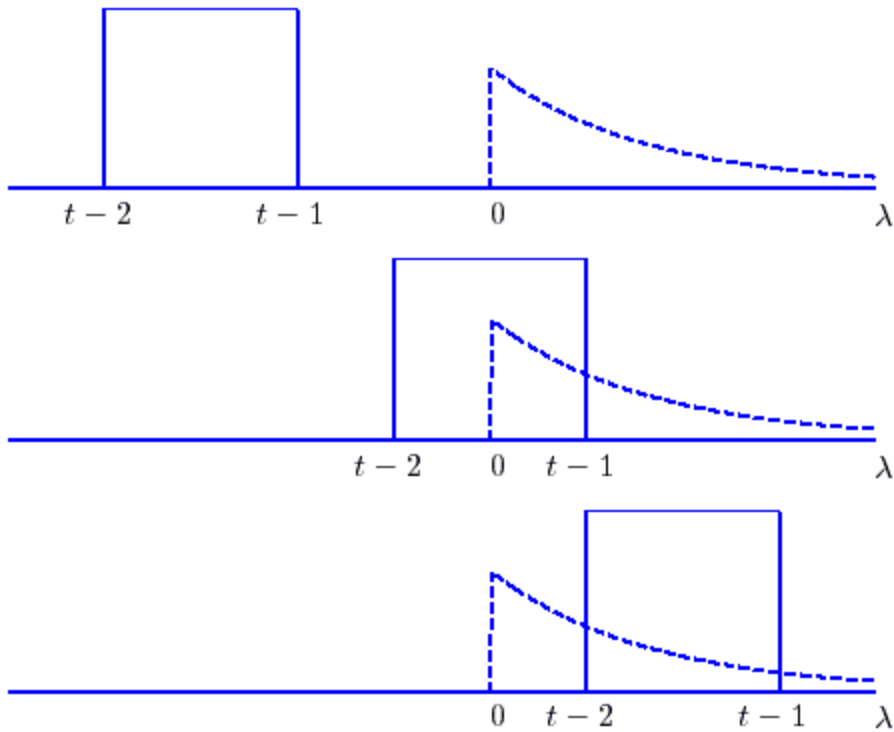
Figure 5.8. Plots of $h(\lambda)$ and $x(t-\lambda)$ (dashed line) for Example 5.5.2 for representative values of t. For these graphs $A=2$, $B=3$, and $\tau = 0.8$.

For $t \geq 2$ we have

$$y(t) = \int_{t-2}^{t-1} Ae^{-\lambda/\tau}Bd\lambda = AB\tau\left[e^{-(t-2)/\tau} - e^{-(t-1)/\tau}\right]$$

Our solution is then $y(t) = \begin{cases} 0 & t \leq 1 \\ AB\tau\left[1-e^{-(t-1)/\tau}\right] & 1 \leq t \leq 2 \\ AB\tau\left[e^{-(t-2)/\tau} - e^{-(t-1)/\tau}\right] & t \geq 2 \end{cases}$

Our solution should be continuous, so we need to check the values at the boundaries. We have

$$y(1) = 0$$
$$y(2) = AB\tau[1-e^{-1/\tau}]$$

While this looks different than our previous answer, it is really the same thing for this range of t. To see this note that from before we had

$$y(t) = \tau AB[1-e^{-(t-1)/\tau}]u(t-1) - \tau AB[1-e^{-(t-2)/\tau}]u(t-2)$$

If $t \geq 2$ both unit step functions are one and we have

$$y(t) = \tau AB[1 - e^{-(t-1)/\tau}] - \tau AB[1 - e^{-(t-2)/\tau}] = AB\tau[e^{-(t-2)/\tau} - e^{-(t-1)/\tau}]$$

The solution is plotted in Figure 5.9.
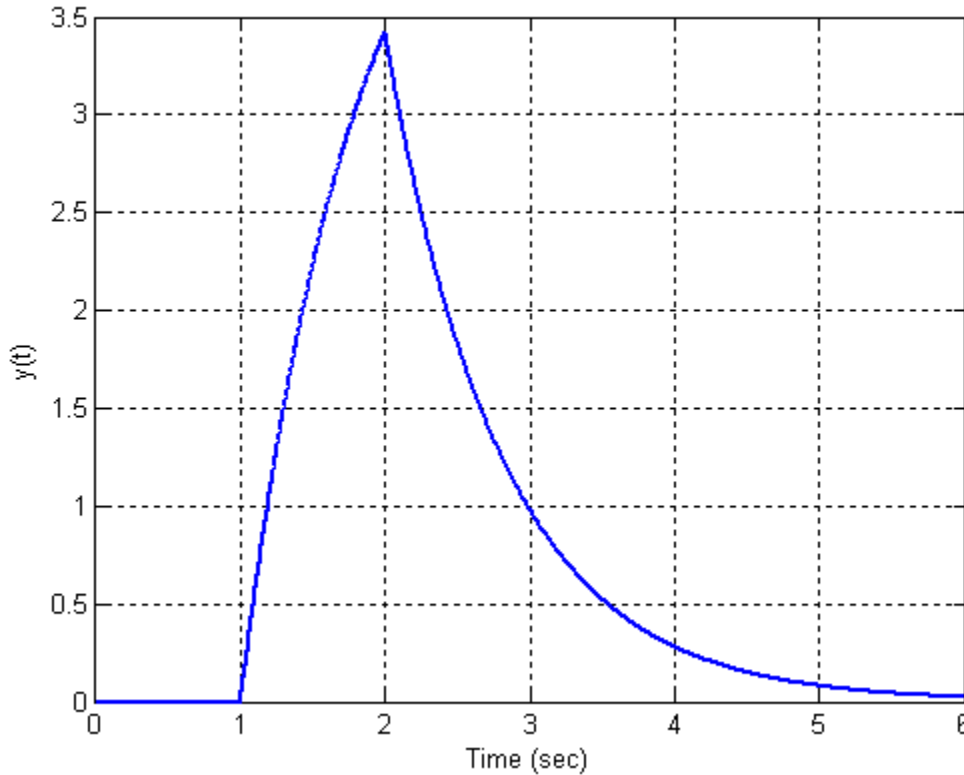
Figure 5.9. Result (output $y(t)$) for Example 5.5.2 assuming $A=2$, $B=3$, and $\tau = 0.8$.

Example 5.5.3. Determine the output of an LTI system with impulse response $h(t) = Ae^{-(t-1)/\tau}u(t-1)$ to input $x(t) = 2u(t) - 2u(t-2) - 3u(t-3)$. Let's use the integral form

$$y(t) = \int_{-\infty}^{\infty} h(t-\lambda)x(\lambda)d\lambda$$

There is really only one marker point of note, that of $h(1)$, which gives $h(1) = h(t-\lambda) \rightarrow \lambda = t-1$.

There are four different graphs we need for this example, $t \leq 1$, $1 \leq t \leq 3$, $3 \leq t \leq 4$, and $t \geq 4$. For $t \leq 1$ the product of the functions $h(t-\lambda)$ and $x(\lambda)$ is zero, so $y(t) = 0$.

For $1 \leq t \leq 3$ we have the situation shown in Figure 5.10. Evaluating the integrals we have

$$y(t) = \int_0^{t-1} e^{-(t-\lambda-1)/\tau}(2)d\lambda = 2e^{-(t-1)/\tau}\int_0^{t-1} e^{\lambda/\tau}d\lambda = 2\tau e^{-(t-1)/\tau}\left[e^{(t-1)/\tau}-1\right] = 2\tau\left[1-e^{-(t-1)/\tau}\right]$$
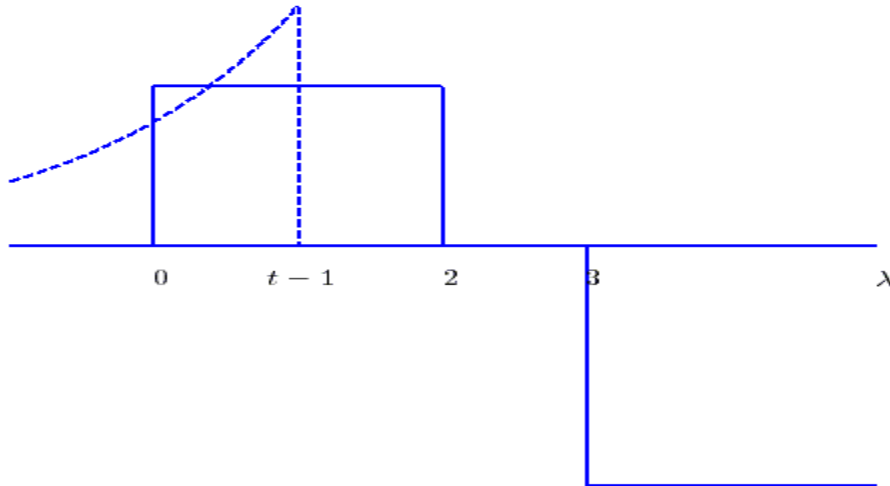
Figure 5.10. First overlapping region for Example 5.5.3 assuming $\tau = 1.5$. This figure is valid for $1 \le t \le 3$

For $3 \le t \le 4$ we have the situation shown in Figure 5.11. Evaluating the integral we have

$$y(t) = \int_0^2 e^{-(t-\lambda-1)/\tau}(2)d\lambda = 2e^{-(t-1)/\tau}\int_0^2 e^{\lambda/\tau}d\lambda = 2\tau e^{-(t-1)/\tau}\left[e^{2/\tau}-1\right] = 2\tau\left[e^{-(t-3)/\tau}-e^{-(t-1)/\tau}\right]$$
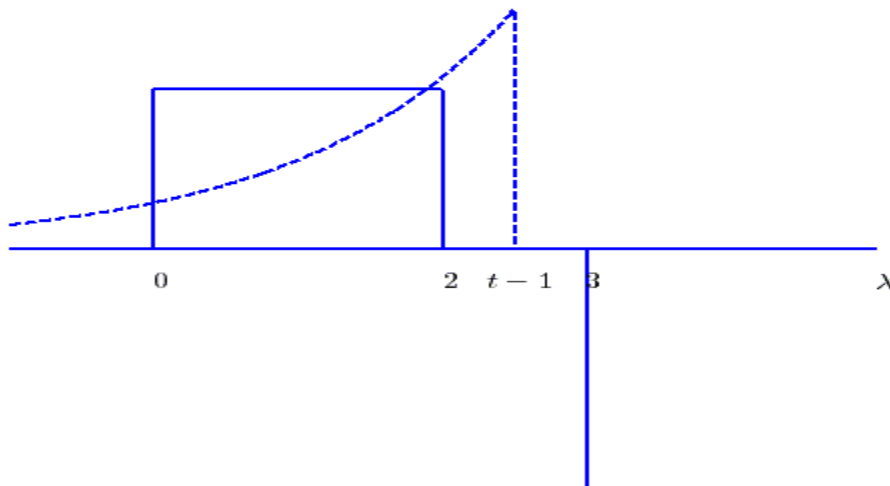
Figure 5.11. Second overlapping region for Example 5.5.3 assuming $\tau = 1.5$. This figure is valid for $3 \le t \le 4$.

For $t \geq 4$ we have the situation shown in Figure 5.12. For this situation we will need two integrals,

$$y(t) = \int_0^2 e^{-(t-\lambda-1)/\tau}(2)d\lambda + \int_3^{t-1} e^{-(t-\lambda-1)/\tau}(-3)d\lambda =$$

$$= 2e^{-(t-1)/\tau}\int_0^2 e^{\lambda/\tau}d\lambda - 3e^{-(t-1)/\tau}\int_3^{t-1} e^{\lambda/\tau}d\lambda$$

$$= 2\tau e^{-(t-1)/\tau}\left[e^{2/\tau}-1\right] - 3\tau e^{-(t-1)/\tau}\left[e^{(t-1)/\tau}-e^{3/\tau}\right]$$

$$= 2\tau\left[e^{-(t-3)/\tau}-e^{-(t-1)/\tau}\right] - 3\tau\left[1-e^{-(t-4)/\tau}\right]$$

In summary we have

$$y(t) = \begin{cases} 0 & t \leq 1 \\ 2\tau\left[1-e^{-(t-1)/\tau}\right] & 1 \leq t \leq 3 \\ 2\tau\left[e^{-(t-3)/\tau}-e^{-(t-1)/\tau}\right] & 3 \leq t \leq 4 \\ 2\tau\left[e^{-(t-3)/\tau}-e^{-(t-1)/\tau}\right] - 3\tau\left[1-e^{-(t-4)/\tau}\right] & t \geq 4 \end{cases}$$
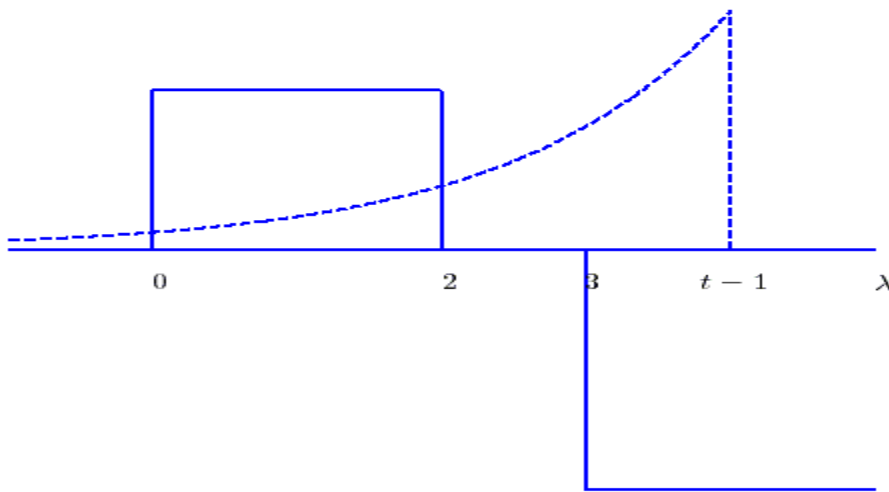
Figure 5.12. Third overlapping region for Example 5.5.3 assuming $\tau = 1.5$. This figure is valid for $t \geq 4$

Checking the values of $y(t)$ at each boundaries we have

$$y(1) = 0$$
$$y(3) = 2\tau[1-e^{-2/\tau}]$$
$$y(4) = 2\tau[e^{1/\tau}-e^{-3/\tau}]$$

The final solution is plotted in Figure 5.13.

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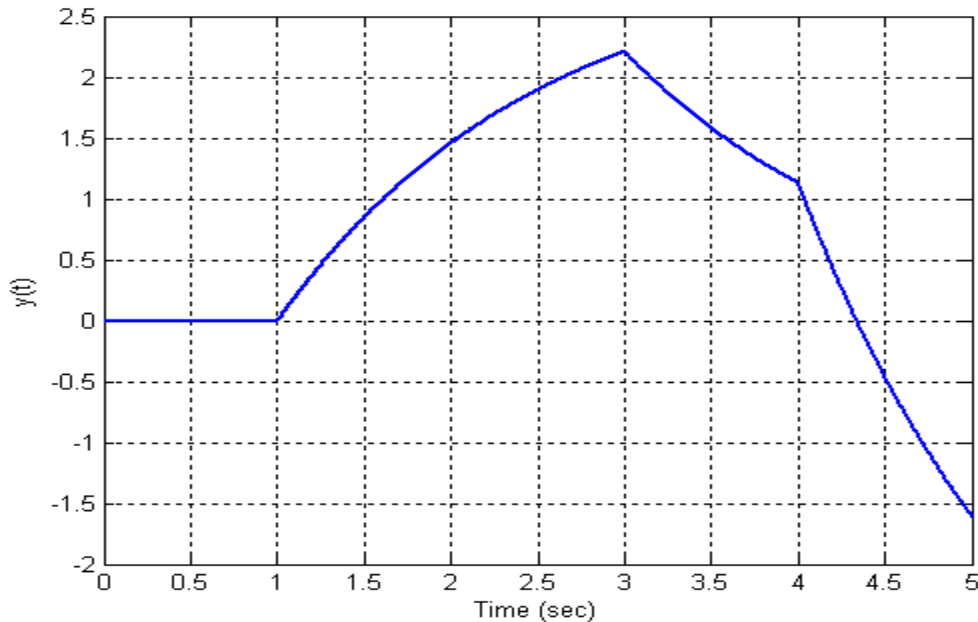
Figure 5.13. Result (output $y(t)$) for Example 5.5.3 assuming $\tau = 1.5$.

Example. 5.5.4. Determine the output of an LTI system with impulse response $h(t) = t[u(t+1) - u(t-1)]$ to input $x(t) = u(t) - u(t-1) + 2u(t-2)$. Let's use the integral form

$$y(t) = \int_{-\infty}^{\infty} h(t-\lambda)x(\lambda)d\lambda$$

There are two marker points of note, that of, $h(-1)$ which gives

$$h(-1) = h(t-\lambda) \rightarrow \lambda = t+1$$
$$h(1) = h(t-\lambda) \rightarrow \lambda = t-1$$

There are six different graphs we need for this example, $t \leq -1, -1 \leq t \leq 0$, $0 \leq t \leq 1$, $1 \leq t \leq 2$, $2 \leq t \leq 3$ and $t \geq 3$.

For $t \leq -1$ the product of the functions $h(t-\lambda)$ and $x(\lambda)$ is zero, so $y(t) = 0$.

For $-1 \leq t \leq 0$ we have the situation depicted in Figure 5.14, and

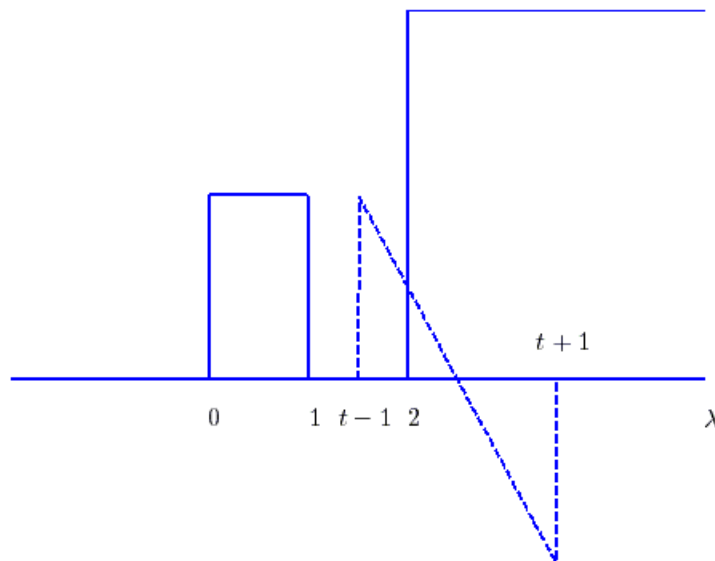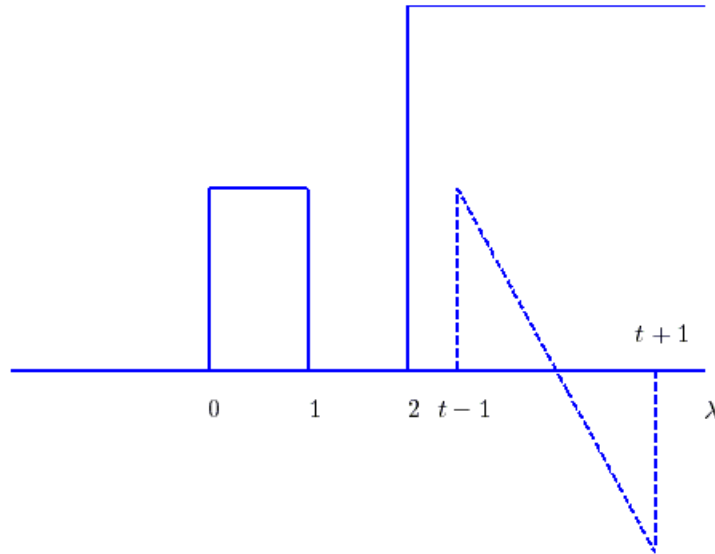$$y(t) = \int_{0}^{t+1} (t-\lambda)(1)d\lambda = \frac{1}{2}(t^2 - 1)$$
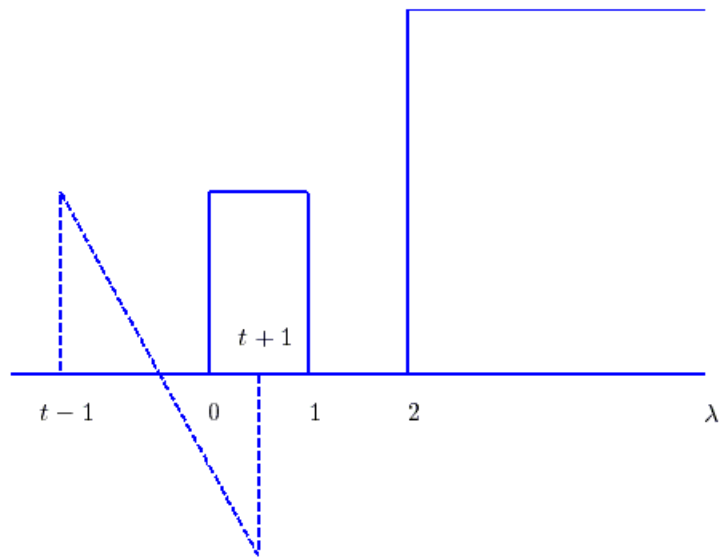
Figure 5.14. Initial overlapping region for Example 5.5.4. This figure is valid for $-1 \le t \le 0$.

For $0 \le t \le 1$ we have the situation depicted in Figure 5.15, and

$$y(t) = \int_0^1 (t - \lambda)(1)d\lambda = t - \frac{1}{2}$$
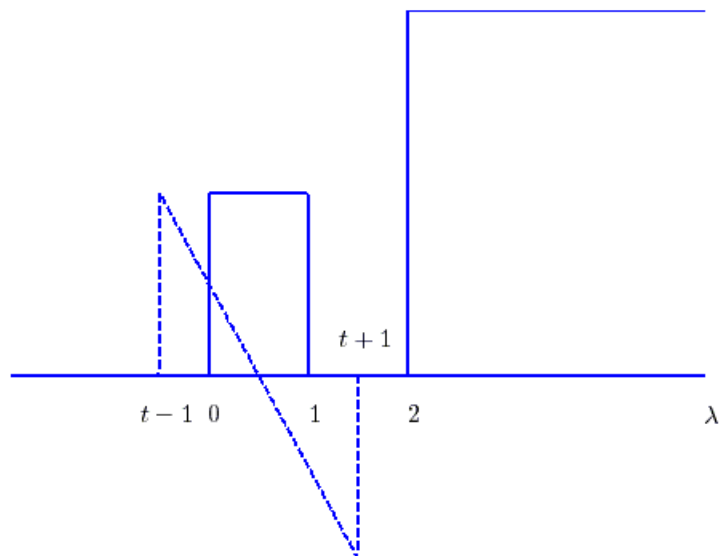
Figure 5.15. Second overlapping region for Example 5.5.4. This figure is valid for $0 \le t \le 1$.

For $1 \le t \le 2$ we have the situation depicted in Figure 5.16, and

$$y(t) = \int_{t-1}^{1} (t-\lambda)(1)d\lambda + \int_{2}^{t+1} (t-\lambda)(2)d\lambda = \frac{1}{2}(t^2 - 6t + 6)$$
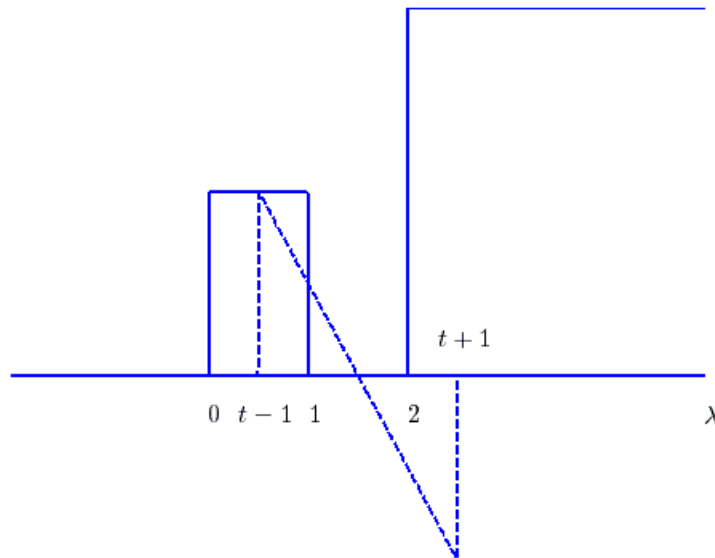
Figure 5.16. Third overlapping region for Example 5.5.4. This figure is valid for $1 \le t \le 2$.

For $2 \le t \le 3$ we have the situation depicted in Figure 5.17, and

$$y(t) = \int_{2}^{t+1} (t-\lambda)(2)d\lambda = t^2 - 4t + 3$$

Figure 5.17. Fourth overlapping region for Example 5.5.4. This figure is valid for $2 \le t \le 3$.

For $t \geq 3$ we have the situation depicted in Figure 5.18, and

$$y(t) = \int_{t-1}^{t+1} (t-\lambda)(2)d\lambda = 0$$

Figure 5.18. Final overlapping region for Example 5.5.4. This figure is valid for $t \geq 3$.

In summary we have

$$y(t) = \begin{cases} 0 & t \leq -1 \\ \dfrac{1}{2}(t^2 - 1) & -1 \leq t \leq 0 \\ t - \dfrac{1}{2} & 0 \leq t \leq 1 \\ \dfrac{1}{2}(t^2 - 6t + 6) & 1 \leq t \leq 2 \\ t^2 - 4t + 3 & 2 \leq t \leq 3 \\ 0 & t \geq 3 \end{cases}$$

Note that although the input to the system starts at time $t = 0$, the output starts at time $t = -1$. Thus the system is noncausal. The system output is graphed in Figure 5.19.
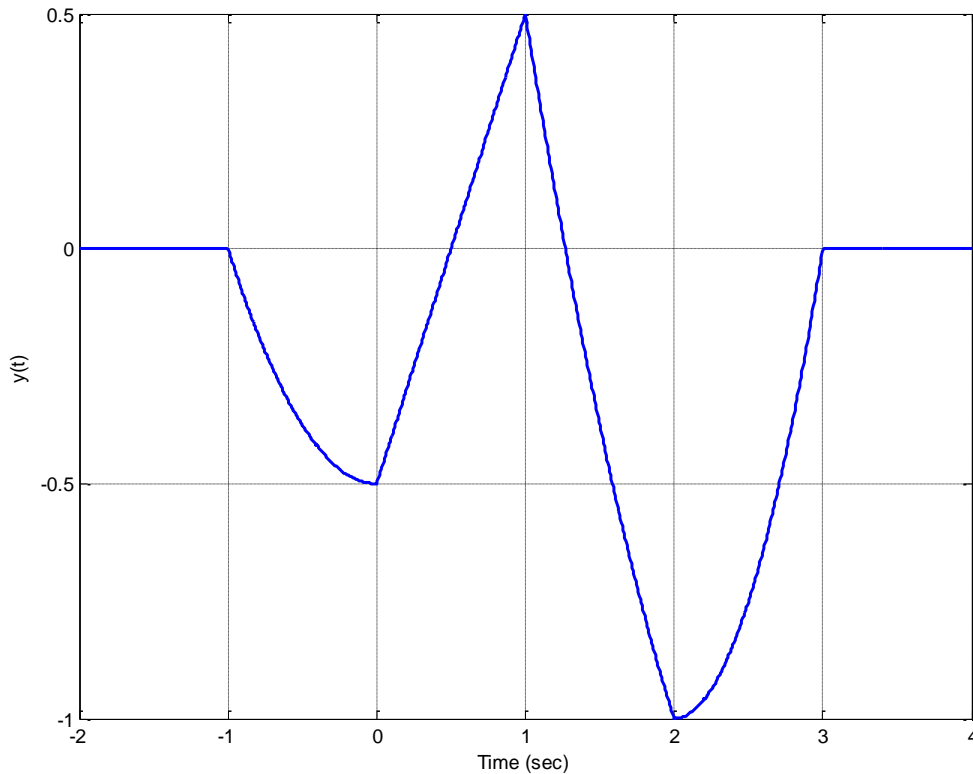
Figure 5.19 Result (output $y(t)$) for Example 5.5.4. Note that the system is not causal since the input starts at $t = 0$ but the output starts at $t = -1$

5.6 Causality and BIBO Stability

Now that we can write the output of an LTI system in terms of the convolution of the input with the impulse response, we can also determine some fairly simple tests to determine if an LTI system is BIBO stable or causal. We have

$$y(t) = \int_{-\infty}^{\infty} h(\lambda)x(t-\lambda)d\lambda$$

If we know the input is bounded, $|x(t)| \leq N$, then we know

$$|y(t)| \leq \left| \int_{-\infty}^{\infty} h(\lambda)x(t-\lambda)d\lambda \right| \leq \int_{-\infty}^{\infty} |h(\lambda)||x(t-\lambda)|d\lambda \leq \int_{-\infty}^{\infty} |h(\lambda)|Nd\lambda = N\int_{-\infty}^{\infty} |h(\lambda)|d\lambda$$

Thus an LTI system will be BIBO stable if

$$\int_{-\infty}^{\infty} |h(\lambda)|d\lambda < \infty$$

Next, let's assume we want to find the output of an LTI system at the time t_0, so we have

$$y(t_0) = \int_{-\infty}^{\infty} h(t_0 - \lambda)x(\lambda)d\lambda$$

We can then break the integral into two parts,

$$y(t_0) = \int_{-\infty}^{t_0} h(t_0 - \lambda)x(\lambda)d\lambda + \int_{t_0}^{\infty} h(t_0 - \lambda)x(\lambda)d\lambda$$

If the system is causal, then the second integral must be zero, since it depends on future values of the input. In order of the second integral to be zero we need

$$h(t_0 - \lambda) = 0 \quad \text{for} \quad \lambda \in (t_0, \infty)$$

Let's assume $\lambda = t_0 + \epsilon, \epsilon > 0$. Substituting this into our expression for the impulse response we have

$$h(t_0 - \lambda) = h(t_0 - [t_0 + \epsilon]) = h(-\epsilon) = 0, \epsilon > 0$$

or

$$h(t) = 0, t < 0$$

This means the impulse response must be zero for any time less than zero in order for the LTI system to be causal.

In summary, an LTI system is BIBO stable if

$$\int_{-\infty}^{\infty} |h(\lambda)| d\lambda < \infty$$

and is causal if

$$h(t) = 0, t < 0$$

Note that these are independent properties, a system can be stable and not causal, or causal and not stable.

5.7 Convolution Properties and Interconnected Systems

There are a number of useful and important properties of convolution. Among the most useful are the following:

Commutative Property: $y(t) = h(t) \star x(t) = x(t) \star h(t)$

Associative Property: $h_2(t) \star [h_1(t) \star x(t)] = [h_2(t) \star h_1(t)] \star x(t)$

Distributive Property: $h(t) \star [x_1(t) + x_2(t)] = h(t) \star x_1(t) + h(t) \star x_2(t)$

The commutative property means that

$$y(t) = \int_{-\infty}^{\infty} x(\lambda)h(t - \lambda)d\lambda = \int_{-\infty}^{\infty} x(t - \lambda)h(\lambda)d\lambda$$

so that we have two equivalent ways of determining the system output, $y(t)$. A convenient method of presenting the relationship between the input, output, and impulse response of a system is depicted in Figure 5.7.
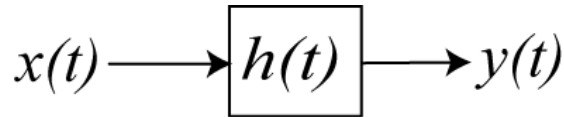
$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

Figure 5.20. Input, impulse response, and output for an LTI system.

If we have two LTI systems in *series*, as shown in Figure 5.21, then we can relate the input to the output as follows:

$$v(t) = x(t) \star h_1(t)$$

Then

$$y(t) = v(t) \star h_2(t) = \left[x(t) \star h_1(t) \right] \star h_2(t)$$

Using the commutative property we can write this as

$$y(t) = [h_1(t) \star h_2(t)] \star x(t)$$

The impulse response between the input and output is then

$$h(t) = h_1(t) \star h_2(t)$$

$$x(t) \longrightarrow \boxed{h_1(t)} \xrightarrow{v(t)} \boxed{h_2(t)} \xrightarrow{y(t)}$$
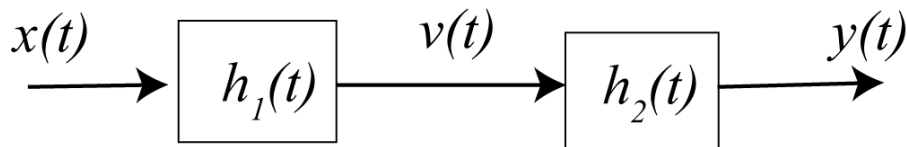
Figure 5.21. Two LTI systems connected in series.

If we have to LTI systems in *parallel*, as shown in Figure 5.22, then we can relate the input to the output as follows

$$v(t) = x(t) \star h_1(t) \text{ and } w(t) = x(t) \star h_2(t)$$

Combining we have

$$y(t) = v(t) + w(t) = x(t) \star h_1(t) = x(t) \star h_2(t)$$

Using the associative and distributive properties we then have

$$y(t) = [h_1(t) + h_2(t)] \star x(t)$$

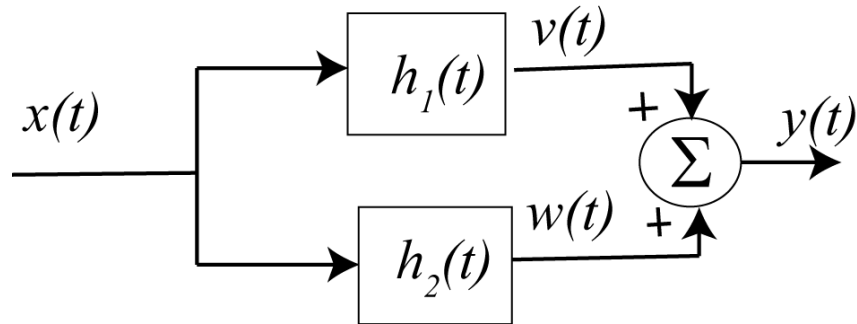Hence the system transfer function is then

$$h(t) = h_1(t) + h_2(t)$$

Figure 5.22. Two LTI systems connected in parallel.

Example 5.7.1. Consider the system shown in Figure 5.23. In this figure we have *negative feedback.*
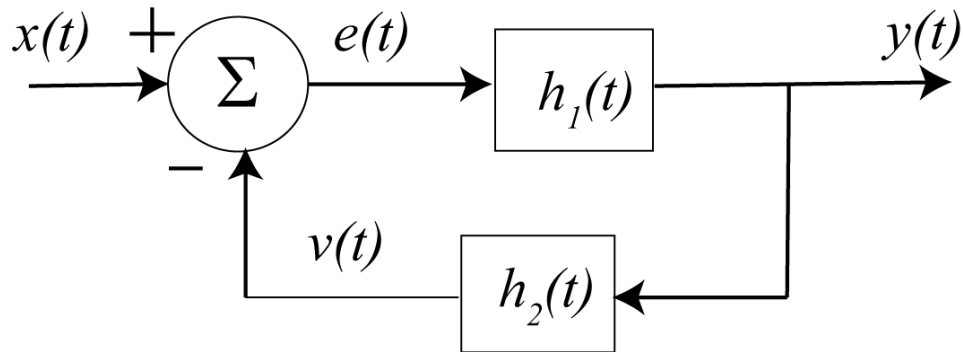

Figure 5-23. System for Example 5.7.1.

For this system we have

$$e(t) = x(t) - v(t) \text{ and } v(t) = y(t) \star h_2(t)$$

Combining we have

$$y(t) = e(t) \star h_1(t) = [x(t) - v(t)] \star h_1(t) =]x(t) - y(t) \star h_2(t)] \star h_1(t)$$

We can rearrange this as

$$y(t) + y(t) \star h_1(t) \star h_2(t) = x(t) \star h_1(t)$$

or

$$y(t) \star [\delta(t) + h_1(t) \star h_2(t)] = x(t) \star [h_1(t)]$$

This is an awkward expression, but at this point we cannot simplify it anymore.

Example 5.7.2. Consider the system shown in Figure 5.24. Again we have negative feedback in this system. Here we have

$$e(t) = x(t) \star h_1(t) - v(t)$$
$$v(t) = y(t) \star h_4(t)$$
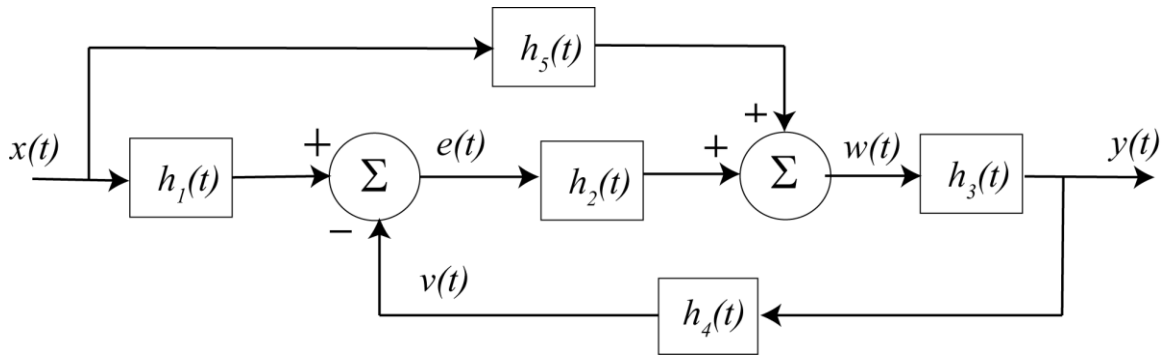$$w(t) = e(t) \star h_2(t) + x(t) \star h_5(t)$$
$$y(t) = w(t) \star h_3(t)$$

Figure 5-24. System for Example 5.7.2.

Starting with our expression for $y(t)$ and working backwards we have

$$y(t) = w(t) \star h_3(t)$$
$$y(t) = [e(t) \star h_2(t) + x(t) \star h_5(t)] \star h_3(t)$$
$$y(t) = [\{x(t) \star h_1(t) - v(t)\} \star h_2(t) + x(t) \star h_5(t)] \star h_3(t)$$
$$y(t) = [\{x(t) \star h_1(t) - (y(t) \star h_4(t))\} \star h_2(t) + x(t) \star h_5(t)] \star h_3(t)$$

Simplifying we get

$$y(t) = x(t) \star h_1(t) \star h_2(t) \star h_3(t) - y(t) \star h_4(t) \star h_2(t) \star h_3(t) + x(t) \star h_5(t) \star h_3(t)$$

Finally

$$y(t) \star [\delta(t) + h_2(t) \star h_3(t) \star h_4(t)] = x(t) \star [h_1(t) \star h_2(t) \star h_3(t) + h_3(t) \star h_5(t)]$$

Trying to determine the system impulse response in this way is very difficult, if not impossible. In addition, if we are trying to modify one of the subsystems system to change the behavior of the overall system, this method of relating the input to the output does not lend itself to any intuition or easy analysis. This is one of the primary reasons Laplace and Fourier transforms were developed. Laplace transforms are hence our next topic.

Chapter 5 Problems

1) Simplify the following expressions as much as possible

a) $g(t) = e^{-t}\delta(t-1)$ **b)** $g(t) = \int_{-2}^{t-3} e^{-(t-\lambda)}\delta(\lambda-1)d\lambda$ **c)** $g(t) = \int_{-2}^{t+2} e^{-(t-\lambda)}\delta(\lambda+1)d\lambda$

d) $g(t) = \int_{t-2}^{2} e^{-(t-\lambda)}\delta(\lambda+1)d\lambda$ **e)** $g(t) = \int_{0}^{t} e^{-(t-\lambda)}\delta(\lambda+1)d\lambda$ **f)** $g(t) = \int_{0}^{t} e^{-(t-\lambda)}\delta(\lambda-2)d\lambda$

g) $g(t) = \int_{-\infty}^{\infty} e^{-3\lambda}\delta(t-\lambda)d\lambda$ **h)** $g(t) = \int_{-\infty}^{\infty} e^{4\lambda}u(\lambda)\delta(t-\lambda+2)d\lambda$ **i)** $g(t) = \int_{-\infty}^{\infty} e^{\lambda}u(\lambda-1)\delta(t-\lambda-3)d\lambda$

Scrambled Answers:
$e^{-(t-1)}u(t-4), 0, e^{-3t}, e^{t-3}u(t-4), e^{4(t+2)}u(t+2), e^{-(t-2)}u(t-2), e^{-(t+1)}u(t+3), e^{-(t+1)}u(1-t), e^{1}\delta(t-1)$

2) Determine the impulse response for each of the following systems

a) $y(t) = 2x(t) + x(t-1)$ **b)** $y(t) = \int_{-\infty}^{t} e^{-2(t-\lambda)}x(\lambda)d\lambda$ **c)** $y(t) = \int_{-\infty}^{\infty} e^{-\lambda}x(t-\lambda)d\lambda$

d) $\dot{y}(t) - \dfrac{1}{2}y(t) = 2\,x(t)$ **e)** $\dot{y}(t) = 2y(t) + x(t+1)$ **f)** $\dot{y}(t) = x(t-1)$

Scrambled Answers: $e^{2(t+1)}u(t+1), e^{-t}, e^{-2t}u(t), 2e^{t/2}u(t), u(t-1), 2\delta(t) + \delta(t-1)$

3) For the following two systems,

a) $y(t) = \int_{-\infty}^{t} e^{-(t-\lambda)}x(\lambda+1)d\lambda$ **b)** $y(t) = \int_{-\infty}^{\infty} e^{2\lambda}u(\lambda)x(t-\lambda)d\lambda$

i) compute the impulse response directly
ii) compute the step response directly
ii) show that the derivative of the step response is the impulse response for these two systems. You should get two terms for the derivative, but one of the terms is zero (you should be able to show why this is true)!
Note that this is a general property of LTI systems, and is important since it is much easier to determine the step respones of a system than it is to determine the impulse response of an actual system.

4) For the following inpulse responses and inputs, determine the system output using analytical evaluation of the convolution integral. Be sure to include any required unit step functions in your answers.

a) $h(t) = e^{-t}u(t)$, $x(t) = u(t\text{-}1)$ 　　　　　 **b)** $h(t) = e^{-t}u(t) + \delta(t)$, $x(t) = e^{-(t-1)}u(t\text{-}1)$

c) $h(t) = \delta(t) + \delta(t-1)$, $x(t) = e^{-t}u(t)$ 　　 **d)** $h(t) = \delta(t-1)$, $x(t) = \delta(t+2)$

Scrambled Answers
$y(t) = \delta(t+1)$, $y(t) = [1-e^{-(t-1)}]u(t-1)$, $y(t) = te^{-(t-1)}u(t-1)$, $y(t) = e^{-t}u(t) + e^{-(t-1)}u(t-1)$:

5) For LTI systems with the following impulse responses, determine if the system is BIBO stable.

a) $h(t) = \delta(t)$ **b)** $h(t) = u(t)$ **c)** $h(t) = e^{-t}u(t)$ **d)** $h(t) = e^{-t^2}u(t)$

Answers: 3 are stable, one is unstable

6) Consider and LTI system with impulse response $h(t)$. We assume that the impulse response begins at some time t_h. That is, the impulse response is zero before this time. Hence we can write $h(t) = \tilde{h}(t)u(t - t_h)$. Next assume we have an input that starts (is zero before) some time t_x, so we can write $x(t) = \tilde{x}(t)u(t - t_x)$. Note that we do not really care what the functions look like here, only their initial times. In doing the graphical evaluation, just make them blobs.

a) Determine, using graphical evaluation, the first possible time at which the output of the system, $y(t)$, can be non-zero, and call this time t_y. This time should be a function of t_h and t_x

b) Show that, in order for the system to be causal, $t_h \geq 0$. That is, the impulse response must be zero for negative times.

7) For the following interconnected systems,

i) determine the overall impulse response (the impulse response between input *x(t)* and output *y(t)*) and

ii) determine if the system is causal.
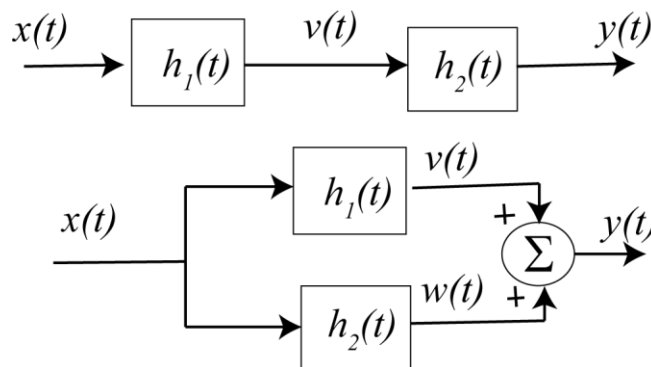
a) $h_1(t) = u(t\text{-}1), \ h_2(t) = u(t+1)$ **b)** $h_1(t) = u(t\text{-}1), \ h_2(t) = \delta(t+2)$
c) $h_1(t) = e^{-(t-1)}u(t\text{-}1), \ h_2(t) = \delta(t) + u(t)$

Scrambled Answers:

$h(t) = u(t-1), h(t) = tu(t), h(t) = u(t+1), h(t) = u(t+1) + u(t-1),$
$h(t) = \delta(t) + u(t) + e^{-(t-1)}u(t-1), h(t) = u(t-1) + \delta(t+2)$

Three systems not causal

8) For the following interconnected systems,

i) determine the overall impulse response (the impulse response between input $x(t)$ and output $y(t)$) and

ii) determine if the system is causal.
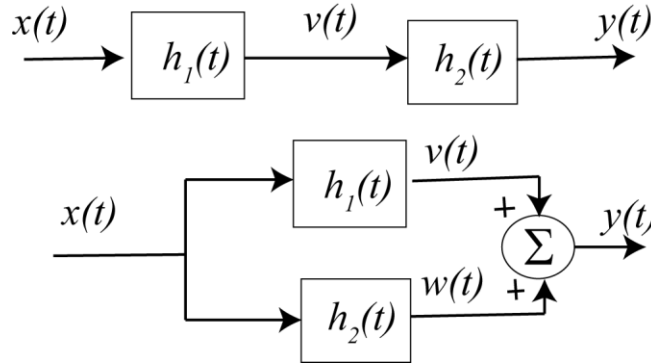
a) $h_1(t) = u(t-1)$, $h_2(t) = u(t+1)$

b) $h_1(t) = e^{-(t-1)}u(t-1)$, $h_2(t) = \delta(t-2)$

Scrambled Answers:

$h(t) = e^{-(t-3)}u(t-3)$, $h(t) = u(t-1) + u(t+1)$

$h(t) = tu(t)$, $h(t) = e^{-(t-1)}u(t-1) + \delta(t-2)$ *One system is not causal*

9) For the following systems, determine the relationship between the input and the output
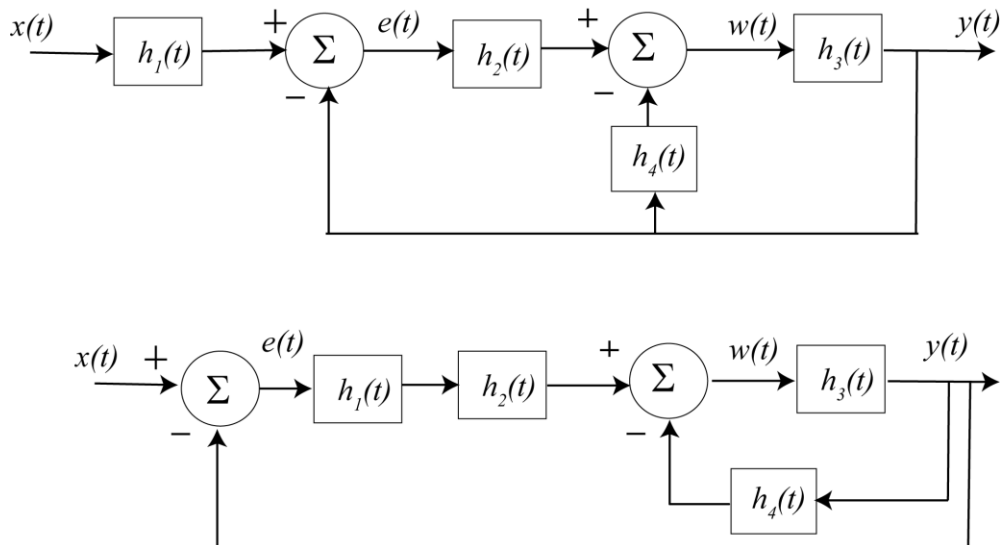
Answers:

$y(t) * [\delta(t) + h_2(t) * h_3(t) + h_4(t) * h_3(t)] = x(t) * [h_1(t) * h_2(t) * h_3(t)]$

$y(t) * [\delta(t) + h_1(t) * h_2(t) * h_3(t) + h_3(t) * h_4(t)] = x * [h_1(t) * h_2(t) * h_3(t)]$

10) Consider a linear time invariant system with impulse response given by
$h(t) = t[u(t+1) - u(t-2)]$ and input $x(t) = 2u(t-2) - 3u(t-3) + u(t-4)$, shown below
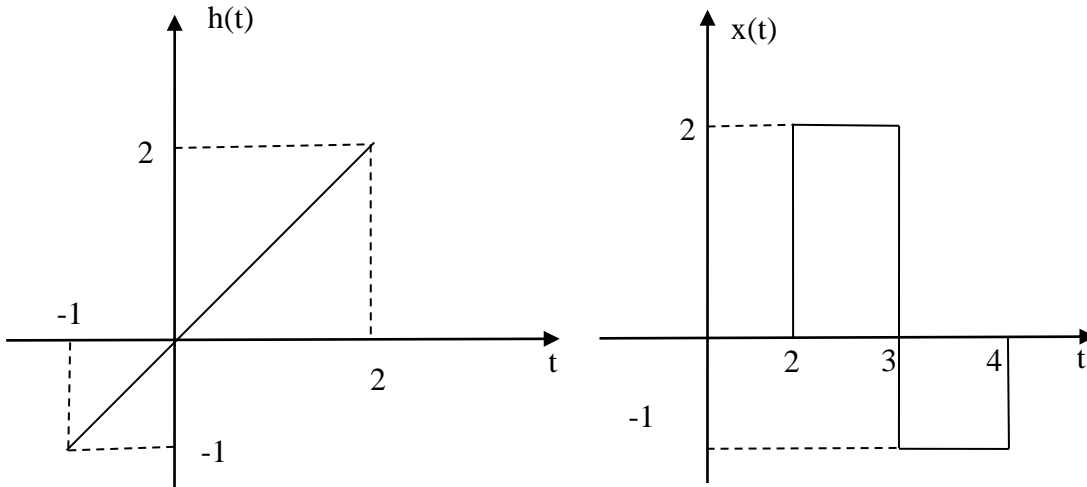
Using **_graphical evaluation_**, determine the output $y(t) = h(t) * x(t)$. Specifically, you must

- Flip and slide $h(t)$, **_NOT_** $x(t)$
- Show graphs displaying both $h(t-\lambda)$ and $x(\lambda)$ for each region of interest
- Determine the range of t for which each part of your solution is valid
- Set up any necessary integrals to compute $y(t)$. Your integrals must be complete, in that they cannot contain the symbols $x(\lambda)$ or $h(t-\lambda)$ but must contain the actual functions.
- Your integrals cannot contain any unit step functions
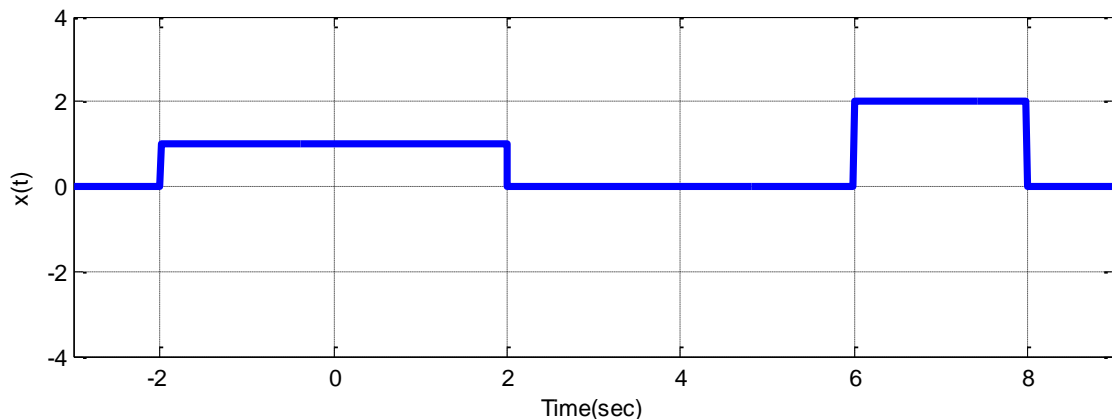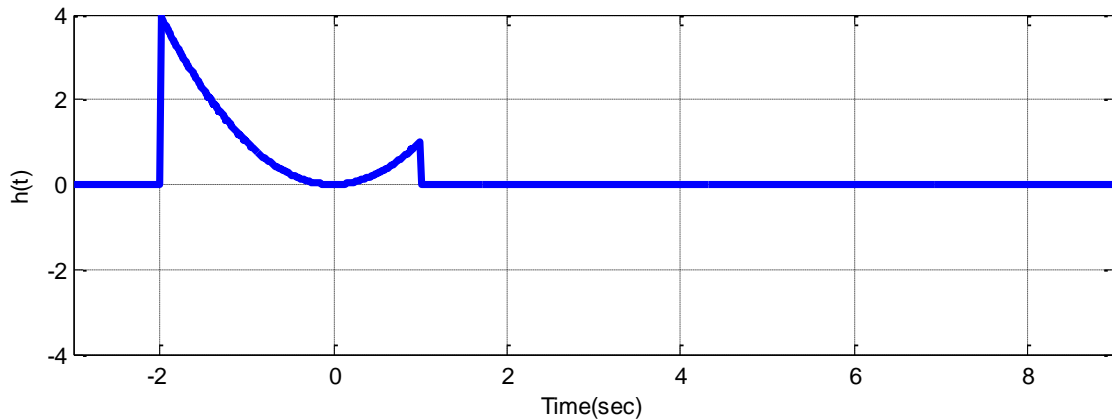- **_DO NOT EVALUATE THE INTEGRALS!!_**

Answer:

$$y(t) = \begin{cases} 0 & t \leq 1 \\ \displaystyle\int_{2}^{t+1} (t-\lambda)(2)d\lambda & 1 \leq t \leq 2 \\ \displaystyle\int_{2}^{3}(t-\lambda)(2)d\lambda + \int_{3}^{t+1}(t-\lambda)(-1)d\lambda & 2 \leq t \leq 3 \\ \displaystyle\int_{2}^{3}(t-\lambda)(2)d\lambda + \int_{3}^{4}(t-\lambda)(-1)d\lambda & 3 \leq t \leq 4 \\ \displaystyle\int_{t-2}^{3}(t-\lambda)(2)d\lambda + \int_{3}^{4}(t-\lambda)(-1)d\lambda & 4 \leq t \leq 5 \\ \displaystyle\int_{t-2}^{4}(t-\lambda)(-1)d\lambda & 5 \leq t \leq 6 \\ 0 & t \geq 6 \end{cases}$$

11) Consider a *noncausal* linear time invariant system with impulse response given by

$$h(t) = [1-(t-1)^2][u(t+1)-u(t-2)]$$

The input to the system is given by

$$x(t) = u(t-1)-u(t-3)-u(t-4)+u(t-6)$$
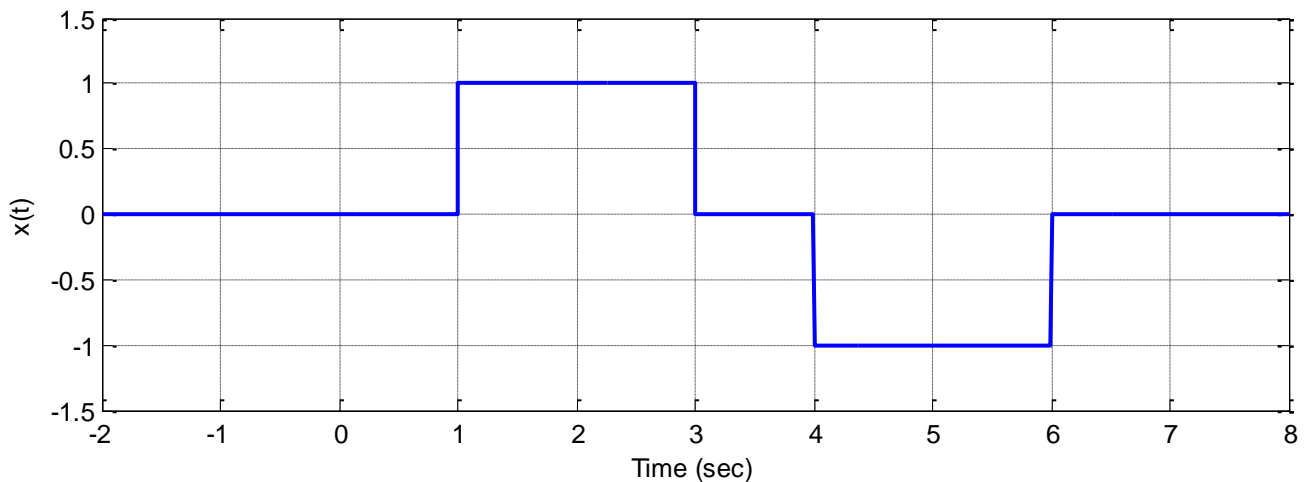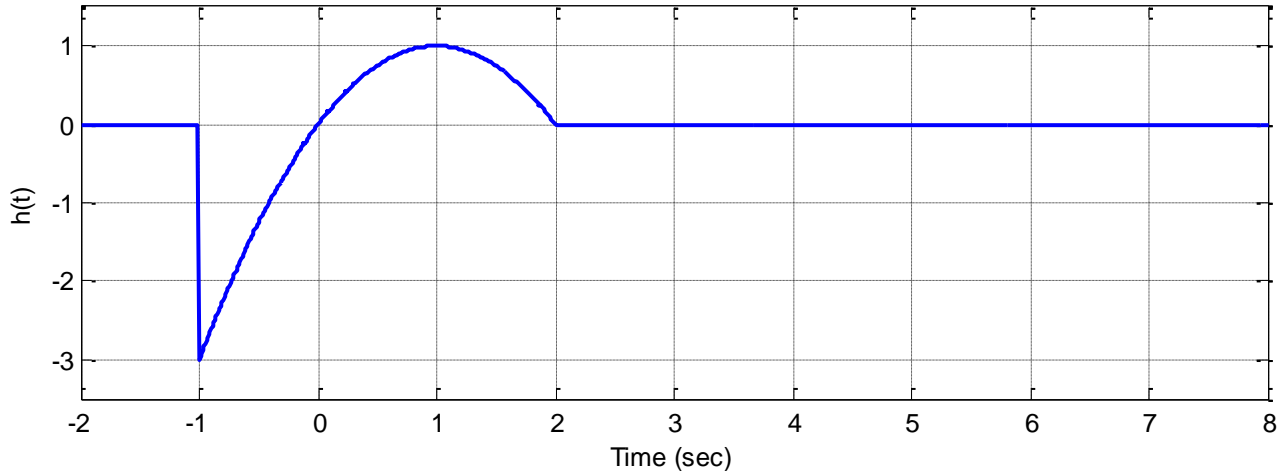
Using ***graphical evaluation***, determine the output $y(t)$. Specifically, you must

- Flip and slide $h(t)$, ***NOT*** $x(t)$
- Show graphs displaying both $h(t-\lambda)$ and $x(\lambda)$ for each region of interest
- Determine the range of t for which each part of your solution is valid
- Set up any necessary integrals to compute $y(t)$. Your integrals must be complete, in that they cannot contain the symbols $x(\lambda)$ or $h(t-\lambda)$ but must contain the actual functions.
- Your integrals cannot contain any unit step functions
- ***DO NOT EVALUATE THE INTEGRALS!!***

Answer

$$y(t) = \begin{cases} 0 & t \leq 0 \\ \int_{1}^{t+1} [1-(t-\lambda-1)^2](1)d\lambda & 0 \leq t \leq 2 \\ \int_{1}^{3} [1-(t-\lambda-1)^2](1)d\lambda & 2 \leq t \leq 3 \\ \int_{t-2}^{3} [1-(t-\lambda-1)^2](1)d\lambda + \int_{4}^{t+1} [1-(t-\lambda-1)^2](-1)d\lambda & 3 \leq t \leq 5 \\ \int_{4}^{6} [1-(t-\lambda-1)^2](-1)d\lambda & 5 \leq t \leq 6 \\ \int_{t-2}^{6} [1-(t-\lambda-1)^2](-1)d\lambda & 6 \leq t \leq 8 \\ 0 & t \geq 8 \end{cases}$$

12) Consider a *noncausal* linear time invariant system with impulse response given by

$$h(t) = e^{-t}[u(t+1) - u(t-2)]$$

The input to the system is given by

$$x(t) = 2u(t-1) - 5u(t-2) + 3u(t-3)$$

Using ***graphical evaluation***, determine the output $y(t)$. Specifically, you must

- Flip and slide $h(t)$, ___NOT___ $x(t)$
- Show graphs displaying both $h(t-\lambda)$ and $x(\lambda)$ for each region of interest
- Determine the range of t for which each part of your solution is valid
- Set up any necessary integrals to compute $y(t)$. Your integrals must be complete, in that they cannot contain the symbols $x(\lambda)$ or $h(t-\lambda)$ but must contain the actual functions.
- Your integrals cannot contain any unit step functions
- ***DO NOT EVALUATE THE INTEGRALS!!***

Answer:

$$y(t) = \begin{cases} 0 & t \leq 0 \\[2ex] \displaystyle\int_1^{t+1} e^{-(t-\lambda)}(2)d\lambda & 0 \leq t \leq 1 \\[2ex] \displaystyle\int_1^2 e^{-(t-\lambda)}(2)d\lambda + \int_2^{t+1} e^{-(t-\lambda)}(-3)d\lambda & 1 \leq t \leq 2 \\[2ex] \displaystyle\int_1^2 e^{-(t-\lambda)}(2)d\lambda + \int_2^3 e^{-(t-\lambda)}(-3)d\lambda & 2 \leq t \leq 3 \\[2ex] \displaystyle\int_{t-2}^2 e^{-(t-\lambda)}(2)d\lambda + \int_2^3 e^{-(t-\lambda)}(-3)d\lambda & 3 \leq t \leq 4 \\[2ex] \displaystyle\int_{t-2}^3 e^{-(t-\lambda)}(-3)d\lambda & 4 \leq t \leq 5 \\[2ex] 0 & t \geq 5 \end{cases}$$

13) Consider a linear time invariant system with impulse response given by

$$h(t) = t^2[u(t+2) - u(t-1)]$$

The input to the system is given by

$$x(t) = [u(t+2) - u(t-2)] + 2[u(t-6) - u(t-8)]$$

Using ***graphical evaluation***, determine the output $y(t)$. Specifically, you must

- Flip and slide $h(t)$, ***NOT*** $x(t)$
- Show graphs displaying both $h(t-\lambda)$ and $x(\lambda)$ for each region of interest
- Determine the range of t for which each part of your solution is valid
- Set up any necessary integrals to compute $y(t)$. Your integrals must be complete, in that they cannot contain the symbols $x(\lambda)$ or $h(t-\lambda)$ but must contain the actual functions.
- Your integrals cannot contain any unit step functions
- ***DO NOT EVALUATE THE INTEGRALS!!***

Answer:

$$
y(t) = \begin{cases}
0 & t \leq -4 \\[6pt]
\displaystyle\int_{-2}^{t+2} (t-\lambda)^2 (1)\,d\lambda & -4 \leq t \leq -1 \\[10pt]
\displaystyle\int_{t-1}^{t+2} (t-\lambda)^2 (1)\,d\lambda & -1 \leq t \leq 0 \\[10pt]
\displaystyle\int_{t-1}^{2} (t-\lambda)^2 (1)\,d\lambda & 0 \leq t \leq 3 \\[10pt]
0 & 3 \leq t \leq 4 \\[6pt]
\displaystyle\int_{6}^{t+2} (t-\lambda)^2 (2)\,d\lambda & 4 \leq t \leq 6 \\[10pt]
\displaystyle\int_{6}^{8} (t-\lambda)^2 (1)\,d\lambda & 6 \leq t \leq 7 \\[10pt]
\displaystyle\int_{t-1}^{8} (t-\lambda)^2 (1)\,d\lambda & 7 \leq t \leq 9 \\[10pt]
0 & t \geq 9
\end{cases}
$$

14) Matlab From the class website download the files **homework5.m** and **convolve.m**. **homework5.m** is a script file that sets up the time arrays and functions, and the invokes the function **convolve.m** to compute the convolution of the two functions. **homework5.m** then plots the two functions to be convolved, and then the resulting convolution of the two functions. Note that we are passing arguments to the function convolve and the function is returning values to the calling program. In **part c** of this problem you will need to complete this code, *don't try to use it until then.*

a) Show(by performing the convolution) that if $h(t) = e^{-t}u(t)$, $x(t) = e^{-(t-T)}u(t-T)$, then
$$y(t) = h(t) \star x(t) = e^{-t}e^{T}(t-T)u(t-T)$$

b) Show (by performing the convolution) that if $h(t) = e^{-t}u(t)$, $x(t) = u(t) - u(t-T)$, then
$$y(t) = h(t) \star x(t) = [1 - e^{-t}]u(t) - [1 - e^{-(t-T)}]u(t-T)$$

c) Read the **Appendix** at the end of this homework and then complete the code for the function **convolve.m.**

d) Use the script **homework5.m** to compute and plot the convolution for the functions of part **a** with $T = 1.5$ (it should default to this). If you have done everything correctly the analytical (true) convolution should be indistinguishable from the estimated (numerical) convolution.

(continued on next page)

e) Comment out your functions from part **d**, and then modify the code in **homework5.m** to compute the convolution for the functions in part b, with $T = 1$. Be sure to also plot the correct (true) solution to check for any errors.

f) The functions in parts **b** and **e** illustrate a capacitor charging and then discharging, something we have seen before. Change the width of the pulse to the pulse is five time constants long, and run the simulation.

Turn in work for parts a and b, and your plots for parts d and e, and your code.

Appendix

Although this is a continuous time course, and Matlab works in discrete-time, we can use Matlab to numerically do convolutions, under certain restrictions. The most important restriction is that the spaces between the time samples be the same for both functions. Another other restriction is that the functions really need to return to zero (or very close to zero) within the time frame we are examining them. Finally, we need fine enough resolution (the sampling interval must be sufficiently small) so that our sampled signals are a good approximation to the continuous signal.

First, we need to have two functions to convolve. Let's assume we want to convolve the functions

$h(t) = e^{-t}u(t)$ and $x(t) = e^{-(t-T)}u(t-T)$

Let's denote the time vector that goes with **h** as **th**, and the time vector that goes with **x** as **tx**. Then we create the functions with something like

```
%
tx = [-1:0.001:8]; % time from -1 to 8 with an increment of 0.001
seconds
th = [-1:0.001:8];
%
h = @(t) 0*(t<0)+exp(-t).*(t>=0);
T = 1.5;
x = @(t) 0*(t<T)+exp(-(t-T)).*(t>=T)
```

Next we will need to determine the time interval between samples. We can determine this as
dt = tx(2)-tx(1);
It doesn't matter if we use **tx** or **th**, since the sample interval must be the same.
Now we can use Matlabs **conv** function to do the convolution. However, since we are trying to do continuous time convolution we need to do some scaling. To understand why, let's look again at convolution:

$$y(t) = \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)d\lambda$$

If we were to try and approximate this integral using discrete-time samples, with sampling interval Δt, we could write

$$y(t) \approx y(k\Delta t)$$
$$x(\lambda) \approx x(n\Delta t)$$
$$h(t-\lambda) \approx h([k-n]\Delta t)$$

We can then approximate the integral as

$$y(t) = \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)d\lambda \approx y(k\Delta t) = \sum_{n=-\infty}^{n=\infty} x(n\Delta t)h([k-n]\Delta t)\Delta t = \Delta t \sum_{n=-\infty}^{n=\infty} x(n\Delta t)h([k-n]\Delta t)$$

The Matlab function **conv** computes the sum $\sum_{n=-\infty}^{n=\infty} x(n\Delta t)h([k-n]\Delta t)$, so to approximate the continuous time integral we need to multiply (or scale) by Δt. Hence

$$y = dt*conv(x1,x2);$$

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Finally, we need to determine a time vector that corresponds to y. To do this we need to determine where the starting point should be. Let's consider the convolution

$$y(t) = \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)d\lambda$$

Let's assume $x(t)$ is zero until time t_1, so we could write $x(t) = \tilde{x}(t)u(t-t_1)$ for some function $\tilde{x}(t)$. Similarly, let's assume $h(t)$ is zero until time t_2, so we could write $h(t) = \tilde{h}(t)u(t-t_2)$ for some function $\tilde{h}(t)$. The convolution integral is then

$$y(t) = \int_{-\infty}^{\infty} \tilde{x}(\lambda)u(\lambda-t_1)\tilde{h}(t-\lambda)u(t-\lambda-t_2)d\lambda = \int_{t_1}^{t-t_2} \tilde{x}(\lambda)\tilde{h}(t-\lambda)d\lambda$$

This integral will be zero unless $t-t_2 \geq t_1$, or $t \geq t_1 + t_2$. Hence we know that $y(t)$ is zero until $t = t_1 + t_2$. This means for our convolution, the initial time of the output is the sum of the initial times:

initial_time = tx(1)+th(1);

Every sample in y is separated by time interval dt. We need to determine how long y is, and create an indexed array of this length

n = [0:length(y)-1];

Finally we can construct the correct time vector (ty) that starts at the correct initial time and runs the correct length

ty = dt*n + initial_time;

6.0 Laplace Transforms

In many applications we have interconnections of LTI systems. We can determine the output of the system by using convolution in the time-domain, but this often proves to be difficult when we have more than just a few interconnecting systems. Sometimes we don't want to just compute the output, but rather we want to be able to determine properties of the system in a simple way. We will utilize Laplace transforms for this, though in some applications the use of Fourier transforms may be more appropriate.

6.1 Laplace Transform Definitions

The *two-sided* Laplace transform of a signal $x(t)$ is defined to be

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

while the *one-sided* Laplace transform is defined as

$$X(s) = \int_{0^-}^{\infty} x(t)e^{-st}dt$$

The only difference between the two definitions is the lower limit, and one and two sided refers to integrating one on side of zero or both sides of zero. In this course we will only use the one-sided transform, and when we refer to the Laplace transform we mean the one sided transform. However, you should be aware the two-sided transform has some use when we are dealing with *noncausal* systems. Looking at the form of the integral, since the exponent must be dimensionless, we can conclude that the variable s has the units of 1/time.

There are a few conventions we need to know about. First of all, the lower limit on the one sided Laplace transform is generally written as 0^-, or starting at a time just before zero. This is particularly useful when determining the Laplace transform of an impulse centered at zero. Secondly, a very common convention is to use lower case letters for time-domain functions, and capital letters for the corresponding transform domain. We usually write $x(t) \leftrightarrow X(s)$ or $\mathcal{L}\{x(t)\} = X(s)$ to show that $x(t)$ and $X(s)$ are transform pairs. We dill denote the Laplace transform operator as $\mathcal{L}$. Finally, the complex variable s is sometimes written in terms of its real and imaginary parts as $s = \sigma + j\omega$. This is particularly useful in determining if the integral is finite (or can be made to be finite) or is infinite.

6.2 Basic Laplace Transforms

Let's start off by determining the Laplace transform of some basic signals. As you will see sometime we need to put conditions on σ to be sure the integral converges. This condition defines the *region of convergence*.

Example 6.2.1. For $x(t) = \delta(t)$ we have

$$X(s) = \int_{0-}^{\infty} \delta(\lambda)e^{-s\lambda}d\lambda = e^{s0}\int_{0^-}^{\infty} \delta(\lambda)d\lambda = 1$$

since the delta function is contained in the region of integration. Hence we have the transform pair

$$x(t) = \delta(t) \leftrightarrow X(s) = 1$$

Example 6.2.2. For $x(t) = \delta(t - t_0)$, where $t_0 \geq 0$, we have

$$X(s) = \int_{0^-}^{\infty} \delta(\lambda - t_0) e^{-s\lambda} d\lambda = e^{-st_0} \int_{0-}^{\infty} \delta(\lambda - t_0) d\lambda = e^{-st_0}$$

since again the delta function is located in the region of integration. Hence we have the transform pair

$$x(t) = \delta(t - t_0) \leftrightarrow X(s) = e^{-st_0}$$

Note that if $t_0 < 0$ the integral will be zero.

Example 6.2.3. For $x(t) = u(t)$ we have

$$X(s) = \int_{0^-}^{\infty} u(\lambda) e^{-s\lambda} d\lambda = \int_{0}^{\infty} e^{-s\lambda} d\lambda = \left. \frac{e^{-s\lambda}}{-s} \right|_{\lambda=0}^{\lambda=\infty}$$

At this point we cannot really evaluate this integral unless we put some conditions on σ. Let's make the substitution $s = \sigma + j\omega$ and we have

$$X(s) = \left. \frac{e^{-(\sigma+j\omega)\lambda}}{-s} \right|_{\lambda=0}^{\lambda=\infty} = \left. \frac{e^{-\sigma\lambda+j\omega\lambda}}{-s} \right|_{\lambda=0}^{\lambda=\infty} = \left. \frac{e^{-\sigma\lambda} e^{j\omega\lambda}}{-s} \right|_{\lambda=0}^{\lambda=\infty}$$

Using Euler's identity

$$e^{j\omega\lambda} = \cos(\omega\lambda) + j\sin(\omega\lambda)$$

We can determine the magnitude of $e^{j\omega\lambda}$ as

$$|e^{j\omega\lambda}| = \sqrt{\cos^2(\omega\lambda) + \sin^2(\omega\lambda)} = 1$$

Hence the term $e^{j\omega\lambda}$ does not contribute to the convergence of the integral. That means that ω does not contribute to the convergence of the integral. That leaves us with σ. If $\sigma > 0$, then the exponent in the exponential is negative, when we evaluate it at the limit of infinity we get zero. Hence we have the transform pair

$$x(t) = u(t) \leftrightarrow X(s) = \frac{1}{s}, \quad \sigma > 0$$

Note that the condition $\sigma > 0$, or the real part of s must be positive, defines the region of convergence. Hence the integral will converge if $\Re\{s\} > 0$ and we can rewrite the transform pair as

$$x(t) = u(t) \leftrightarrow X(s) = \frac{1}{s}, \quad \Re\{s\} > 0$$

Example 6.2.4. For $x(t) = u(t - t_0)$ we have

$$X(s) = \int_{0^-}^{\infty} u(\lambda - t_0) e^{-s\lambda} d\lambda = \int_{t_0}^{\infty} e^{-s\lambda} d\lambda = \left. \frac{e^{-s\lambda}}{-s} \right|_{\lambda=t_0}^{\lambda=\infty} = \frac{e^{-st_0}}{s}, \quad \Re\{s\} > 0$$

The region of convergence is the same as for the non-delayed unit step. So we have the transform pair

$$x(t) = u(t - t_0) \leftrightarrow X(s) = \frac{e^{-st_0}}{s}, \quad \Re\{s\} > 0$$

Example 6.2.5. For $x(t) = e^{-at}u(t)$ we have

$$X(s) = \int_{0^-}^{\infty} e^{-a\lambda}u(\lambda)e^{-s\lambda}d\lambda = \int_0^{\infty} e^{-(s+a)\lambda}d\lambda = \frac{e^{-(s+a)\lambda}}{-(s+a)}\bigg|_{\lambda=0}^{\lambda=\infty}$$

Now in order for the integral to converge, we need $\Re\{s+a\} > 0$. We can rewrite this as $\Re\{s\} + \Re\{a\} > 0$ or $\Re\{s\} > -\Re\{a\}$. Hence the region of convergence is defined as $\Re\{s\} > -\Re\{a\}$ and we have the Laplace transform pair

$$x(t) = e^{-at}u(t) \leftrightarrow X(s) = \frac{1}{s+a}, \Re\{s\} > -\Re\{a\}$$

Example 6.2.6. For $x(t) = \cos(\omega_0 t)u(t)$ we will need to use Euler's identity in the form

$$\cos(\omega_0 t) = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$$

Computing the Laplace transform we have

$$X(s) = \int_{0^-}^{\infty} \cos(\omega_0 \lambda)u(\lambda)e^{-s\lambda}d\lambda = \int_0^{\infty} \frac{1}{2}\left[e^{j\omega_0\lambda} + e^{-j\omega_0\lambda}\right]e^{-s\lambda}d\lambda$$

$$= \frac{1}{2}\int_0^{\infty} e^{(j\omega_0 - s)\lambda}d\lambda + \frac{1}{2}\int_0^{\infty} e^{-(j\omega_0 + s)\lambda}d\lambda = \frac{1}{2}\frac{e^{(j\omega_0 - s)\lambda}}{j\omega_0 - s}\bigg|_{\lambda=0}^{\lambda=\infty} - \frac{1}{2}\frac{e^{-(j\omega_0 + s)\lambda}}{(j\omega_0 + s)}\bigg|_{\lambda=0}^{\lambda=\infty}$$

Both integrals will converges if $\Re\{s\} > 0$, and then we have

$$X(s) = \frac{1}{2}\frac{1}{s - j\omega_0} + \frac{1}{2}\frac{1}{s + j\omega_0} = \frac{1}{2}\frac{s + j\omega_0 + s - j\omega_0}{s^2 + \omega_0^2} = \frac{s}{s^2 + \omega_0^2}$$

The Laplace transform pair is then

$$x(t) = \cos(\omega_0 t)\,u(t) \leftrightarrow X(s) = \frac{s}{s^2 + \omega_0^2}, \Re\{s\} > 0$$

Example 6.2.7. Let's assume we want to find the Laplace transform of the derivative of $x(t)$. Then we have

$$\mathcal{L}\left\{\frac{dx(t)}{dt}\right\} = \int_{0^-}^{\infty} \frac{dx(\lambda)}{d\lambda}e^{-s\lambda}d\lambda$$

In order to evaluate this we will need to use integration by parts. For two functions $u(\lambda)$ and $v(\lambda)$, we can write $d(uv) = vdu + udv$. Rearranging this we get the usual form for integration by parts, $\int udv = uv - \int vdu$. For our integral we have

$$\frac{dv(\lambda)}{d\lambda} = \frac{dx(\lambda)}{d\lambda}$$

so $dv = dx$ or $v(\lambda) = x(\lambda)$. We also have $u(\lambda) = e^{-s\lambda}$, so

$$\frac{du(\lambda)}{d\lambda} = -se^{-s\lambda}$$

or

$$du = -se^{-s\lambda}d\lambda$$

Combining these we have

$$\mathcal{L}\left\{\frac{dx(t)}{dt}\right\} = \int_{0^-}^{\infty} \frac{dx(\lambda)}{d\lambda}e^{-s\lambda}d\lambda = x(\lambda)e^{-s\lambda}\Big|_{\lambda=0^-}^{\lambda=\infty} + s\int_{0^-}^{\infty} x(\lambda)e^{-s\lambda}d\lambda = -x(0^-) + sX(s)$$

So we have the transform pair

$$\mathcal{L}\left\{\frac{dx(t)}{dt}\right\} = sX(s) - x(0^-)$$

Table 6.1 summarizes some common Laplace transforms.

6.3 Laplace Transforms of RLC Circuits

In this section we will determine the Laplace transforms of resistors, capacitors, and inductors, and then use these relationships in some examples to determine the output of some RLC circuits in the Laplace domain. In the next sections we will review the use of partial fractions to go back from the Laplace domain to the time domain.

Consider a resistor element shown in the left panel of Figure 6.1. If $v(t)$ is the voltage across the resistor, $i(t)$ is the current through the resistor, and R is the resistance, then we have by Ohm's law $v(t) = i(t)R$. Taking Laplace transforms of both sides of this equation we have $\mathcal{L}\{v(t)\} = \mathcal{L}\{i(t)R\}$ or $V(s) = I(s)R$. Thus, in the Laplace domain, the equivalent impedance is still just R and the resistive circuit element in the left panel of Figure 6.3.1 is replaced with the circuit element in the right panel of Figure 6.1.

$$\mathcal{L}\{\delta(t)\} = 1$$

$$\mathcal{L}\{u(t)\} = \frac{1}{s}$$

$$\mathcal{L}\{tu(t)\} = \frac{1}{s^2}$$

$$\mathcal{L}\left\{\frac{t^{m-1}}{(m-1)!}u(t)\right\} = \frac{1}{s^m}$$

$$\mathcal{L}\{e^{-at}u(t)\} = \frac{1}{s+a}$$

$$\mathcal{L}\{te^{-at}u(t)\} = \frac{1}{(s+a)^2}$$

$$\mathcal{L}\left\{\frac{t^{(m-1)}}{(m-1)!}e^{-at}u(t)\right\} = \frac{1}{(s+a)^m}$$

$$\mathcal{L}\{\cos(\omega_0 t)u(t)\} = \frac{s}{s^2+\omega_0^2}$$

$$\mathcal{L}\{\sin(\omega_0 t)u(t)\} = \frac{\omega_0}{s^2+\omega_0^2}$$

$$\mathcal{L}\{e^{-\alpha t}\cos(\omega_0 t)u(t)\} = \frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}$$

$$\mathcal{L}\{e^{-\alpha t}\sin(\omega_0 t)u(t)\} = \frac{\omega_0}{(s+\alpha)^2+\omega_0^2}$$

$$\mathcal{L}\left\{\frac{dx(t)}{dt}\right\} = sX(s) - x(0^-)$$

$$\mathcal{L}\left\{\frac{d^2 x(t)}{dt^2}\right\} = s^2 X(s) - sx(0^-) - \dot{x}(0^-)$$

$$\mathcal{L}\{x(t-a)\} = e^{-as}X(s)$$

$$\mathcal{L}\{e^{-at}x(t)\} = X(s+a)$$

$$\mathcal{L}\left\{x\left(\frac{t}{a}\right), a > 0\right\} = aX(as)$$

Table 6.1. Some common Laplace transform pairs.
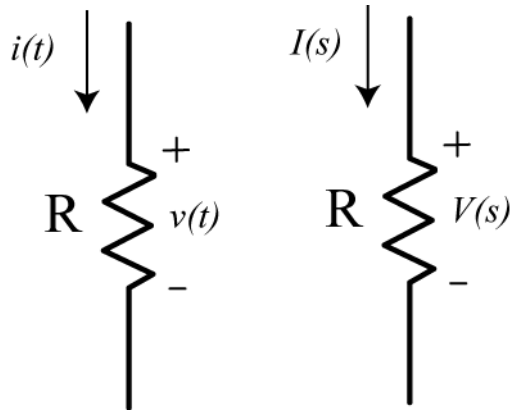
Figure 6.1. Resistive element in the time domain (left) and Laplace domain (right).

Consider the capacitive circuit element shown in the left panel of Figure 6.2. If $v(t)$ is the voltage across the capacitor, $i(t)$ is the current through the capacitor, and C is the capacitance, then we have by Ohm's law $i(t) = C\dfrac{dv(t)}{dt}$. Taking Laplace transforms of both sides of this equation we have

$$\mathcal{L}\{i(t)\} = I(s) = \mathcal{L}\left\{C\frac{dv(t)}{dt}\right\} = C\mathcal{L}\left\{\frac{dv(t)}{dt}\right\} = C\{sV(s) - v(0^-)\}$$

or

$$I(s) = CsV(s) - Cv(0^-)$$

Rewriting this we have

$$V(s) = I(s)\frac{1}{Cs} + \frac{v(0^-)}{s}$$

If for now we ignore the initial conditions we have

$$V(s) = I(s)\frac{1}{Cs}$$

which means the capacitor has an equivalent impedance of $\dfrac{1}{Cs}$. If we make the substitution $s = j\omega$, we have the equivalent impedance $\dfrac{1}{j\omega C}$ which is identical to what you used in sinusoidal (phasor) analysis.

Thus, in the Laplace domain, the capacitor circuit element in the left panel of Figure 6.2 is replaced with the two circuit elements in the right panel of Figure 6.2.
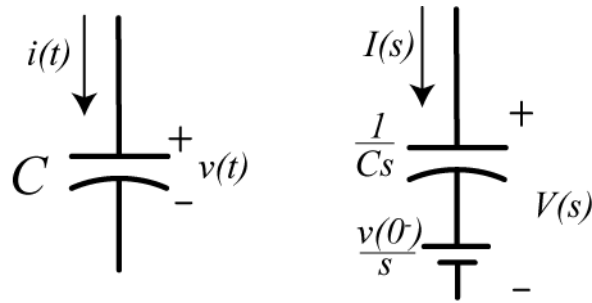
Figure 6.2. Capacitive element in the time domain (left) and Laplace domain (right).

Consider the inductive circuit element shown in the left panel of Figure 6.3. If $v(t)$ is the voltage across the inductor, $i(t)$ is the current through the inductor, and L is the inductance, then we have by Ohm's law. $v(t) = L\dfrac{di(t)}{dt}$ Taking Laplace transforms of both sides of this equation we have

$$\mathcal{L}\{v(t)\} = V(s) = \mathcal{L}\left\{L\frac{di(t)}{dt}\right\} = L\mathcal{L}\left\{\frac{di(t)}{dt}\right\} = L\{sI(s) - i(0^-)\}$$

or

$$V(s) = LsI(s) - Li(0^-)$$

We can rearrange this equation into its more common form as

$$I(s) = V(s)\frac{1}{Ls} + \frac{i(0^-)}{s}$$

Again, if we ignore the initial condition term we have $I(s) = V(s)\dfrac{1}{Ls}$ which means the inductor has the equivalent impedance sL. Making the substitution $s = j\omega$ we have the equivalent impedance $j\omega L$ which is identical the what you used in sinusoidal analysis. Thus, in the Laplace domain, the inductor circuit element in the left panel of Figure 6.3 is replaced with the circuit elements in the right panel of Figure 6.3.
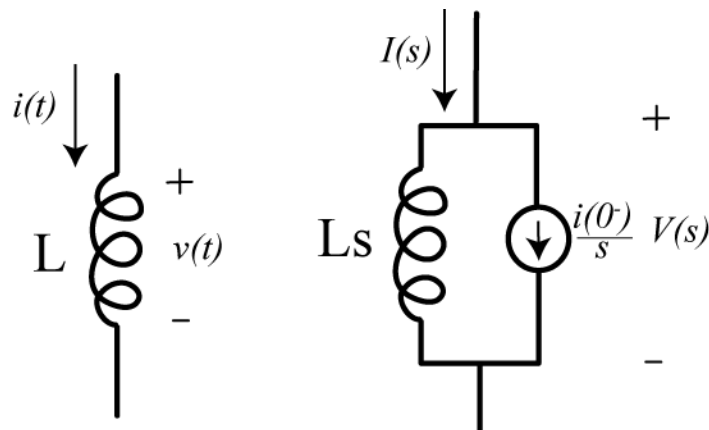
Figure 6.3. Inductive element in the time domain (left) and Laplace domain (right).

Note that the substitution $s = j\omega$ and ignoring the initial conditions gives the same equivalent impedances for these circuit elements as you used before in your sinusoidal steady state analysis. We will have more to say about this substitution when we talk about frequency response. However, this is only one possible value of s.

In the following examples we use Laplace transforms of the circuit elements to write the output of the circuit in terms of the input and initial conditions

Example 6.3.1. Consider the RC circuit shown in Figure 6.4. The output voltage is the voltage across the resistor, and the input is voltage $v_{in}(t)$. We assume $i(t)$ is the current flowing in the circuit. We want to write the output of the circuit in terms of the input and initial voltage on the capacitor, $v(0^-)$, in the Laplace domain. The circuit is redrawn in the Laplace domain in Figure 6.5. Going around the loop we have

$$V_{in}(s) - I(s)R - I(s)\frac{1}{Cs} = \frac{v(0^-)}{s}$$

Solving for the current we have

$$I(s) = \frac{V_{in}(s) - \dfrac{v(0^-)}{s}}{R + \dfrac{1}{Cs}}$$

The system output is the voltage across the resistor, so we have

$$V_{out}(s) = I(s)R = \frac{\left[V_{in}(s) - \dfrac{v(0^-)}{s}\right]R}{R + \dfrac{1}{Cs}}$$

Finally we can write the output as the sum of two different parts. The ***Zero State Response*** (ZSR) is the response of the system to the input alone, assuming no initial conditions. The ***Zero Input Response*** (ZIR) is the response of the system to the initial conditions alone, assuming there is no input. Hence our final solution is

$$V_{out}(s) = \frac{V_{in}(s)R}{R + \dfrac{1}{Cs}} - \frac{\dfrac{v(0^-)}{s}R}{R + \dfrac{1}{Cs}} = \underbrace{\left[\frac{V_{in}(s)RCs}{RCs+1}\right]}_{ZSR} + \underbrace{\left[-\frac{v(0^-)RC}{RCs+1}\right]}_{ZIR}$$
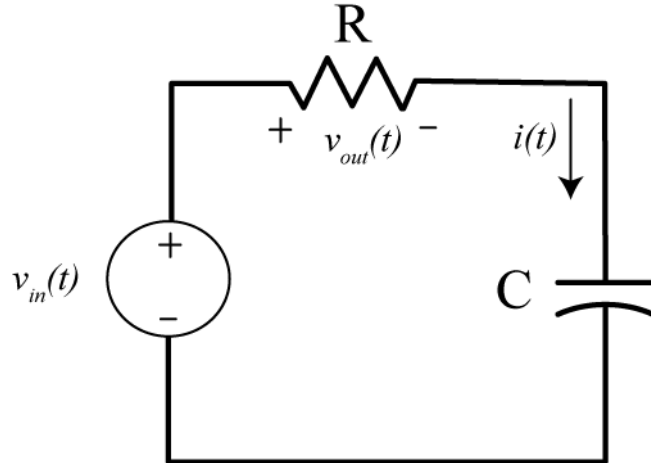
Figure 6.4. Circuit for Example 6.3.1.
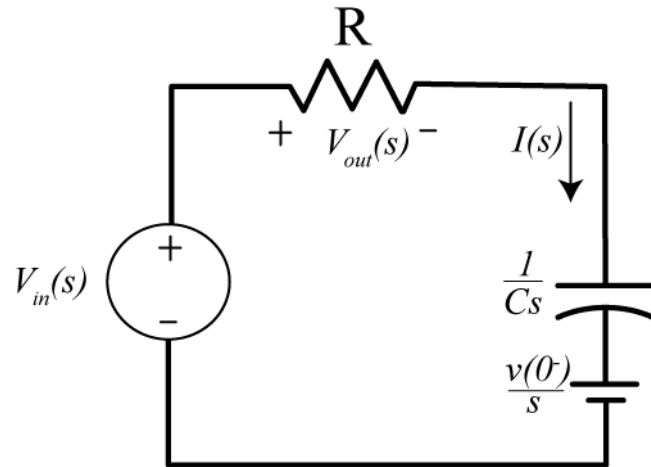
Figure 6.5. Circuit from Example 6.3.1 in the Laplace domain.

Example 6.3.2. Consider the RL circuit shown in Figure 6.6. We again assume the output of the system is the voltage across the resistor, and want to determine the output of the system in terms of the input voltage and the initial current in the inductor $i(0^-)$ in the Laplace domain. In Figure 6.7 the circuit has been redrawn in the Laplace domain. To analyze this circuit, let's define the voltage across the inductor as $V_L(s)$, so we have $V_L(s) = V_{in}(s) - V_{out}(s)$. Equating currents we then have

$$\frac{V_{out}(s)}{R} = \frac{V_L(s)}{Ls} + \frac{i(0^-)}{s} = \frac{V_{in}(s) - V_{out}(s)}{Ls} + \frac{i(0^-)}{s}$$

Rearranging we get

$$V_{out}(s)\left[\frac{1}{R} + \frac{1}{Ls}\right] = V_{out}(s)\left[\frac{R + Ls}{RLs}\right] = \frac{V_{in}(s)}{Ls} + \frac{i(0^-)}{s}$$

and finally

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$$V_{out}(s) = \underbrace{\left[\frac{R}{R+Ls} V_{in}(s) \right]}_{ZSR} + \underbrace{\left[\frac{RL}{R+Ls} i(0^-) \right]}_{ZIR}$$
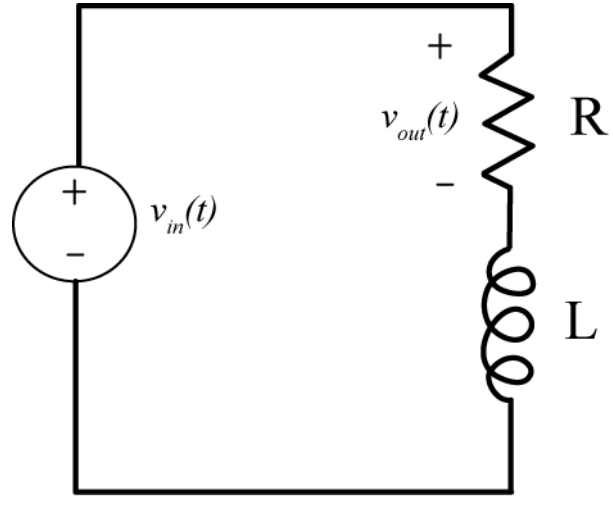
Figure 6.6. Circuit for Example 6.3.2.
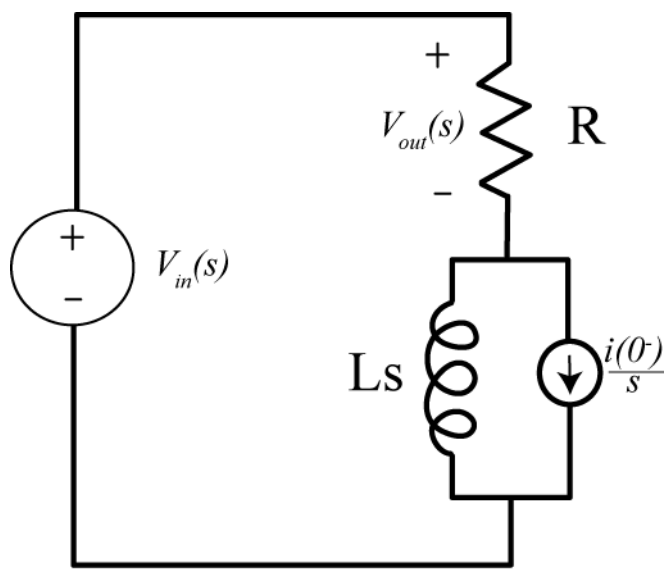
Figure 6.7. Circuit from Example 6.3.2 in the Laplace domain.

Example 6.3.3. Consider the RLC circuit shown in Figure 6.8. We again assume the output of the system is the voltage across the resistor, and want to determine the output of the system in terms of the input current and both the initial voltage across the capacitor $v(0^-)$ and the initial current in the inductor $i(0^-)$ in the Laplace domain. In Figure 6.9 the circuit has been redrawn in the Laplace domain. To analyze this circuit we need to equate all of the currents, as follows:

$$I_{in}(s) = \frac{V_{out}(s)}{R} + \frac{\left[V_{out}(s) - \frac{v(0^-)}{s}\right]}{\frac{1}{Cs}} + \frac{V_{out}(s)}{Ls} + \frac{i(0^-)}{s}$$

Combining terms we have

$$I_{in}(s) = V_{out}(s)\left[\frac{1}{R} + Cs + \frac{1}{Ls}\right] - Cv(0^-) + \frac{i(0^-)}{s}$$

or

$$I_{in}(s) = V_{out}(s)\left[\frac{RLCs^2 + Ls + R}{RLs}\right] - Cv(0^-) + \frac{i(0^-)}{s}$$

Finally we get

$$V_{out}(s) = \underbrace{\left[\frac{I_{in}(s)RLs}{RLCs^2 + Ls + R}\right]}_{ZSR} + \underbrace{\left[\frac{-RLi(0^-) + RLCsv(0^-)}{RLCs^2 + Ls + R}\right]}_{ZIR}$$
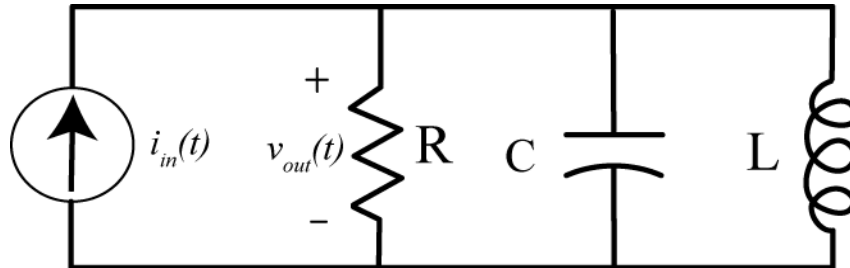
Figure 6.8. Circuit for Example 6.3.3.
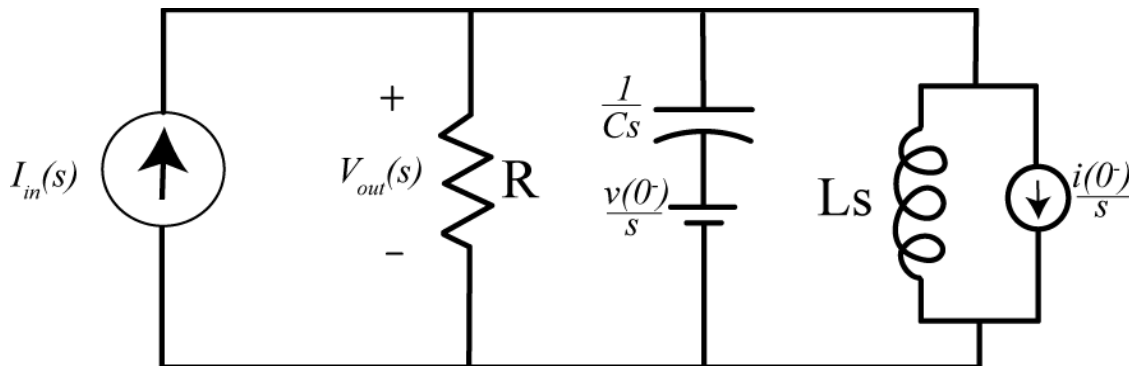
Figure 6.9. Circuit from Example 6.3.3 in the Laplace domain.

6.4 Transfer Functions and the Impulse Response

Assume we have and LTI system that is initially at rest (all initial conditions are zero). We define the *transfer function* as the ratio of the Laplace transform of the output divided by the Laplace transform of the input, or

$$H(s) = \frac{\mathcal{L}\{output(t)\}}{\mathcal{L}\{input(t)\}}$$

Here we have denoted the transfer function as $H(s)$, though obviously other letters are possible. For example if we have and LTI system with input $x(t)$ and output and have the Laplace transform pairs $x(t) \leftrightarrow X(s)$ and $y(t) \leftrightarrow Y(s)$, then the transfer function is defined as

$$H(s) = \frac{Y(s)}{X(s)}$$

We can rearrange this relationship to be $\quad Y(s) = H(s)X(s)$

Example 6.4.1. In Example 6.3.1 the transfer function is

$$H(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{RCs}{RCs+1}$$

Example 6.4.2. In Example 6.3.2 the transfer function is

$$H(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{R}{R+Ls}$$

Example 6.4.3. In Example 6.3.3 the transfer function is

$$H(s) = \frac{V_{out}(s)}{I_{in}(s)} = \frac{RLs}{RLCs^2 + Ls + R}$$

Note that the transfer function comes from the Zero State Response (ZSR), since the Zero Input Response (ZIR) includes the initial conditions, and we assume all initial conditions are zero when determining the transfer function.

If we know a system is LTI, then we know that the output is the convolution of the input with the impulse response,

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\lambda)x(t-\lambda)d\lambda$$

Next, let's assume the input is causal and the system is causal, so both are zero for $t < 0$. Then we have

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\lambda)x(t-\lambda)d\lambda = \int_{0}^{\infty} h(\lambda)x(t-\lambda)d\lambda$$

Since this is an equality, we can take the Laplace transform of each side of the equation,

$$Y(s) = \mathcal{L}\{y(t)\} = Y(s) = \int_0^\infty y(t)e^{-st}dt = \mathcal{L}\{h(t) * \mathrm{x}(t)\} = \int_0^\infty \left[\int_0^\infty h(\lambda)x(t-\lambda)d\lambda\right]e^{-st}dt$$

Rearranging the order of integration we have

$$\int_0^\infty \left[\int_0^\infty h(\lambda)x(t-\lambda)d\lambda\right]e^{-st}dt = \int_0^\infty h(\lambda)\left[\int_0^\infty x(t-\lambda)e^{-st}dt\right]d\lambda$$

Next, let's let $\sigma = t - \lambda$ in the innermost integral. As far as this integral is concerned, λ is just a constant parameter so $d\sigma = dt$. The integral then becomes

$$\int_0^\infty h(\lambda)\left[\int_0^\infty x(t-\lambda)e^{-st}dt\right]d\lambda = \int_0^\infty h(\lambda)\left[\int_0^\infty x(\sigma)e^{-s(\sigma+\lambda)}d\sigma\right]d\lambda = \int_0^\infty h(\lambda)e^{-s\lambda}d\lambda\int_0^\infty x(\sigma)e^{-s\sigma}d\sigma = H(s)X(s)$$

Hence we have derived the convolution property of Laplace transforms,

$$y(t) = h(t) * x(t)$$
$$Y(s) = H(s)X(s)$$

We have also derived an important relationship between the impulse response $h(t)$ and the transfer function $H(s)$. These are Laplace transform pairs,

$$h(t) \leftrightarrow H(s)$$

6.5   Poles, Zeros, and Pole-Zero Plots

In most instances, the transfer function is of the form

$$H(s) = \frac{N(s)}{D(s)} = \frac{A(s-z_1)(s-z_2)...(s-z_m)}{(s-p_1)(s-p_2)...(s-p_n)}$$

where $N(s)$ and $D(s)$ are polynomials in s. The _zeros_ of the transfer function , $z_1, z_2, ..., z_m$, are the roots of values $N(s)$, i.e., the values of s that make $N(s)$ zero. The _poles_ of the transfer function, $p_1, p_2, ..., p_n$, are the roots of $D(s)$, i.e., the values of s that make $D(s)$ zero. If the degree of the numerator is less than the degree of the denominator, or $m < n$, then the transfer function is _strictly proper_, while if the degree of the numerator is less than or equal to the degree of the denominator, or $m \le n$, the transfer function is _proper_. Clearly a transfer function that is strictly proper is also proper. Note also that if the transfer function has real valued coefficients, then _the poles or zeros of the transfer function must occur as complex conjugate pairs._

Sometimes it is useful to plot the poles and zeros of a transfer function. This _pole-zero plot_ is an alternative way of presenting the information contained in the algebraic representation of $H(s)$. As you will see later, the pole-zero plot allows us to easily visually determine the response of an LTI system to a

sinusoid of different frequencies. The pole-zero plot is just a plot of poles (represented by **x**'s) and zeros (represented by **o**'s) in the complex plane, where the horizontal axis represents real values, the vertical axis represents imaginary values. Thus any complex number can be represented in this plane.

Example 6.5.1. Determine the pole-zero plot for

$$H(s) = \frac{10(s-1)(s+2+j)(s+2-j)}{s(s+1)(s+4+2j)(s+4-2j)}$$

Note that this is a strictly proper transfer function. The zeros of the transfer function are at 1, -2-j, and -2+j, while the poles are at 0, -1, -4-2j, and -4+2j. The pole-zero plot for this transfer function is shown in Figure 6.10.

6.6 Partial Fractions for Computing Inverse Laplace Transforms

The first thing we need to do before using the partial fraction technique to find the inverse Laplace transform is to be sure the transfer function is a ratio of polynomials. If there is a time delay (e^{-st_0}) we remove this and account for it when we are done. In addition, we need a *strictly proper* ratio of polynomials. This means that the order of the numerator polynomial must be less than the order of the denominator polynomial. If this is not the case, then we use long division and use partial fractions on the remainder. Finally, we need to be sure the denominator polynomial is *monic*. This means the leading coefficient in the denominator polynomial is a one.
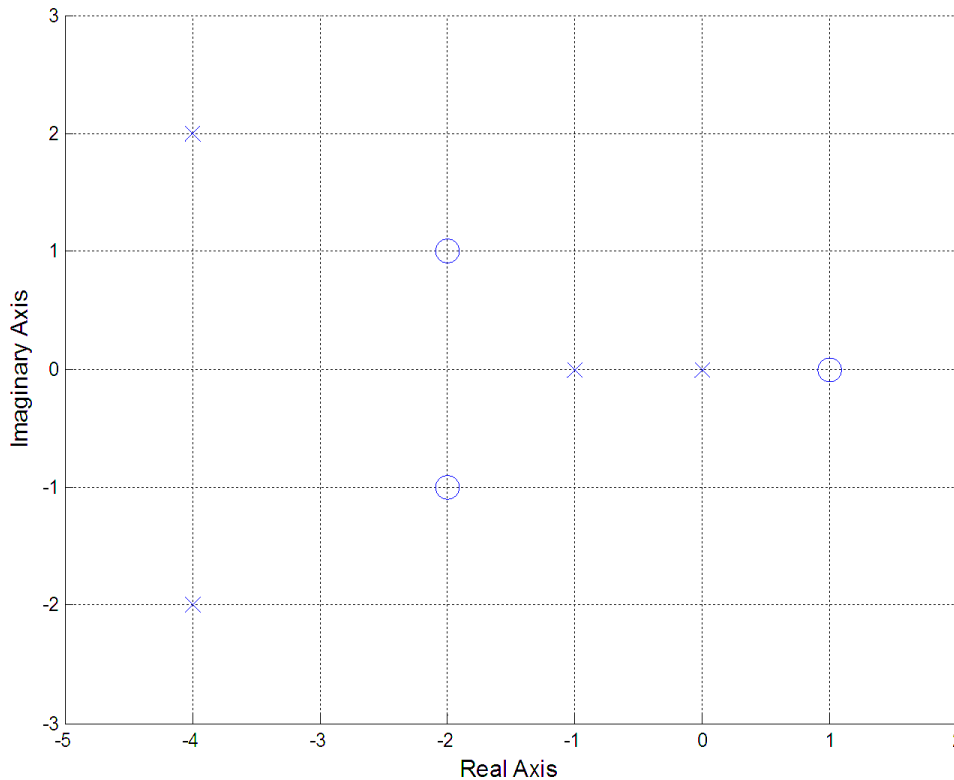
Figure 6.10. Pole-zero plot for transfer function of Example 6.5.1.

Example 6.6.1. Prepare the transfer function

$$H(s) = \frac{e^{-2s}(s+1)}{(s+3)(s+4)}$$

so we can use partial fractions to determine the inverse Laplace transform. The first thing we need to do is to remove the time delay term, and write

$$H(s) = e^{-2s}G(s)$$

where

$$G(s) = \frac{s+1}{(s+3)(s+4)}$$

Since $G(s)$ is a strictly proper ratio of polynomials and the denominator polynomial is monic, we are ready for partial fractions now.

Example 6.6.2. Prepare the transfer function

$$H(s) = \frac{s^2+1}{(s+2)^2}$$

for partial fraction expansion. Here there is no time-delay, but the transfer function is not strictly proper so we need to do some long division. We have then

$$H(s) = \frac{s^2+1}{(s+2)^2} = \frac{s^2+1}{s^2+4s+4} = 1 - \frac{4s+3}{s^2+4s+4} = 1 - G(s)$$

where

$$G(s) = \frac{4s+3}{(s+2)^2}$$

Since $G(s)$ is a strictly proper ratio of polynomials and the denominator polynomial is monic, we are ready for partial fractions now.

Example 6.6.3. Prepare the transfer function

$$H(s) = \frac{e^{-s}(s^2+2s+2)}{(2s+1)(s+2)}$$

for partial fraction expansion. Here we have a time delay, the ratio of polynomials is not strictly proper, and the denominator is not monic.

First we pull out the time-delay part and do the long division, with the result

$$H(s) = \frac{e^{-s}(s^2+2s+2)}{(2s+1)(s+2)} = e^{-s}\frac{s^2+2s+2}{2s^2+5s+2} = e^{-s}\left[0.5 - \frac{0.5s-1}{2s^2+5s+2}\right]$$

Next we scale the denominator so it is monic

$$H(s) = e^{-s}\left[0.5 - \frac{0.25s - 0.5}{s^2 + 2.5s + 1}\right] = e^{-s}\left[0.5 - \frac{0.25s - 0.5}{(s+0.5)(s+2)}\right]$$

Finally we have

$$H(s) = e^{-s}\left[0.5 - G(s)\right]$$

where

$$G(s) = \frac{0.25s - 0.5}{(s+0.5)(s+2)}$$

Partial Fractions with Distinct Poles

Let's assume we have a strictly proper transfer function

$$H(s) = \frac{N(s)}{D(s)} = \frac{K(s - z_1)(s - z_2)\cdots(s - z_m)}{(s - p_1)(s - p_2)\cdots(s - p_n)}$$

The poles of the system are at $p_1, p_2, ..., p_n$ and the zeros of the system are at. $z_1, z_2, ..., z_m$ Since we have distinct poles, we know that $p_i \neq p_j$ for $i \neq j$. We also assume that $N(s)$ and $D(s)$ have no common factors so there is no pole/zero cancellation. We would like to find the corresponding *impulse response* $h(t)$. To do this we assume

$$H(s) = a_1 \frac{1}{s - p_1} + a_2 \frac{1}{s - p_2} + ... + a_n \frac{1}{s - p_n}$$

If we can find the a_i then it will be easy to determine $h(t)$ since we know

$$\frac{1}{s - p_i} \leftrightarrow e^{p_i t} u(t)$$

To find a_1 we first multiply by $s - p_1$,

$$(s - p_1)H(s) = a_1 + a_2 \frac{(s - p_1)}{s - p_2} + ... + a_n \frac{(s - p_1)}{s - p_n}$$

Next take the limit as $s \to p_1$. Since all of the poles are distinct all of the terms on the right hand side of the equation are zero, except for the first term. Hence we have

$$a_1 = \lim_{s \to p_1}(s - p_1)H(s)$$

Similarly we get

$$a_2 = \lim_{s \to p_2}(s - p_2)H(s)$$

And in general

$$a_i = \lim_{s \to pi}(s - p_i)H(s)$$

Example 6.6.4. For the transfer function

$$H(s) = \frac{s+1}{(s+2)(s+3)}$$

determine the corresponding impulse response. We have

$$H(s) = a_1 \frac{1}{s+2} + a_2 \frac{1}{s+3}$$

Then

$$a_1 = \lim_{s \to -2}(s+2)H(s) = \lim_{s \to -2} \frac{(s+2)(s+1)}{(s+2)(s+3)} = \lim_{s \to -2} \frac{(s+1)}{(s+3)} = -1$$

$$a_2 = \lim_{s \to -3}(s+3)H(s) = \lim_{s \to -3} \frac{(s+3)(s+1)}{(s+2)(s+3)} = \lim_{s \to -3} \frac{(s+1)}{(s+2)} = 2$$

So

$$H(s) = \frac{-1}{s+2} + \frac{2}{s+3}$$

and

$$h(t) = e^{-2t}u(t) + 2e^{-3t}u(t)$$

It is often unnecessary to write out all of these steps, since we know in advance there will be a cancellation with one of the poles. In particular, when we want to find a_i we know we will have a cancellation between the $s - p_i$ term in the numerator and the $s - p_i$ term in the denominator. In fact, when we to find a_i, we can just ignore (or *cover up)* the $s - p_i$ term in the denominator. This is usually just called the "cover up" method. For our previous example, we have

$$a_1 = \lim_{s \to -2} \frac{(s+1)}{\cancel{(s+2)}(s+3)} = -1$$

$$a_2 = \lim_{s \to -3} \frac{(s+1)}{(s+2)\cancel{(s+3)}} = 2$$

This can often just be done in your head!

Example 6.6.5. Consider the transfer function from Example 6.6.1. We have

$$H(s) = \frac{e^{-2s}(s+1)}{(s+3)(s+4)} = e^{-2s}G(s)$$

where

$$G(s) = \frac{(s+1)}{(s+3)(s+4)}$$

We now do partial fractions for $G(s)$ and use the fact that the e^{-2s} terms corresponds to a time delay of 2 units. We have then

$$G(s) = \frac{s+1}{(s+3)(s+4)} = a_1 \frac{1}{s+3} + a_2 \frac{1}{s+4}$$

$$a_1 = \lim_{s \to -3} \frac{(s+1)}{\cancel{(s+3)}(s+4)} = -2$$

$$a_2 = \lim_{s \to -4} \frac{(s+1)}{(s+3)\cancel{(s+4)}} = 3$$

Then

$$g(t) = -2e^{-3t}u(t) + 3e^{-4t}u(t)$$

Finally we take the time delay into account to get the final answer

$$h(t) = g(t-2) = -2e^{-3(t-2)}u(t-2) + 3e^{-4(t-2)}u(t-2)$$

Example 6.6.6. Consider the transfer function from Example 6.6.3. We have

$$H(s) = \frac{e^{-s}(s^2 + 2s + 2)}{(2s+1)(s+2)} = e^{-s}\left[0.5 - \frac{0.25s - 0.5}{(s+0.5)(s+2)}\right] = e^{-s}\left[0.5 - G(s)\right]$$

where

$$G(s) = \frac{0.25s - 0.5}{(s+0.5)(s+2)}$$

We then have

$$G(s) = \frac{0.25s - 0.5}{(s+0.5)(s+2)} = a_1 \frac{1}{s+0.5} + a_2 \frac{1}{s+2}$$

$$a_1 = \lim_{s \to -0.5} \frac{0.25s - 0.5}{\cancel{(s+0.5)}(s+2)} = -\frac{5}{12}$$

$$a_2 = \lim_{s \to -2} \frac{0.25s - 0.5}{(s+0.5)\cancel{(s+2)}} = \frac{2}{3}$$

So

$$g(t) = -\frac{5}{12}e^{-0.5t}u(t) + \frac{2}{3}e^{-2t}u(t)$$

If we then define

$$F(s) = 0.5 - G(s)$$

then

$$f(t) = 0.5 - g(t) = 0.5 + \frac{5}{12}e^{-0.5t}u(t) - \frac{2}{3}e^{-2t}u(t)$$

Finally we take the time delay into account,

$$H(s) = e^{-s}F(s)$$

$$h(t) = f(t-1) = 0.5 + \frac{5}{12}e^{-0.5(t-1)}u(t-1) - \frac{2}{3}e^{-2(t-1)}u(t-1)$$

Example 6.6.7. Consider the circuit in Example 6.3.1. Here we have the relationship

$$V_{out}(s) = \frac{V_{in}(s)RCs}{RCs+1} - \frac{v(0^-)RC}{RCs+1}$$

Instead of determining the impulse response, let's determine the system output for a step input of amplitude A. Hence we have $v_{in}(t) = Au(t)$, or $V_{in}(s) = \frac{A}{s}$. Then we can write the output as

$$V_{out}(s) = \frac{ARCs}{s(RCs+1)} - \frac{v(0^-)RC}{RCs+1}$$

Next we need to make the denominator polynomials monic, so we have

$$V_{out}(s) = \frac{As}{s\left(s+\dfrac{1}{RC}\right)} - \frac{v(0^-)}{s+\dfrac{1}{RC}} = \frac{A}{s+\dfrac{1}{RC}} - \frac{v(0^-)}{s+\dfrac{1}{RC}}$$

This problem is already in the correct for, and we have

$$v_{out}(t) = Ae^{-t/RC}u(t) - v(0^-)e^{-t/RC}u(t) = \left[A - v(0^-)\right]e^{-t/RC}u(t)$$

If we look at the circuit we have modeling (in Figure 6.4), this answer makes sense. The initial voltage across the resistor is the difference between the initial applied voltage and the initial voltage across the capacitor. In addition, the voltage across the resistor should approach zero as time increases, since eventually the voltage on the capacitor will reach the applied voltage. Finally, it is clear that the time constant for this simple circuit is $\tau = RC$ and this is the time constant of our results.

Example 6.6.8. Consider the circuit in Example 6.3.2. Here we have the relationship

$$V_{out}(s) = \frac{R}{R+Ls}V_{in}(s) + \frac{RL}{R+Ls}i(0^-)$$

Again let's determine the output when the input is a step of amplitude A. Then we have

$$V_{out}(s) = \frac{R}{R+Ls}\frac{A}{s} + \frac{RL}{R+Ls}i(0^-)$$

We need to make the denominators monic, so we rewrite this as

$$V_{out}(s) = \frac{\dfrac{AR}{L}}{s\left(s+\dfrac{R}{L}\right)} + \frac{R}{s+\dfrac{R}{L}}i(0^-)$$

Applying partial fraction to the first term we have

$$V_{out}(s) = \frac{A}{s} - \frac{A}{s + \dfrac{R}{L}} + \frac{R}{s + \dfrac{R}{L}} i(0^-)$$

In the time-domain we then have

$$v_{out}(t) = A(1 - e^{-\frac{R}{L}t})u(t) + Ri(0^-)e^{-\frac{R}{L}t}u(t)$$

Partial Fractions with Repeated Poles

If there are repeated poles with no other poles, then the inverse Laplace transform is very straightforward, using the formula

$$\frac{t^{m-1}}{(m-1)!}e^{-pt}u(t) \leftrightarrow \frac{1}{(s+p)^m}$$

If instead of isolated repeated poles and nonrepeated poles, then we need to use a different form for the partial fractions for the repeated poles. If we assume we have a pole at $-p$ that is of order m, then for this pole we will use an expansion of the form

$$a_1 \frac{1}{(s+p)} + a_2 \frac{1}{(s+p)^2} + \cdots + a_m \frac{1}{(s+p)^m}$$

We will also need a new approach for determining some of the expansion coefficients for the repeated poles. There are two common approaches

- Multiply both sides of the equation by s and taking the limit as $s \rightarrow \infty$
- Select convenient values for s and evaluate both sides of the equation for these values of s

These techniques are probably most easily explained by the use of examples. In each of these examples you should note that *there are always as many unknowns as there are poles!*

Example 6.6.9. Find the impulse response that corresponds to the transfer function

$$H(s) = \frac{1}{(s+1)(s+2)^2}$$

To do this we look for a partial fraction expansion of the form

$$H(s) = a_1 \frac{1}{s+1} + a_2 \frac{1}{s+2} + a_3 \frac{1}{(s+2)^2}$$

For the distinct pole we have

$$a_1 = \lim_{s \rightarrow -1} \frac{1}{\cancel{(s+1)}(s+2)^2} = 1$$

Next we determine the coefficient for the highest power of the repeated root. In this case we multiply both sides of our partial fraction expansion by $(s+2)^2$ and then take the limit as $s \rightarrow -2$

$$a_3 = \lim_{s \rightarrow -2} \frac{1}{(s+1)\cancel{(s+2)^2}} = -1$$

Finally, to get the value of a_2 we must resort to a different method. If we multiply both sides by s and take the limit as $s \rightarrow \infty$ we have

$$\lim_{s \to \infty} sH(s) = \lim_{s \to \infty} \frac{s}{(s+1)(s+2)^2} = \lim_{s \to \infty} \left[a_1 \frac{s}{s+1} + a_2 \frac{s}{s+2} + a_3 \frac{s}{(s+2)^2} \right]$$

Simplifying this we have

$$0 = a_1 + a_2$$

or

$$a_2 = -a_1 = -1$$

Hence we have

$$H(s) = \frac{1}{s+1} - \frac{1}{s+2} - \frac{1}{(s+2)^2}$$

with corresponding impulse response

$$h(t) = e^{-t}u(t) - e^{-2t}u(t) - te^{-2t}u(t)$$

As an alternative method for determining a_2, we start with

$$H(s) = \frac{1}{(s+1)(s+2)^2} = \frac{1}{s+1} + a_2 \frac{1}{s+2} - \frac{1}{(s+2)^2}$$

This expression must be true for all values of s **_as long as both sides remain finite._** Let's choose a convenient value of s, like $s = 0$. Then we have

$$H(0) = \frac{1}{4} = 1 + a_2 \frac{1}{2} - \frac{1}{4}$$

or

$$-\frac{1}{2} = a_2 \frac{1}{2}$$

which again give us $a_2 = -1$.

Example 6.6.10. Determine the impulse response that corresponds to the transfer function

$$H(s) = \frac{s+1}{s^2(s+2)(s+3)}$$

The partial fraction expansion we need is of the form

$$H(s) = a_1 \frac{1}{s} + a_2 \frac{1}{s^2} + a_3 \frac{1}{s+2} + a_4 \frac{1}{s+3}$$

First we find the coefficients that correspond to the distinct poles and the coefficient that goes with the highest power of the repeated pole.

$$a_2 = \lim_{s \to 0} \frac{s+1}{\cancel{s^2}(s+2)(s+3)} = \frac{1}{6}$$

$$a_3 = \lim_{s \to -2} \frac{s+1}{s^2 \cancel{(s+2)}(s+3)} = -\frac{1}{4}$$

$$a_4 = \lim_{s \to -3} \frac{s+1}{s^2(s+2)\cancel{(s+3)}} = \frac{2}{9}$$

At this point we have

$$H(s) = \frac{s+1}{s^2(s+2)(s+3)} = a_1 \frac{1}{s} + \frac{1}{6}\frac{1}{s^2} - \frac{1}{4}\frac{1}{s+2} + \frac{2}{9}\frac{1}{s+3}$$

To find the unknown coefficient, we will multiply by s and let $s \to \infty$,

$$\lim_{s \to \infty} sH(s) = \lim_{s \to \infty} \frac{s(s+1)}{s^2(s+2)(s+3)} = \lim_{s \to \infty}\left[a_1 \frac{s}{s} + \frac{1}{6}\frac{s}{s^2} - \frac{1}{4}\frac{s}{s+2} + \frac{2}{9}\frac{s}{s+3}\right]$$

or

$$0 = a_1 + 0 - \frac{1}{4} + \frac{2}{9}$$

This simplifies to

$$a_1 = \frac{1}{36}$$

Finally we have

$$H(s) = \frac{s+1}{s^2(s+2)(s+3)} = \frac{1}{36}\frac{1}{s} + \frac{1}{6}\frac{1}{s^2} - \frac{1}{4}\frac{1}{s+2} + \frac{2}{9}\frac{1}{s+3}$$

which corresponds to the impulse response

$$h(t) = \frac{1}{36}u(t) + \frac{1}{6}tu(t) - \frac{1}{4}e^{-2t}u(t) + \frac{2}{9}e^{-3t}u(t)$$

As an alternative to taking limits in the expression

$$H(s) = \frac{s+1}{s^2(s+2)(s+3)} = a_1 \frac{1}{s} + \frac{1}{6}\frac{1}{s^2} - \frac{1}{4}\frac{1}{s+2} + \frac{2}{9}\frac{1}{s+3}$$

we can make the substitution $s = -1$ in both sides of the expression (note that we cannot use $s = 0, s = -2$, or $s = -3$, since these substitutions make the function infinite). The we have

$$0 = -a_1 + \frac{1}{6} - \frac{1}{4} + \frac{1}{9} = -a_1 + \frac{6-9+4}{36} = -a_1 + \frac{1}{36}$$

which again yields

$$a_1 = \frac{1}{36}$$

Example 6.6.11. Determine the impulse response for the transfer function

$$H(s) = \frac{s+3}{s(s+1)^2(s+2)^2}$$

The appropriate partial fraction expansion is of the form

$$H(s) = \frac{s+3}{s(s+1)^2(s+2)^2} = a_1 \frac{1}{s} + a_2 \frac{1}{s+1} + a_3 \frac{1}{(s+1)^2} + a_4 \frac{1}{s+2} + a_5 \frac{1}{(s+2)^2}$$

Again we first find the easy coefficients, a_1, a_3 and a_5.

$$a_1 = \lim_{s \to 0} \frac{s+3}{\cancel{s}(s+1)^2(s+2)^2} = \frac{3}{4}$$

$$a_3 = \lim_{s \to -1} \frac{s+3}{s\cancel{(s+1)^2}(s+2)^2} = -2$$

$$a_5 = \lim_{s \to -2} \frac{s+3}{s(s+1)^2\cancel{(s+2)^2}} = -\frac{1}{2}$$

Next we use limits

$$\lim_{s \to \infty} sH(s) = \lim_{s \to \infty} \frac{s(s+3)}{s(s+1)^2(s+2)^2} = \lim_{s \to \infty} \left[\frac{3}{4} \frac{s}{s} + a_2 \frac{s}{s+1} - 2 \frac{s}{(s+1)^2} + a_4 \frac{s}{s+2} - \frac{1}{2} \frac{1}{(s+2)^2} \right]$$

or

$$0 = \frac{3}{4} + a_2 + a_4$$

We need another equation, so let's let $s = -3$, in the expression

$$H(s) = \frac{s+3}{s(s+1)^2(s+2)^2} = \frac{3}{4} \frac{1}{s} + a_2 \frac{1}{s+1} - 2 \frac{1}{(s+1)^2} + a_4 \frac{1}{s+2} - \frac{1}{2} \frac{1}{(s+2)^2}$$

which yields

$$0 = -\frac{1}{4} - a_2 \frac{1}{2} - \frac{1}{2} - a_4 - \frac{1}{2}$$

or

$$a_2 \frac{1}{2} - a_4 = \frac{5}{4}$$

Solving these two equations yields $a_2 = 1$ and $a_4 = -\frac{7}{4}$. Finally, we have

$$H(s) = \frac{s+3}{s(s+1)^2(s+2)^2} = \frac{3}{4} \frac{1}{s} + \frac{1}{s+1} - 2 \frac{1}{(s+1)^2} - \frac{7}{4} \frac{1}{s+2} - \frac{1}{2} \frac{1}{(s+2)^2}$$

which corresponds to the impulse response

$$h(t) = \frac{3}{4}u(t) + e^{-t}u(t) - 2te^{-t}u(t) - \frac{7}{4}e^{-2t}u(t) - \frac{1}{2}te^{-2t}u(t)$$

Partial Fractions with Complex Conjugate Poles

The transform pairs we are primarily going to use with complex conjugate poles are

$$e^{-at}\cos(bt)u(t) \leftrightarrow \frac{s+a}{(s+a)^2 + b^2}$$

$$e^{-at}\sin(bt)u(t) \leftrightarrow \frac{b}{(s+a)^2 + b^2}$$

Note that complex conjugate poles always result in sines and cosines (or a single sine/cosine with a phase angle). We will try and make terms with complex conjugate poles look like these terms by completing the square in the denominator. That is, we need to be able to write the denominator as

$$D(s) = (s+a)^2 + b^2$$

Note that if we cannot write the denominator in this form, the poles are not complex conjugates! To determine the correct values of *a* and *b*, use the fact that the coefficient of *s* should be *2a*. Once we find *a*, the value for *b* is easy to find. A few examples will make this clear.

Example 6.6.12. Assume $D(s) = s^2 + s + 2$ and we want to write this in the correct form. First we recognize that the coefficient of *s* is *1*, so that *2a = 1*, or $a = \frac{1}{2}$. We then have

$$D(s) = s^2 + s + 2 = (s + \frac{1}{2})^2 + b^2 = s^2 + s + \frac{1}{4} + b^2$$

So then $2 = \frac{1}{4} + b^2$ and we can determine $b = \frac{\sqrt{7}}{2}$. Thus we have

$$D(s) = s^2 + s + 1 = (s + \frac{1}{2})^2 + \left(\frac{\sqrt{7}}{2}\right)$$

Example 6.6.13. Assume $D(s) = s^2 + 3s + 5$ and we want to write this in the correct form. The coefficient of *s* is *3*, so we have $2a = 3$ and $a = \frac{3}{2}$. We then have

$$D(s) = s^2 + 3s + 5 = (s + \frac{3}{2})^2 + b^2 = s^2 + 3s + \frac{9}{4} + b^2$$

Then $5 = \dfrac{9}{4} + b^2$ and we can determine $b = \dfrac{\sqrt{11}}{4}$.

Now that we know how to complete the square we will look at two simple examples of complex conjugate poles, then at more complicated examples.

Example 6.6.14. Assuming $H(s) = \dfrac{1}{s^2 + s + 2}$, determine the corresponding impulse response $h(t)$.

From our previous example we know

$$H(s) = \frac{1}{(s+\frac{1}{2})^2 + \left(\dfrac{\sqrt{7}}{2}\right)^2}$$

This almost has the form we want, which is

$$\frac{b}{(s+a)^2 + b^2} \leftrightarrow e^{-at} \sin(bt) u(t)$$

However, to use this form we need a b in the numerator. To achieve this we will multiply and divide by $b = \dfrac{\sqrt{7}}{2}$,

$$H(s) = \frac{1}{\dfrac{\sqrt{7}}{2}} \frac{\dfrac{\sqrt{7}}{2}}{(s+\frac{1}{2})^2 + \left(\dfrac{\sqrt{7}}{2}\right)^2} = \frac{2}{\sqrt{7}} \frac{\dfrac{\sqrt{7}}{2}}{(s+\frac{1}{2})^2 + \left(\dfrac{\sqrt{7}}{2}\right)^2}$$

and we can determine

$$h(t) = \frac{2}{\sqrt{7}} e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{7}}{2} t\right) u(t)$$

Example 6.6.15. Assuming $H(s) = \dfrac{s}{s^2 + 3s + 5}$, determine the corresponding impulse response $h(t)$.

From our previous example we know

$$H(s) = \frac{s}{\left(s + \dfrac{3}{2}\right)^2 + \left(\dfrac{\sqrt{11}}{2}\right)^2}$$

This is almost the form we want, which is

$$H(s) = \frac{s+a}{(s+a)^2 + b^2}$$

However, to use this form, we will add and subtract 3/2 in our transfer function,

$$H(s) = \frac{s + \frac{3}{2} - \frac{3}{2}}{\left(s + \frac{3}{2}\right)^2 + \left(\frac{\sqrt{11}}{2}\right)^2} = \frac{s + \frac{3}{2}}{\left(s + \frac{3}{2}\right)^2 + \left(\frac{\sqrt{11}}{2}\right)^2} - \frac{\frac{3}{2}}{\left(s + \frac{3}{2}\right)^2 + \left(\frac{\sqrt{11}}{2}\right)^2}$$

The first term is now what we want, but we need to scale the second term,

$$H(s) = \frac{s + \frac{3}{2}}{\left(s + \frac{3}{2}\right)^2 + \left(\frac{\sqrt{11}}{2}\right)^2} - \frac{3}{2} \frac{2}{\sqrt{11}} \frac{\frac{\sqrt{11}}{2}}{\left(s + \frac{3}{2}\right)^2 + \left(\frac{\sqrt{11}}{2}\right)^2} = \frac{s + \frac{3}{2}}{\left(s + \frac{3}{2}\right)^2 + \left(\frac{\sqrt{11}}{2}\right)^2} - \frac{3}{\sqrt{11}} \frac{\frac{\sqrt{11}}{2}}{\left(s + \frac{3}{2}\right)^2 + \left(\frac{\sqrt{11}}{2}\right)^2}$$

So we finally have the impulse response

$$h(t) = e^{-\frac{3}{2}t} \cos(\frac{\sqrt{11}}{2}t)u(t) - \frac{3}{\sqrt{11}} e^{-\frac{3}{2}t} \sin(\frac{\sqrt{11}}{2}t)u(t)$$

The next two examples are much more involved, but they have the same general approach. In general, if there are complex conjugate poles we will look for partial fractions of the form

$$\frac{c(s+a)}{(s+a)^2 + b^2} + \frac{d(b)}{(s+a)^2 + b^2}$$

where c and *d* are the parameters to be determined.

Example 6.6.16. Assume

$$Y(s) = \frac{1}{s+2} \frac{1}{s^2 + s + 1}$$

Use partial fractions to determine the corresponding time domain function. We will have

$$Y(s) = \frac{1}{s+2} \frac{1}{s^2+s+1} = \frac{1}{(s+2)\left[\left(s+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2\right]} = \frac{A}{s+2} + \frac{c(s+\frac{1}{2})}{\left(s+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} + \frac{d\frac{\sqrt{3}}{2}}{\left(s+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \text{ We}$$

need to determine the three coefficients A, c, and d. To determine A we use the cover-up method as before

$$A = \lim_{s \to -2} \frac{1}{\cancel{s+2}} \frac{1}{s^2 + s + 1} = \frac{1}{4 - 2 + 1} = \frac{1}{3}$$

To determine c, multiply both sides by s and let $s \to \infty$,

$$\lim_{s \to \infty} \frac{s}{(s+2)(s^2+s+1)} = \lim_{s \to \infty} \left[\frac{sA}{s+2} + \frac{sc(s+\frac{1}{2})}{\left(s+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} + \frac{sd\frac{\sqrt{3}}{2}}{\left(s+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \right]$$

or

$$0 = A + c = \frac{1}{3} + c$$

So we have $c = -\frac{1}{3}$. Finally, we choose a convenient value for s, and evaluate both sides. Let's choose $s = 0$, so we have

$$\lim_{s \to 0} Y(s) = \lim_{s \to 0} \frac{1}{(s+2)(s^2+s+1)} = \lim_{s \to 0} \left[\frac{A}{s+2} + \frac{c(s+\frac{1}{2})}{\left(s+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} + \frac{d\frac{\sqrt{3}}{2}}{\left(s+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \right]$$

or

We can simplify this to

$$\frac{1}{2} = \frac{A}{2} + c(\frac{1}{2}) + d(\frac{\sqrt{3}}{2}) = \frac{1}{6} - \frac{1}{6} + d(\frac{\sqrt{3}}{2})$$

which yields $d = \frac{1}{\sqrt{3}}$. Finally the time-domain result is

$$y(t) = \frac{1}{3} e^{-2t} u(t) - \frac{1}{3} e^{-\frac{1}{2}t} \cos(\frac{\sqrt{3}}{2}t) u(t) + \frac{1}{\sqrt{3}} e^{-\frac{1}{2}t} \sin(\frac{\sqrt{3}}{2}t) u(t)$$

Example 6.6.17. Find the step response of the system with transfer function

$$H(s) = \frac{1}{s^2 + 2s + 2}$$

The step response for this system will be given by

$$Y(s) = H(s)\frac{1}{s} = \frac{1}{s(s^2+2s+2)} = \frac{A}{s} + \frac{c(s+1)}{(s+1)^2+1} + \frac{d}{(s+1)^2+1}$$

Using the cover-up method we get $A = \frac{1}{2}$. To get c, multiply both sides by s and let $s \to \infty$,

$$\lim_{s \to \infty} sY(s) = \frac{s}{s(s^2+2s+2)} = \lim_{s \to \infty} \left[\frac{sA}{s} + \frac{sc(s+1)}{(s+1)^2+1} + \frac{sd}{(s+1)^2+1} \right]$$

or

$$0 = A + c = \frac{1}{2} + c$$

which gives us $c = -\frac{1}{2}$. Finally, we set s to a convenient value and equate both sides. In this case $s = -1$ is a good choice.

$$\lim_{s \to -1} Y(s) = \lim_{s \to -1} \frac{1}{s(s^2 + 2s + 2)} = \lim_{s \to -1} \left[\frac{A}{s} + \frac{c(s+1)}{(s+1)^2 + 1} + \frac{d}{(s+1)^2 + 1} \right]$$

$$-1 = -A + d = -\frac{1}{2} + d$$

So we can conclude that $d = -\frac{1}{2}$ and the complete answer is

$$y(t) = \frac{1}{2}u(t) - \frac{1}{2}e^{-t}cos(t)u(t) - \frac{1}{2}e^{-t}\sin(t)u(t)$$

6.7 Block Diagrams, Signal Flow Graphs, and Mason's Rule

For an LTI system, we know that we can relate the input and output using convolution and the impulse response, $y(t) = h(t) * x(t)$. Using Laplace transforms we have the equivalent relationship $Y(s) = H(s)X(s)$. Here $H(s)$ is called the transfer function, and is the Laplace transform of the impulse response $h(t)$. We can represent interconnected systems in the time domain, as shown in Figure 6.11, where we have the input/output relationships

$$y(t) = h_1(t) * h_2(t) * x(t)$$
$$y(t) = \left[h_1(t) + h_2(t) \right] * x(t)$$
$$y(t) * [\delta(t) + g_1(t) * g_2(t) * h_1(t)] = x(t) * [g_0(t) * g_1(t) * g_2(t)]$$

As expected, these relationships are in terms of convolution.
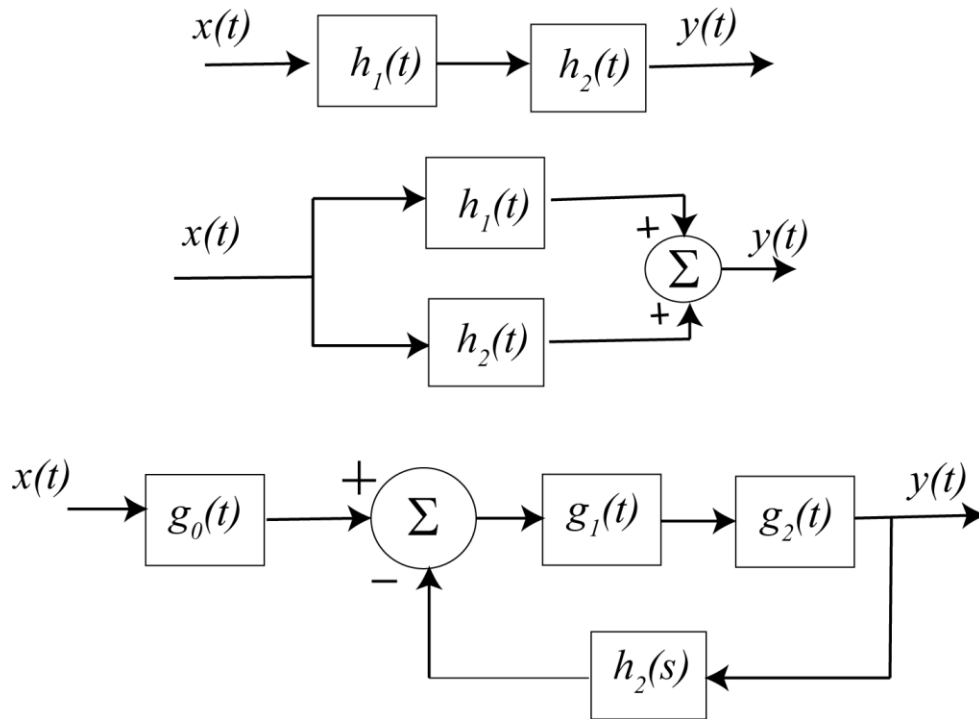
Figure 6.11. Time domain interconnection of systems.

Instead of dealing with time-domain functions and convolutions, we can deal with transfer functions and multiplications. The transfer function representation of the systems in Figure 6.11 can be represented in terms of transfer functions as shown in Figure 6.12. In the transfer function domain we have

$$Y(s) = H_1(s)H_2(s)X(s)$$

$$Y(s) = [H_1(s) + H_2(s)]X(s)$$

$$Y(s) = \frac{G_0(s)G_1(s)G_2(s)}{1 + G_1(s)G_2(s)H_1(s)}$$

Now we have replaced convolution with multiplication and division, which is easier to work with. As you become more familiar with transfer functions, you will realize that you can learn a great deal about the time-domain performance without actually performing the inverse Laplace transforms.
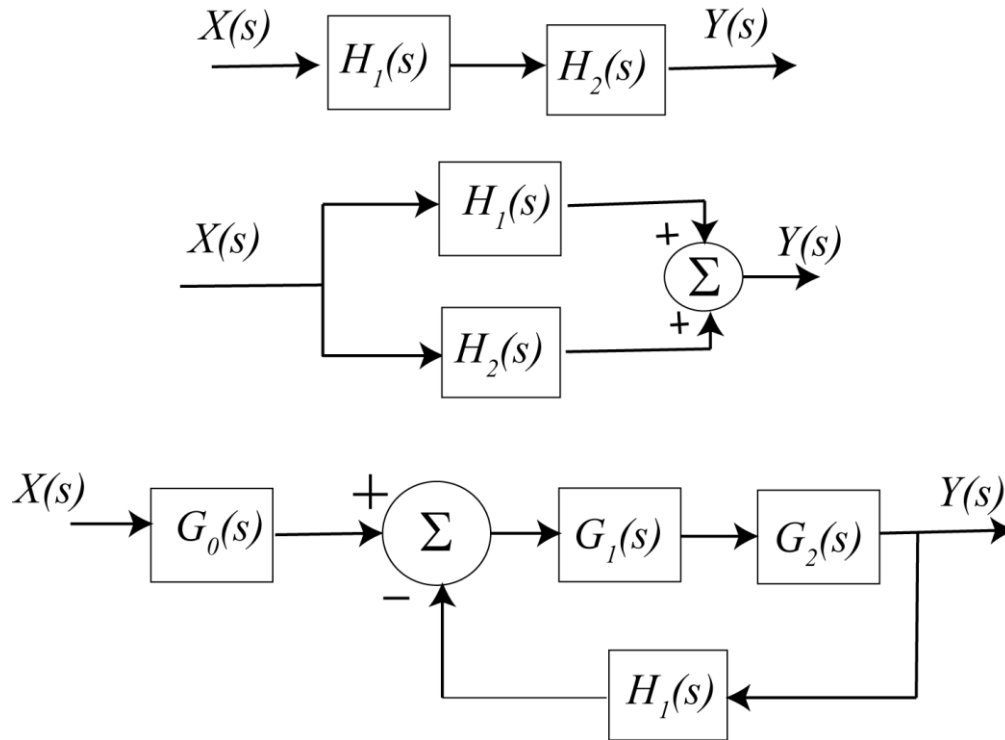
Figure 6.12. Transform domain representation of systems from Figure 6.11.

As an alternative to block diagrams, we can represent the interconnection between subsystems utilizing signal flow graphs. A signal flow graph is composed of *directed branches* and *nodes*. Just like block diagrams, these graphs show the flow of signals throughout an interconnected system but rather than shown transfer functions in blocks, the transfer functions (or transmittances) are written over the directed branches. Finally, in a signal flow graph all of the branches into a node are summed. Hence, if you want negative feedback you need to include the negative sign in one of the transfer functions. Before getting into too much detail, you should examine the following examples of block diagrams and the equivalent signal flow diagrams. In the following signal flow graphs, a '1' indicates the transfer function is just unity.
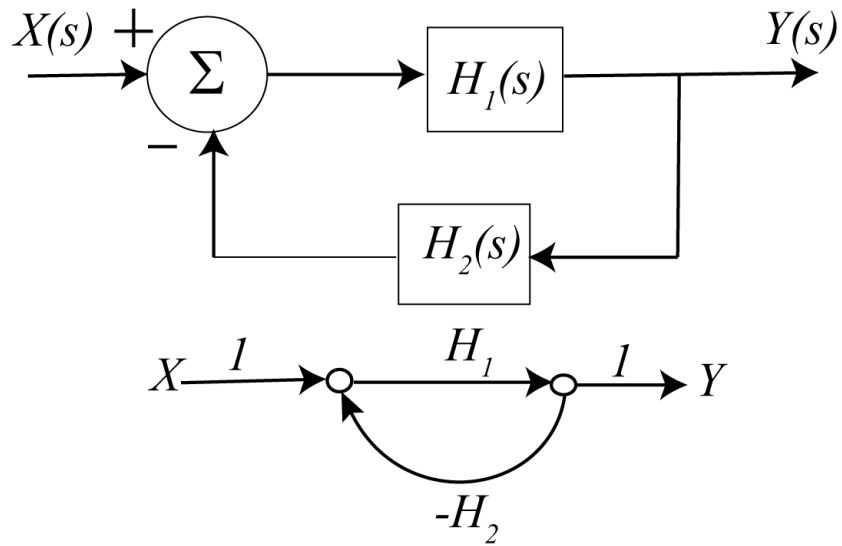
Figure 6.13. Signal flow graph of a simple feedback system.
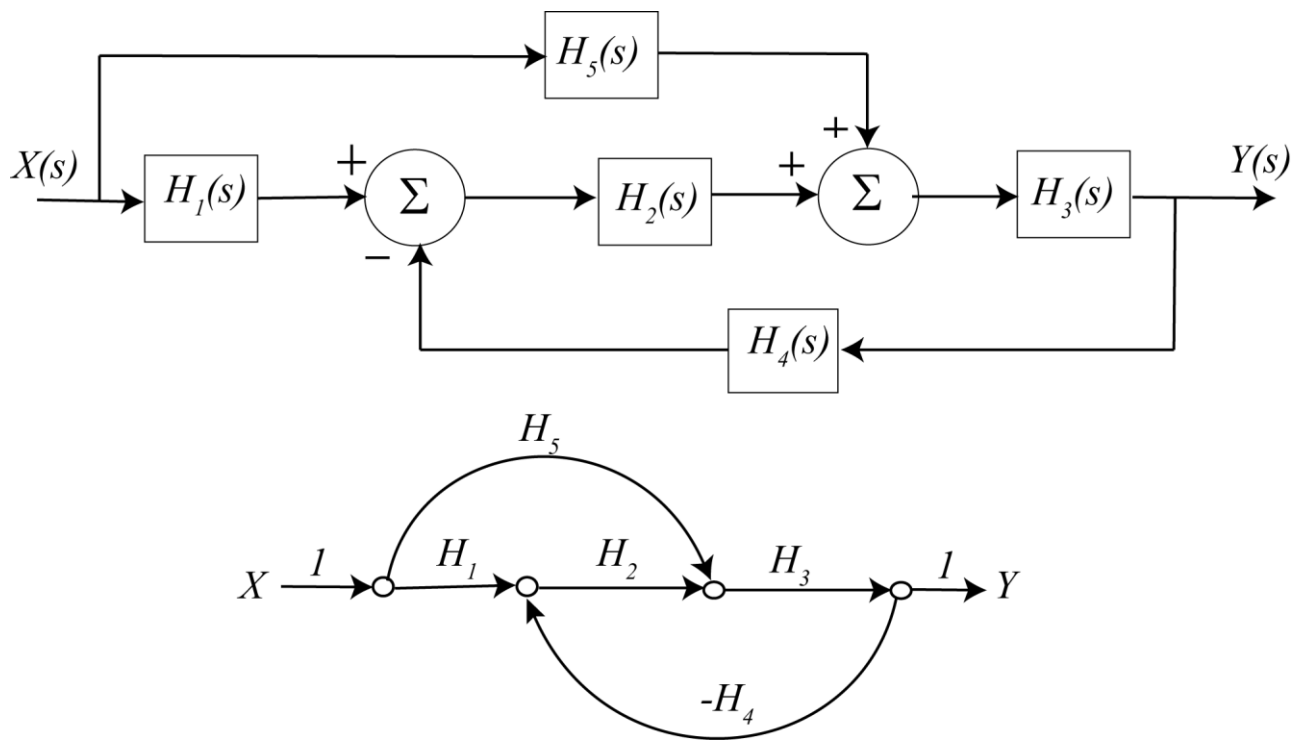
Figure 6.14. Second example of a block diagram and the equivalent signal flow graph.
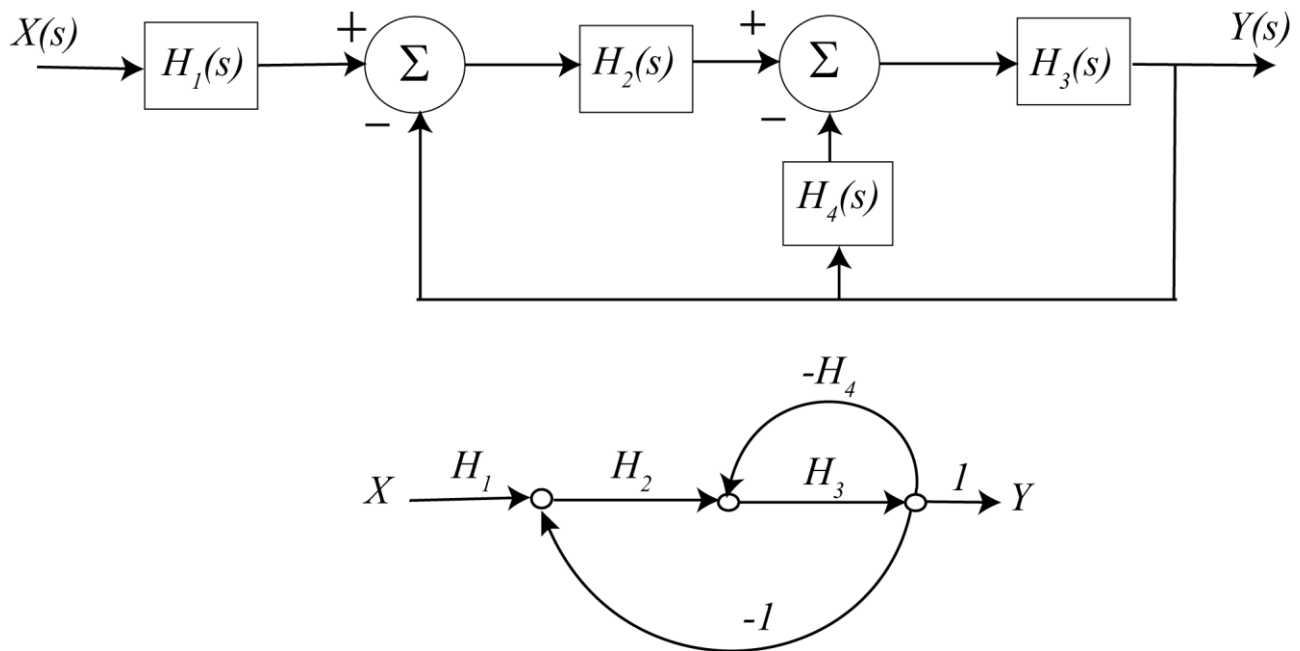
Figure 6.15. Third example of a block digraph and the equivalent signal flow graph.

We would like to be able to determine the overall system transfer function form the input to the output of a system represented by a signal flow diagram. In order to do this we will need to introduce some definitions and apply them to an example problem. It is usually helpful to label some of the *paths* and *loops* on the signal flow graph to keep track of them. As you are going through the following definitions and steps, try to label these on the diagram. Let's assume we have the (fairly complicated) signal flow graph shown in Figure 6.16.
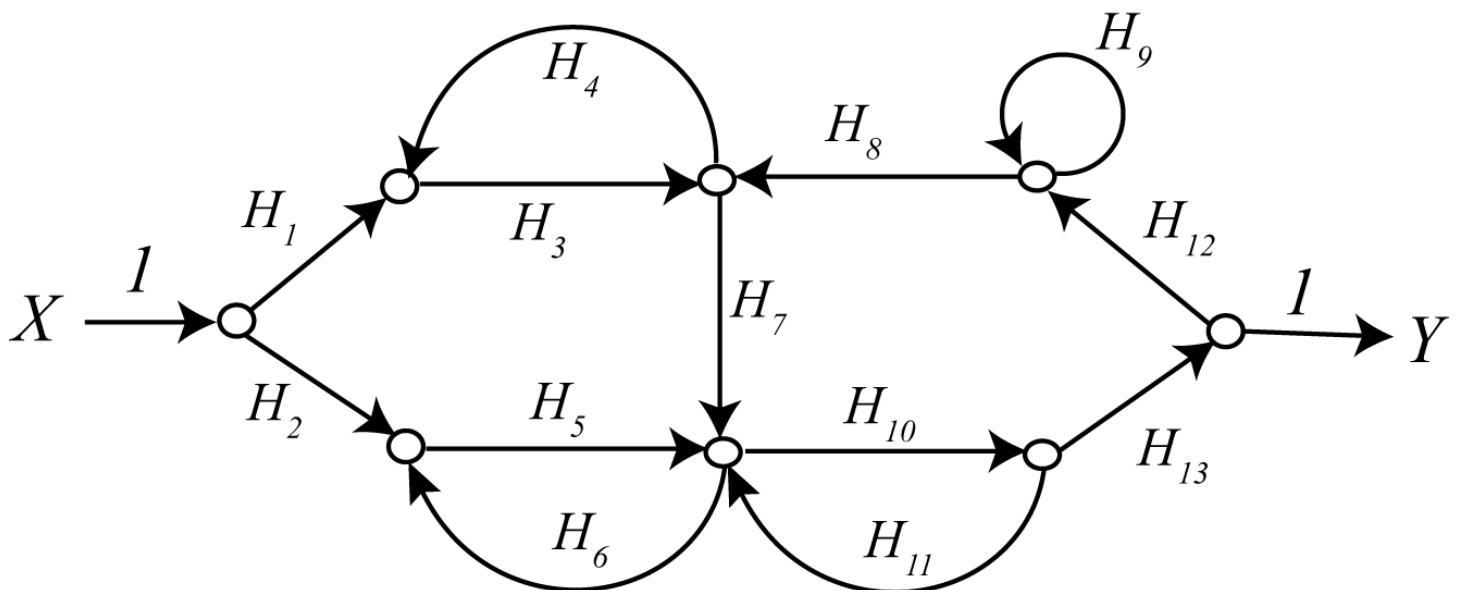
Figure 6.16. Example signal flow graph used in the definitions example.

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Definition: A *path* is any succession of branches, from input to output, in the direction of the arrows, which does not pass any node more than once. A *path gain* is the product of the transfer functions (transmittances) of the branches comprising the path.

For the signal flow graph in Figure 6.16, the path gains are (arbitrarily numbered):

$$P_1 = H_1 H_3 H_7 H_{10} H_{13}, \ P_2 = H_2 H_5 H_{10} H_{13}$$

Definition: A *loop* is any closed succession of branches, in the direction of the arrows, which does not pass any node more than once. The *loop gain* is the product of the transfer functions of the branches comprising the loops.

For the signal flow graph in Figure 6.16, the loop gains are (arbitrarily numbered):

$$L_1 = H_3 H_4, \ L_2 = H_5 H_6, \ L_3 = H_{10} H_{11}, \ L_4 = H_9, \ L_5 = H_{10} H_{13} H_{12} H_8 H_7$$

Definition: To loops are *touching* if they have any node in common. A path and a loop are *touching* if they have any node in common.

Definition: The *determinant* of a signal flow graph is denoted by Δ, and is computed by the formula

$\Delta = 1 -$ (sum of all loop gains) + (sum of all products of gains of all combinations of 2 non touching loops) − (sum of products of gains of all combinations of 3 nontouching loops) + …

For the signal flow graph in Figure 6.16, the determinant is

$$\begin{aligned} \Delta = 1 - (L_1 + L_2 + L_3 + L_4 + L_5) \\ + (L_1 L_2 + L_1 L_3 + L_1 L_4 + L_2 L_4 + L_3 L_4) - (L_1 L_2 L_4 + L_1 L_3 L_4) \end{aligned}$$

Definition: The *cofactor of a path* is the determinant of the signal flow graph formed by deleting all loops touching the path. To get the cofactor for each path, you write out the determinant and cross off all loops touching that particular path.

For the signal flow graph in Figure 6.16,

$$\begin{aligned} path \ P_1, \ \Delta_1 = 1 - L_4 \\ path \ P_2, \ \Delta_2 = 1 - (L_1 + L_4) + (L_1 L_4) \end{aligned}$$

Definition: The *transfer function* of the signal flow graph is given by the formula

$$H_{system} = \frac{P_1 \Delta_1 + P_2 \Delta_2 + P_3 \Delta_3 + \cdots}{\Delta}$$

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For the signal flow graph in Figure 6.16, the system transfer function is then

$$H_{system} = \frac{P_1 \Delta_1 + P_2 \Delta_2}{\Delta}$$

In most cases our systems are not as complicated as this one, but this method will work for any system provided we are systematic. Let's do some examples now. You should try these and then check your answers.

Example 6.7.1. For the signal flow graph in Figure 6.13 we have

$$P_1 = H_1, L_1 = H_1(-H_2) = -H_1 H_2, \Delta = 1 + H_1 H_2, \Delta_1 = 1$$
$$H_{system} = \frac{H_1}{1 + H_1 H_2}$$

Example 6.7.2. For the signal flow graph in Figure 6.14 we have

$$P_1 = H_5 H_3, P_2 = H_1 H_2 H_3, L_1 = -H_2 H_3 H_4, \Delta = 1 - L_1, \Delta_1 = 1, \Delta_2 = 1$$
$$H_{system} = \frac{H_3 H_5 + H_1 H_2 H_3}{1 + H_2 H_3 H_4}$$

Example 6.7.3. For the signal flow graph shown in Figure 6.15, we have

$$P_1 = H_1 H_2 H_3, L_1 = -H_2 H_3, L_2 = -H_3 H_4, \Delta = 1 - L_1 - L_2, \Delta_1 = 1$$
$$H_{system} = \frac{H_1 H_2 H_3}{1 + H_2 H_3 + H_3 H_4}$$

.

Example 6.7.4. For the block diagram and corresponding signal flow graph from Figure 6.17, the transfer function is

$$H_{system} = \frac{H_1 + H_3}{1 + H_1 H_2}$$

Example 6.7.5. For the block diagram and corresponding signal flow graph from Figure 6.18 the transfer function is

$$H_{system} = \frac{H_3(1 + H_1 H_2) + H_1}{1 + H_1 H_2}$$

Example 6.7.6. For the block diagram and corresponding signal flow graph from Figure 6.19, the transfer function is

$$H_{system} = \frac{H_1 H_4 + H_1 H_2 H_3}{1 - H_1 H_2 H_6 - H_2 H_3 H_5 - H_4 H_5 + H_1 H_2 H_3 + H_1 H_4}$$
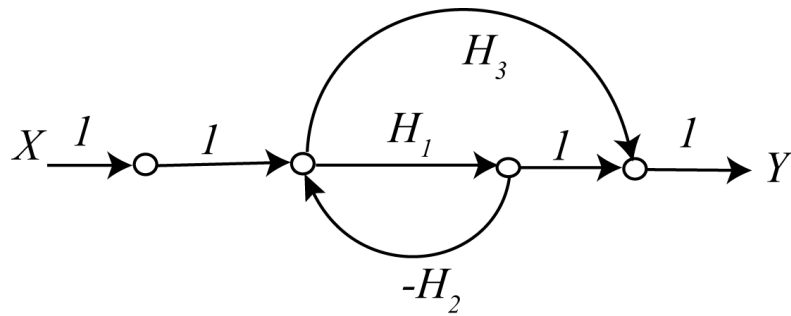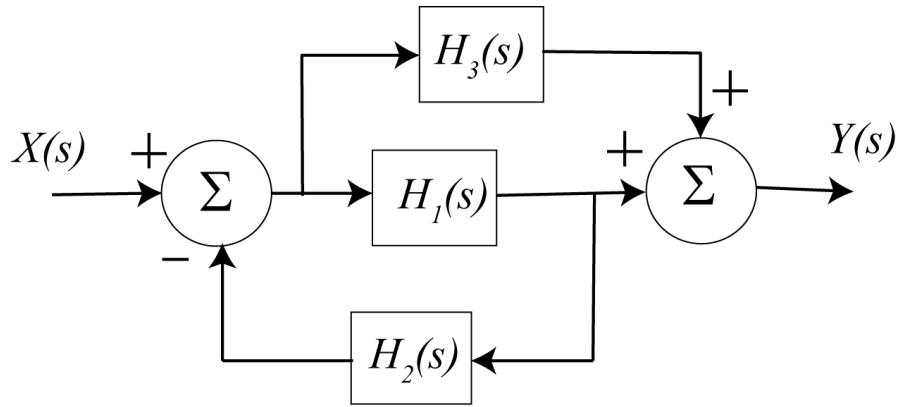
Figure 6.17. Block diagram and signal flow graph for Example 6.7.4.
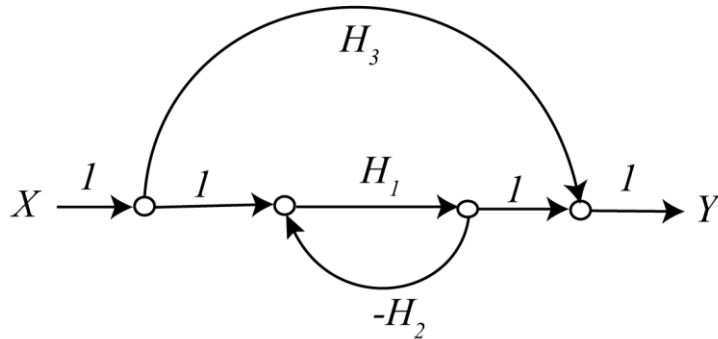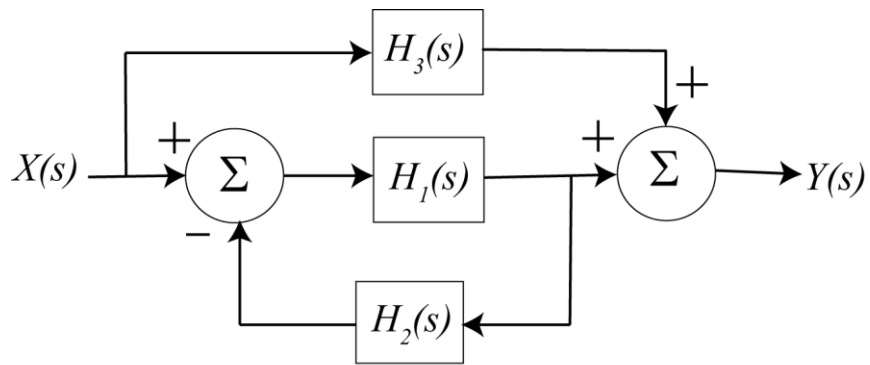
Figure 6.18. Block diagram and signal flow graph for Example 6.7.5.
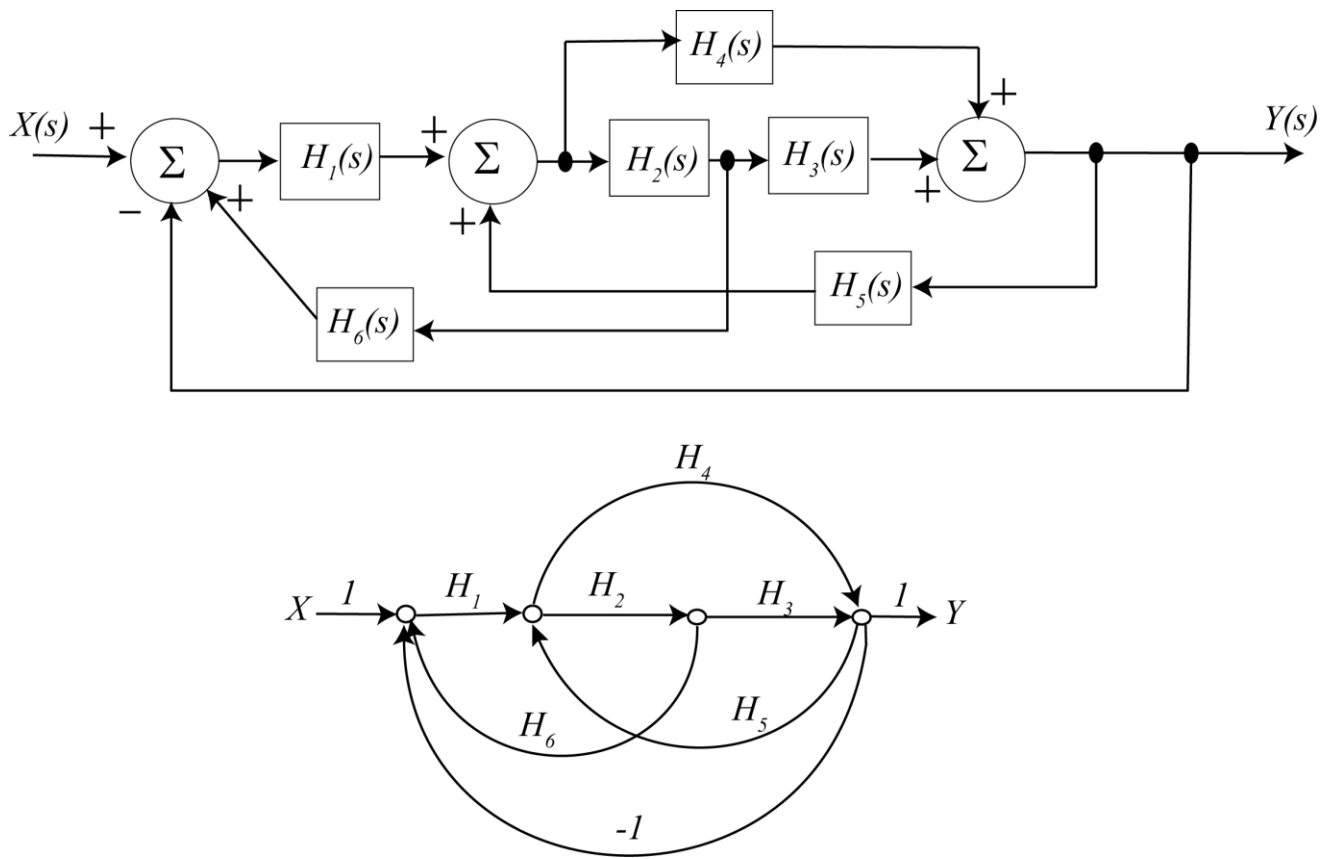
Figure 6.19. Block diagram and signal flow graph for Example 6.7.6.

Chapter 6 Problems

1) In this problem we will derive some useful properties of Laplace transforms starting from the basic relationship

$$X(s) = \int_{0^-}^{\infty} x(t)e^{-st}\,dt$$

a) Let's assume $x(t)$ is a causal signal (it is zero for $t < 0$). We can then write $x(t) = x(t)u(t)$ to emphasize the fact that $x(t)$ is zero before time zero. If there is a delay in the signal and it starts at time t_0, then we can write the signal as $x(t-t_0) = x(t-t_0)u(t-t_0)$ to emphasize the fact that the signal is zero before time t_0.

Using the definition of the Laplace transform and a simple change of variable in the integral, show that $\mathcal{L}\{x(t-t_0)u(t-t_0)\} = X(s)e^{-st_0}$

b) Using the results from part **a**, determine the inverse Laplace transform of $X(s) = \dfrac{e^{-3s}}{(s+2)(s+4)}$

Answer: $x(t) = \dfrac{1}{2}\left[e^{-2(t-3)} - e^{-4(t-3)}\right]u(t-3)$

c) Starting from the definition of the Laplace transform, show that $\mathcal{L}\{tx(t)\} = -\dfrac{dX(s)}{ds}$.

d) Using the result from part c, and the transform pair $x(t) = e^{-at}u(t) \leftrightarrow X(s) = \dfrac{1}{s+a}$, and some simple calculus, show that

$$\mathcal{L}\{te^{-at}u(t)\} = \frac{1}{(s+a)^2}, \ \mathcal{L}\{t^2 e^{-at}u(t)\} = \frac{2}{(s+a)^3}, \ \mathcal{L}\{t^3 e^{-at}u(t)\} = \frac{6}{(s+a)^4}$$

2) For the following circuits, determine the transfer function and the corresponding impulse responses.
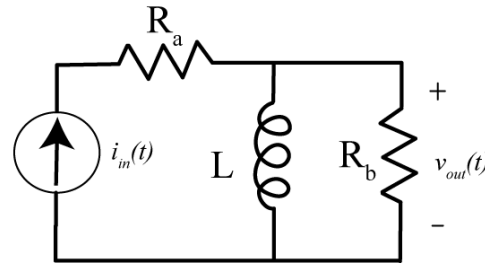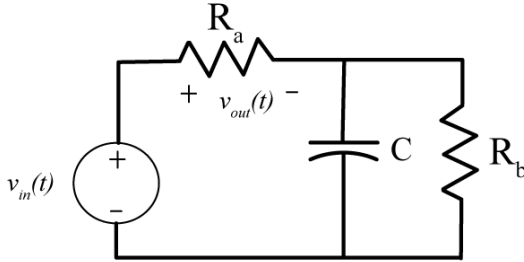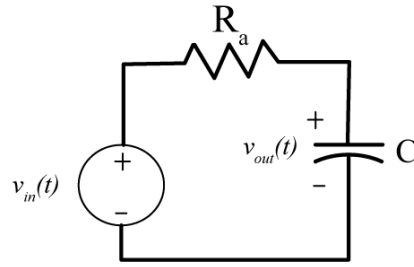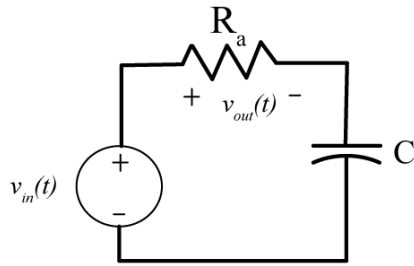
Scrambled Answers:

$$h(t) = R_b \delta(t) - \frac{R_b^2}{L} e^{-tR_b/L} u(t), \; h(t) = \delta(t) - \frac{1}{R_a C} e^{-t/R_a C} u(t), \; h(t) = \frac{1}{R_a C} e^{-t/R_a C} u(t), \; h(t) = [\delta(t) - \frac{1}{R_a C} e^{-t\frac{R_a+R_b}{CR_a R_b}}] u(t)$$

3) For the following impulse responses and inputs, compute the system output using transfer functions.

a) $h(t) = e^{-t} u(t)$, $x(t) = u(t)$ **b)** $h(t) = e^{-2t} u(t)$, $x(t) = \delta(t)$ **c)** $h(t) = e^{-2(t-1)} u(t-1)$, $x(t) = e^{-2t} u(t)$

d) $h(t) = e^{-t} u(t)$, $x(t) = (t-1) u(t-1)$ **e)** $h(t) = e^{-2t} u(t)$, $x(t) = u(t) - u(t-1)$

f) $h(t) = e^{-2(t-1)} u(t-1)$, $x(t) = t e^{-3t} u(t)$

Scrambled Answers :

$$y(t) = \frac{1}{2}\left[1 - e^{-2t}\right] u(t) - \frac{1}{2}\left[1 - e^{-2(t-1)}\right] u(t-1), \; y(t) = \left(1 - e^{-t}\right) u(t), \; y(t) = (t-1) e^{-2(t-1)} u(t-1)$$

$$y(t) = e^{-2t} u(t), \; y(t) = [-1 + (t-1) + e^{-(t-1)}] u(t-1),$$

$$y(t) = [e^{-2(t-1)} - e^{-3(t-1)} - (t-1) e^{-3(t-1)}] u(t-1)$$

4) For the following circuits, determine and expression for the output $V_{out}(s)$ in terms of the ZSR and ZIR. Do not assume the initial conditions are zero. Also determine the system transfer function.
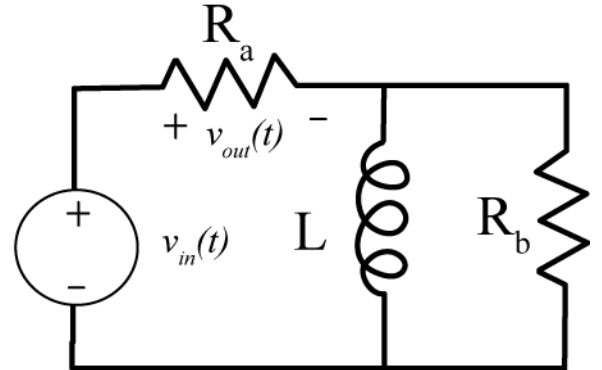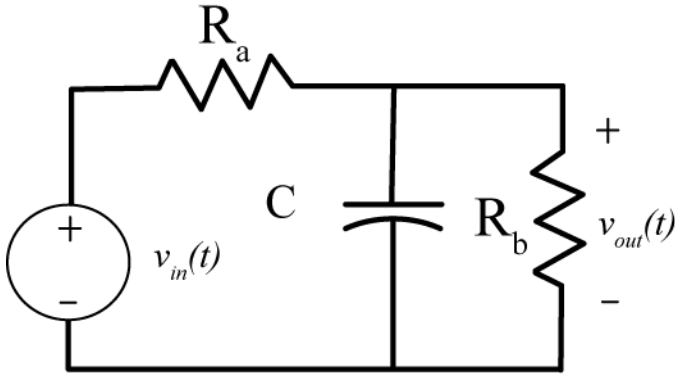
Answers:

$$V_{out}(s) = \left[\frac{R_b}{R_a R_b Cs + R_a + R_b}\right] V_{in}(s) + \left[\frac{R_a R_b C}{R_a R_b Cs + R_a + R_b}\right] v(0^-)$$

$$V_{out}(s) = \left[\frac{R_a(R_b + Ls)}{(R_a + R_b)Ls + R_a R_b}\right] V_{in}(s) + \left[\frac{R_a R_b L}{(R_a + R_b)Ls + R_a R_b}\right] i(0^-)$$

5) For the following transfer functions

$$H(s) = \frac{2}{s^2 + 2s + 2} \quad H(s) = \frac{3}{s^2 + 4s + 6} \quad H(s) = \frac{5}{s^2 + 6s + 10}$$

$$H(s) = \frac{4}{s^2 - 4s + 7} \quad H(s) = \frac{1}{s^2 + 4}$$

By computing the inverse Laplace transform show that the step responses are given by

$$y(t) = \left[1 - e^{-t}\cos(t) - e^{-t}\sin(t)\right]u(t) \quad y(t) = \left[\frac{1}{2} - \frac{1}{\sqrt{2}}e^{-2t}\sin(\sqrt{2}t) - \frac{1}{2}e^{-2t}\cos(\sqrt{2}t)\right]u(t)$$

$$y(t) = \left[\frac{1}{2} - \frac{3}{2}e^{-3t}\sin(t) - \frac{1}{2}e^{-3t}\cos(t)\right]u(t) \quad y(t) = \left[\frac{4}{7} + \frac{8\sqrt{3}}{21}e^{2t}\sin(\sqrt{3}t) - \frac{4}{7}e^{2t}\cos(\sqrt{3}t)\right]u(t)$$

$$y(t) = \left[\frac{1}{4} - \frac{1}{4}\cos(2t)\right]u(t)$$

6) For the following signal flow diagrams determine the system transfer function. *You may use Maple.*
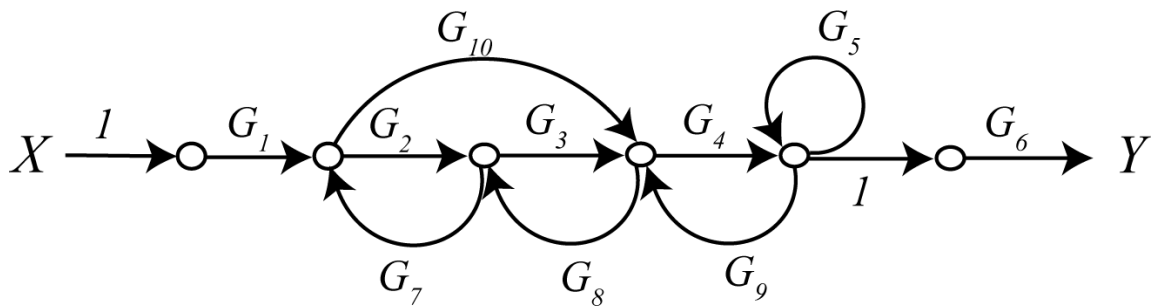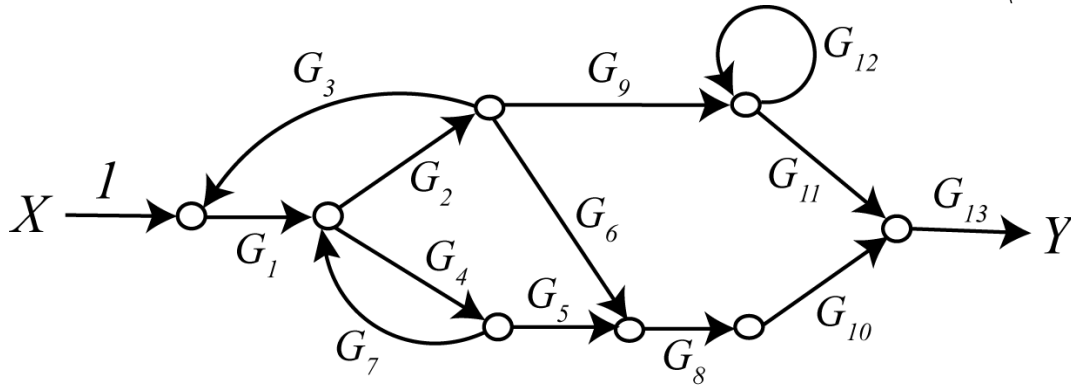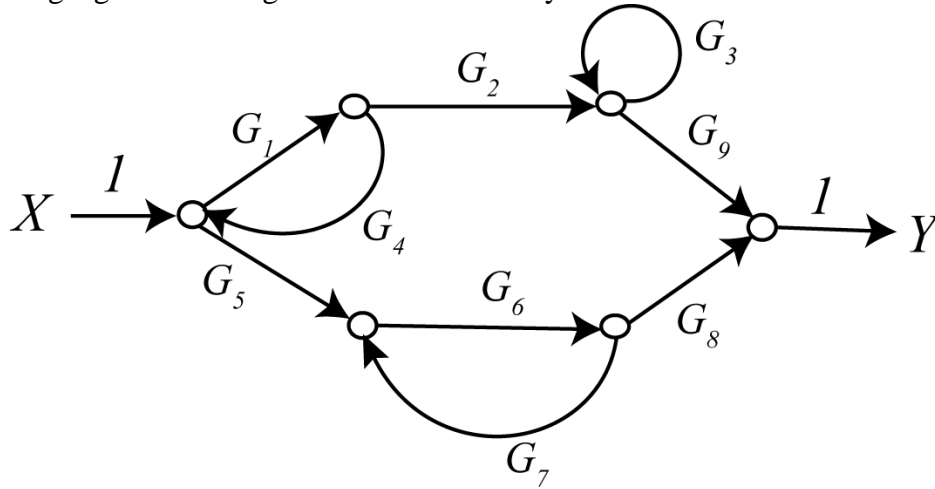
Answers:

$$\frac{Y}{X} = \frac{G_1 G_2 G_9 (1 - G_6 G_7) + G_5 G_6 G_8 (1 - G_3)}{1 - G_1 G_4 - G_6 G_7 - G_3 + G_1 G_4 G_6 G_7 + G_1 G_3 G_4 + G_3 G_6 G_7 - G_1 G_3 G_4 G_6 G_7}$$

$$\frac{Y}{X} = \frac{G_1 G_2 G_3 G_4 G_6 + G_1 G_4 G_6 G_{10}}{1 - G_2 G_7 - G_3 G_8 - G_4 G_9 - G_7 G_8 G_{10} - G_5 + G_2 G_4 G_7 G_9 + G_2 G_5 G_7 + G_3 G_5 G_8 + G_5 G_7 G_8 G_{10}}$$

$$\frac{Y}{X} = \frac{G_1 G_2 G_9 G_{11} G_{13} + G_1 G_2 G_6 G_8 G_{10} G_{13}(1 - G_{12}) + G_1 G_4 G_5 G_8 G_{10} G_{13}(1 - G_{12})}{1 - G_1 G_2 G_3 - G_{12} - G_4 G_7 + G_1 G_2 G_3 G_{12} + G_4 G_7 G_{12}}$$

7) We can also use Mason's rule for systems with multiple inputs and multiple outputs. To do this, we use superposition and assume only one input is non-zero at a time. The only things that changes are the paths (which depends on the input and the output) and the cofactors (which depends on the path). The determinant does not change, since it is intrinsic to the system. For the following systems, determine the transfer functions from all inputs (R) to all outputs (Y). *You may use Maple.*
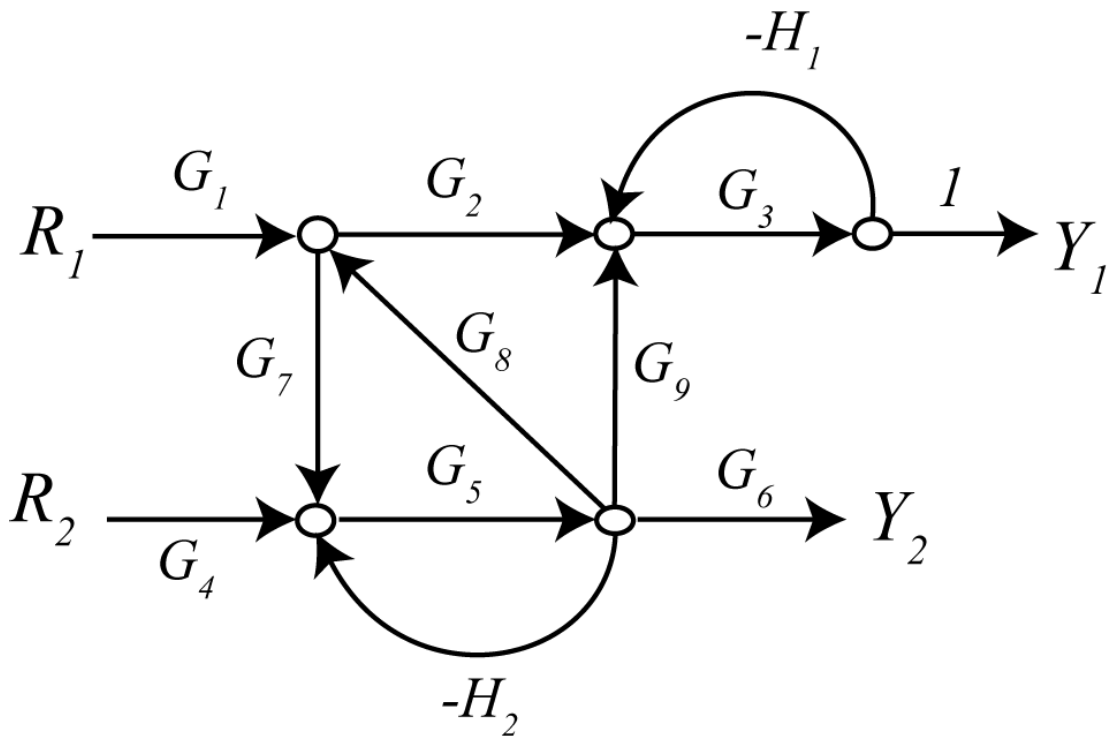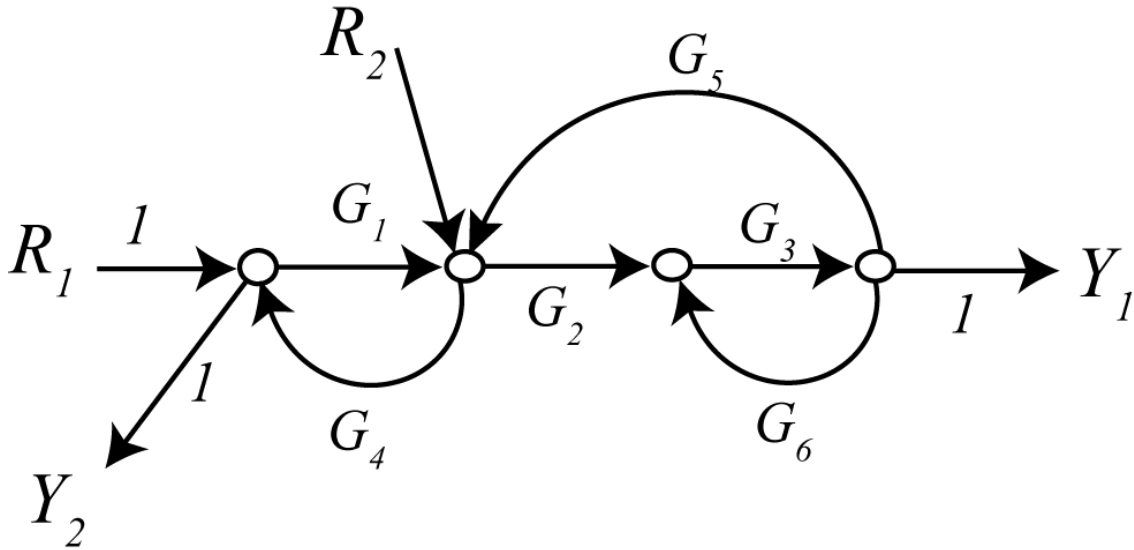
Answers:

$$\frac{Y1}{R1} = \frac{G_1 G_2 G_3}{1 - G_1 G_4 - G_2 G_3 G_5 - G_3 G_6 + G_1 G_3 G_4 G_6},$$

$$\frac{Y2}{R1} = \frac{1 - G_2 G_3 G_5 - G_3 G_6}{1 - G_1 G_4 - G_2 G_3 G_5 - G_3 G_6 + G_1 G_3 G_4 G_6}$$

$$\frac{Y1}{R2} = \frac{G_2 G_3}{1 - G_1 G_4 - G_2 G_3 G_5 - G_3 G_6 + G_1 G_3 G_4 G_6},$$

$$\frac{Y2}{R2} = \frac{G_4 (1 - G_3 G_6)}{1 - G_1 G_4 - G_2 G_3 G_5 - G_3 G_6 + G_1 G_3 G_4 G_6}$$

$$\frac{Y1}{R1} = \frac{G_1 G_2 G_3 (1 + G_5 H_2) + G_1 G_3 G_5 G_7 G_9}{1 + G_5 H_2 - G_5 G_7 G_8 + H_1 G_3 + G_3 G_5 H_1 H_2 - H_1 G_3 G_5 G_7 G_8},$$

$$\frac{Y2}{R1} = \frac{G_1 G_5 G_6 G_7 (1 + G_3 H_1)}{1 + G_5 H_2 - G_5 G_7 G_8 + H_1 G_3 + G_3 G_5 H_1 H_2 - H_1 G_3 G_5 G_7 G_8}$$

$$\frac{Y1}{R2} = \frac{G_3 G_4 G_5 G_9 + G_2 G_3 G_4 G_5 G_8}{1 + G_5 H_2 - G_5 G_7 G_8 + H_1 G_3 + G_3 G_5 H_1 H_2 - H_1 G_3 G_5 G_7 G_8},$$

$$\frac{Y2}{R2} = \frac{G_4 G_5 G_6 (1 + H_1 G_3)}{1 + G_5 H_2 - G_5 G_7 G_8 + H_1 G_3 + G_3 G_5 H_1 H_2 - H_1 G_3 G_5 G_7 G_8}$$

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7.0 Laplace Transform Applications

In this chapter we will examine many applications of Laplace transforms. While it is possible to go back to the time-domain to determine properties of a system, it is often more convenient to be able to determine these properties in the s-domain directly.

7.1 Characteristic Polynomial, Characteristic Modes, and the Impulse Response

Consider a transfer function of the form

$$H(s) = \frac{N(s)}{D(s)}$$

where $N(s)$ and $D(s)$ are polynomials in s with no common factors. $D(s)$ is called the *characteristic polynomial of the system*. The poles of the system are determined from $D(s)$ and these give us most of the information we need to completely characterize the system. The time-domain functions that correspond to the poles of the transfer function are called the *characteristic modes of the system*. To determine the characteristic modes of a system it is often easiest to think of doing a partial fraction expansion and determining the resulting time functions. Finally, the *impulse response* is a linear combination of characteristic modes. A few examples will help.

Example 7.1.1. Consider the transfer function

$$H(s) = \frac{s+2}{s^s(s+1)(s+3)} = a_1 \frac{1}{s} + a_2 \frac{1}{s^2} + a_3 \frac{1}{s+1} + a_4 \frac{1}{s+3}$$

The characteristic polynomial is $D(s) = s^2(s+1)(s+3)$ and the characteristic modes are $u(t), tu(t), e^{-t}u(t)$, and $e^{-3t}u(t)$. The impulse response is a linear combination of these characteristic modes, $h(t) = a_1 u(t) + a_2 tu(t) + a_3 e^{-t}u(t) + a_4 e^{-3t}u(t)$.

Example 7.1.2. Consider the transfer function

$$H(s) = \frac{s-3}{s(s+1)^2(s+3)} = a_1 \frac{1}{s} + a_2 \frac{1}{s+1} + a_3 \frac{1}{(s+1)^2} + a_4 \frac{1}{s+3}$$

The characteristic polynomial is $D(s) = s(s+1)^2(s+3)$ and the characteristic modes are $u(t), e^{-t}u(t), te^{-t}u(t)$, and $e^{-3t}u(t)$. The impulse response is a linear combination of these characteristic modes, $h(t) = a_1 u(t) + a_2 e^{-t}u(t) + a_3 te^{-t}u(t) + a_4 e^{-3t}u(t)$.

Example 7.1.3. Consider the transfer function

$$H(s) = \frac{1}{s^2 + s + 1} = \frac{1}{\left(s + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = a_1 \frac{s + \frac{1}{2}}{\left(s + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} + a_2 \frac{\frac{\sqrt{3}}{2}}{\left(s + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

The characteristic polynomial is $D(s) = s^2 + s + 1$ and the characteristic modes are $e^{-\frac{1}{2}t} \cos\left(\frac{\sqrt{3}}{2}t\right)u(t)$

and $e^{-\frac{1}{2}t} \sin\left(\frac{\sqrt{3}}{2}t\right)u(t)$. The impulse response is the linear combination

$$h(t) = a_1 e^{-\frac{1}{2}t} \cos\left(\frac{\sqrt{3}}{2}t\right)u(t) + a_2 e^{-\frac{1}{2}t} \sin\left(\frac{\sqrt{3}}{2}t\right)u(t).$$

There are a few things to keep in mind when finding the characteristic modes of a system:

- There are as many characteristic modes as there are poles in the transfer function

- For complex conjugate poles of the form $-\sigma \pm j\omega_d$ the characteristic modes will be of the form $e^{-\sigma t}\cos(\omega_d t)u(t)$ and $e^{-\sigma t}\sin(\omega_d t)u(t)$. Note that we could combine these into the form $e^{-\sigma t}\cos(\omega_d t + \theta)u(t)$

Example 7.1.4. If a transfer function has poles at $-1, -1, -2 \pm 3j,$ and $-5 \pm 2j$ the characteristic modes will be $e^{-t}u(t), te^{-t}u(t), e^{-2t}\cos(3t)u(t), e^{-2t}\sin(3t)u(t)$, $e^{-5t}\cos(2t)u(t)$, and $e^{-5t}\sin(2t)u(t)$.

7.2 Asymptotic Stability

We have previously introduced the concept of *Bounded Input Bounded Output*, or BIBO, stability. As we have seen, an LTI system is BIBO stable if

$$\int_{-\infty}^{\infty} |h(\lambda)| \, d\lambda < \infty$$

Another useful definition of stability, which is used often in control systems, is that of *asymptotic stability*. A system is defined to be *asymptotically stable* if all of its characteristic modes go to zero as $t \to \infty$, or equivalently, if $\lim_{t\to\infty} h(t) = 0$. A system is defines to be *asymptotically marginally stable* is all of its characteristic modes are bounded as $t \to \infty$, or equivalently, if $\lim_{t\to\infty} |h(t)| \le M$ for some constant M. If a system is neither asymptotically stable or asymptotically marginally stable, the system is *asymptotically unstable*. In determining asymptotic stability, the following mathematical truths should be remembered:

$$\lim_{t \to \infty} t^n e^{-at} = 0 \qquad \textit{for all } n > 0 \textit{ and } a > 0$$

$$\lim_{t \to \infty} e^{-at} \cos(\omega_d t + \theta) = 0 \qquad \textit{for all } a > 0$$

$$u(t) \qquad \textit{is bounded}$$

$$\cos(\omega_d t + \theta) \qquad \textit{is bounded}$$

$$\sin(\omega_d t + \theta) \qquad \textit{is bounded}$$

Example 7.2.1. Assume a system has poles at -1, 0. and -2. Is the system asymptotically stable? The characteristic modes for this system are $e^{-t}u(t), u(t)$ and $e^{-2t}u(t)$. Both $e^{-t}u(t)$ and $e^{-2t}u(t)$ go to zero as $t \to \infty$. $u(t)$ does not go to zero as $t \to \infty$, but it is bounded. Hence the system is *asymptotically marginally stable*.

Example 7.2.2. Assume a system has poles at -1, 1, and $-2 \pm 3j$. Is the system asymptotically stable? The characteristic modes for this system are $e^{-t}u(t)$, $e^{t}u(t)$, $e^{-2t}\cos(3t)u(t)$, and $e^{-2t}\sin(3t)u(t)$. All of the modes go to zero as $t \to \infty$ except for $e^{t}u(t)$, which goes to infinity. Hence the system is *asymptotically unstable*.

Example 7.2.3. Assume a system has poles at -1, -1, $-2 \pm j$. Is the system asymptotically stable? The characteristic modes for this system are $e^{-t}u(t)$, $te^{-t}u(t)$, $e^{-2t}\cos(t)u(t)$, and $e^{-2t}\sin(t)u(t)$. All of the characteristic modes of the system go to zero as $t \to \infty$, so the system *is asymptotically stable*.

From these examples, if should be clear that *a system will be asymptotically stable if all of the poles of the system are in the left half plane (all of the poles have negative real parts).* This is a very easy test to remember.

7.3 Settling Time and Dominant Poles

Once we think about representing the impulse response as a linear combination of characteristic modes, we can define asymptotic stability in terms of the way these modes behave as $t \to \infty$. Another benefit of this approach is that we can think of the settling time of a system, i.e., the time the system takes to reach 2% of its final value, in terms of the settling time of each of its characteristic modes. When we talk about the settling time of a system, we assume
- the system is asymptotically stable
- the poles of the system are distinct (no repeated poles)
- the input to the system is a step

Let's assume our system has transfer function $H(s)$ with corresponding impulse response
$$h(t) = a_1 \phi_1(t) + a_2 \phi_2(t) + \cdots + a_n \phi_n(t)$$
Here the a_i are the coefficients we determine using the partial fraction expansion, and the $\phi_i(t)$ are the characteristic modes, i.e.,

$$\frac{a_i}{s + p_i} \leftrightarrow a_1 \phi_i(t)$$

Now let's assume the input to our system is a step of amplitude A and we want to use partial fractions to determine the output,

$$Y(s) = H(s)\frac{A}{s} = B\frac{1}{s} + b_1\frac{1}{s+p_1} + b_2\frac{1}{s+p_2} + \cdots + b_n\frac{1}{s+p_n}$$

In the time-domain this will have the form

$$y(t) = Bu(t) + b_1\phi_1(t) + b_2\phi_2(t) + \cdots + b_n\phi_n(t)$$

The primary difference between this and the impulse response of the system is the term $Bu(t)$, which represents the final value of the system due to the step, and a different linear combination of the characteristic modes (the a_i are now b_i, but the $\phi_i(t)$ remain the same.)

Recall that the 2% settling time for an exponential function $\phi_i(t) = e^{-t/\tau_i}u(t)$ is equivalent to four time constants, $T_s = 4\tau$. In order to determine the settling of a system with multiple characteristic modes, we determine the settling time corresponding to each characteristic mode. *The longest settling time determines the settling time of the system.*

Note that this is not an exact formula, since the actual settling time of the system is also a function of the coefficients. However, for most instances this gives a reasonably good first estimate. *The pole that produces the longest settling time is called the dominant pole. If a system has two complex conjugate poles that produce the longest settling time, those poles are the dominant poles.*

Example 7.3.1. Consider a system with transfer function

$$H(s) = \frac{20}{(s+1)(s+5)(s+10)}$$

and corresponding impulse response

$$h(t) = \frac{5}{9}e^{-t}u(t) - e^{-5t}u(t) + \frac{4}{9}e^{-10t}u(t)$$

The unit step response of the system is

$$y(t) = \frac{2}{5}u(t) - \frac{5}{9}e^{-t}u(t) + \frac{1}{5}e^{-5t}u(t) - \frac{2}{45}e^{-10t}u(t)$$

Note that except for the first term, which comes from the input, each of the other time functions is the same as for the impulse response. The time constant for each of the characteristic modes is 1, 0.2, and 0.1 with corresponding settling times of 4, 0.8, and 0.4 seconds. Hence we estimate the settling time of the system to be the largest of these, or $T_s \approx 4$ seconds. Figure 7.1 plots the system output and each of the characteristic modes. As the plot indicates, the settling time is approximately 4 seconds, though this is not exact.
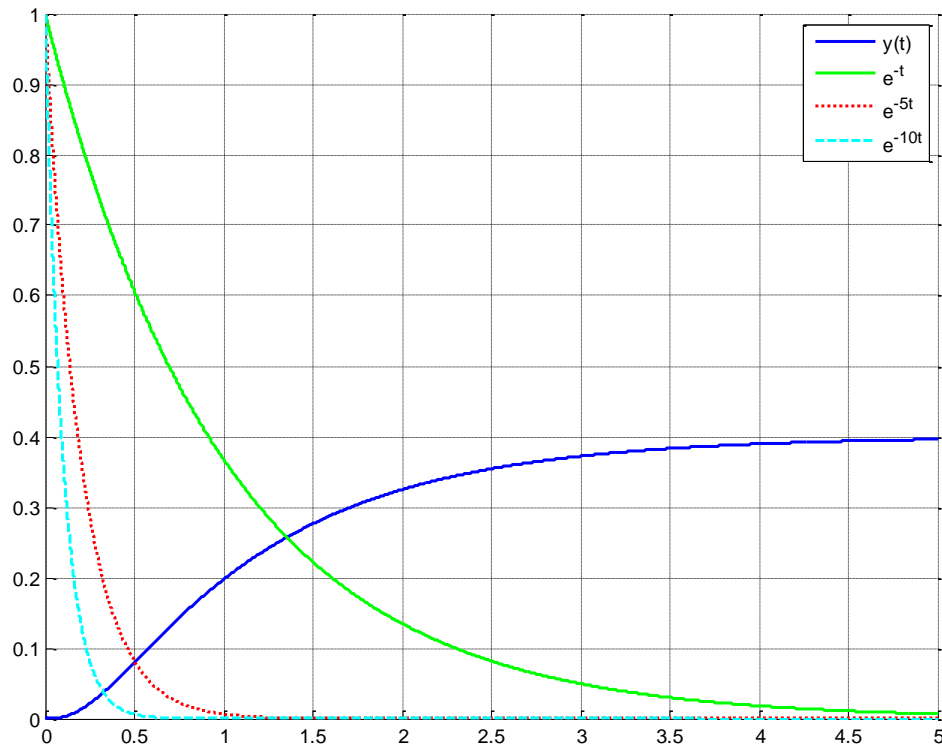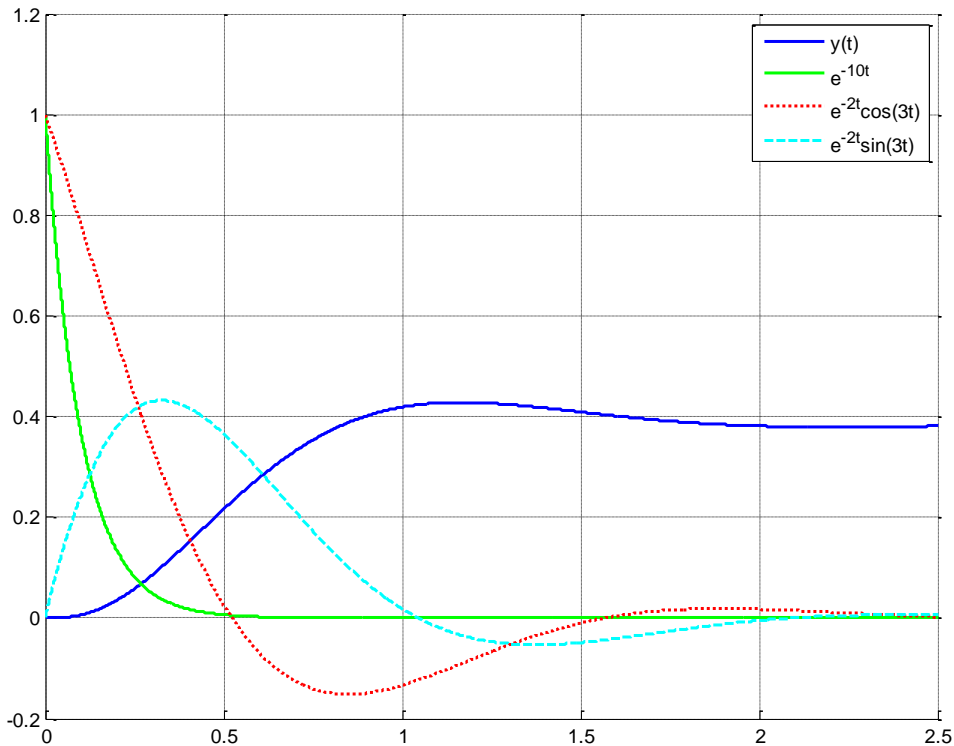
Figure 7.1. Response of system from Example 7.3.1.

Example 7.3.2. Consider a system with transfer function

$$H(s) = \frac{50}{(s+10)\left[(s+2)^2 + 9\right]}$$

and corresponding impulse response

$$h(t) = 0.68493e^{-10t}u(t) - 0.68493e^{-2t}\cos(3t)u(t) + 1.82648e^{-2t}\sin(3t)u(t)$$

The unit step response of the system is given by

$$y(t) = 0.38462u(t) - 0.06849e^{-10t}u(t) - 0.31613e^{-2t}\cos(3t)u(t) - 0.43905e^{-2t}\sin(3t)u(t)$$

The time constant for each of the characteristic modes is 0.1, 0.5, and 0.5 with corresponding settling times of 0.4, 2.0, and 2.0 seconds. Hence we estimate the settling time of the system to be the largest of these, or $T_s \approx 2$ seconds. Figure 7.2 plots the system output and each of the characteristic modes. As the plot indicates, the settling time is approximately 2 seconds, though this is not exact.

Figure 7.2. Response of system from Example 7.3.2.

Example 7.3.3. Consider a system with a transfer function with poles at -2, -4, and -6. Estimate the settling time of the system. We do not have to actually determine the characteristic functions of the impulse response to do this problem. The corresponding time constants will be ½. ¼, and 1/6. The estimated settling times corresponding to each of these time constants is 2, 1, and 0.667 seconds. The largest settling time is 2 seconds, so that is our estimate of the settling time of the system. In this case, the dominant pole is the pole at -2, since it leads to the largest settling time.

Example 7.3.4. Consider a system with a transfer function with poles at $-4, -6$, and $-2 \pm 5j$. Estimate the settling time of the system. The time constants that correspond to these pole locations are ¼, 1/6, and ½ seconds. The settling time associate with each of these time constants is 1, 0.667, and 2 seconds. Hence the estimated settling time is again 2 seconds, and the dominant poles are at $-2 \pm 5j$.

Example 7.3.5. Determine the estimated settling time for a transfer function with poles at $-4 \pm 5j$, $-8 \pm j$, and -10. The corresponding time constants are 0.25, 0.125, and 0.1 seconds. The associated settling times are 1, 0.5, and 0.4 seconds. Hence the estimated settling time is 1 second and the dominant poles are at $-4 \pm 5j$.

It should be obvious by now that the dominant poles are those poles with the real part closest to the $j\omega$ axis.

7.4 Initial and Final Value Theorems

It is often necessary to determine the initial ($t = 0$) or final ($t \rightarrow \infty$) value of a function represented in the *s*-domain. While it is possible to perform partial fractions and determine the time domain representation, this is often tedious and we would like to be able to perform this computation directly in the *s*-domain. Instead we generally use the initial and final value Theorems, which are stated below:

Initial Value Theorem: If $x(t) \leftrightarrow X(s)$ and $X(s)$ is asymptotically stable, then
$$\lim_{t \to 0^+} x(t) = \lim_{s \to \infty} sX(s)$$

Final Value Theorem: If $x(t) \leftrightarrow X(s)$ and $X(s)$ is asymptotically stable, then
$$\lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s)$$
Note that these final value theorems are also valid if there is a single pole at the origin.

Example 7.4.1. Consider the transform pair
$$X(s) = \frac{1}{(s+1)(s+3)} \leftrightarrow x(t) = \frac{1}{2}e^{-t}u(t) - \frac{1}{2}e^{-3t}u(t)$$

Clearly all of the poles are in the left half plane, so we can use the initial value theorem,
$$\lim_{t \to 0^+} x(t) = \frac{1}{2} - \frac{1}{2} = 0$$
$$\lim_{s \to \infty} sX(s) = 0$$

Example 7.4.2. Consider the transform pair
$$X(s) = \frac{s+1}{(s+3)(s+4)} \leftrightarrow x(t) = -2e^{-3t}u(t) + 3e^{-4t}u(t)$$

Clearly all of the poles are in the left half plane, so we can use the initial value theorem,
$$\lim_{t \to 0^+} x(t) = 2 - 3 = 1$$
$$\lim_{s \to \infty} sX(s) = 1$$

Example 7.4.3. Consider the transform pair
$$X(s) = \frac{3s^2 + 16s + 71}{(s+1)\left[(s+3)^2 + 25\right]} \leftrightarrow x(t) = 2e^{-t}u(t) + e^{-3t}\cos(5t)u(t)$$

Clearly all of the poles are in the left half plane, so we can use the initial value theorem,

$$\lim_{t \to 0^+} x(t) = 2 + 1 = 3$$

$$\lim_{s \to \infty} sX(s) = 3$$

Example 7.4.4. Consider the transform pair

$$X(s) = \frac{1}{s} \leftrightarrow x(t) = u(t)$$

Clearly all of the poles are in the left half plane, with the exception of a single pole at the origin, so we can use the final value theorem,

$$\lim_{t \to \infty} x(t) = 1$$

$$\lim_{s \to 0} sX(s) = 1$$

Example 7.4.5. Consider the transform pair

$$X(s) = \frac{1}{s(s+3)} \leftrightarrow x(t) = \frac{1}{3}u(t) - \frac{1}{3}e^{-3t}u(t)$$

Clearly all of the poles are in the left half plane, with the exception of a single pole at the origin, so we can use the final value theorem,

$$\lim_{t \to \infty} x(t) = \frac{1}{3}$$

$$\lim_{s \to 0} sX(s) = \frac{1}{3}$$

Example 7.4.6. Consider the transform pair

$$X(s) = \frac{-2s^2 + 26}{s\left[(s+3)^2 + 4\right]} \leftrightarrow x(t) = 2u(t) - 4e^{-3t}\cos(2t)u(t)$$

Clearly all of the poles are in the left half plane, with the exception of a single pole at the origin, so we can use the final value theorem,

$$\lim_{t \to \infty} x(t) = 2$$

$$\lim_{s \to 0} sX(s) = 2$$

As you will see as we go through this chapter, the initial value theorem is often used to determine the initial amount of "effort" needed for a system, while the final value theorem is used for determining the final value of a system and the steady state error.

7.5 Static Gain

The *static gain* of a system is basically the steady state gain of a system when the input is a step function. Obviously the concept of the static gain of a system only makes sense for systems that are asymptotically stable. Since the static gain of the system is measured during steady state, all of the transients have died out.

The final value theorem is generally used to find the static gain of a system. Assume we have an asymptotically stable system with transfer function $H(s)$ and the input $x(t)$ to our system is a step of amplitude A, $x(t) = Au(t)$. Then we can determine the output of the system as

$$Y(s) = H(s)X(s) = H(s)\frac{A}{s}$$

Using the final value theorem, we can then determine the steady state value of the output as

$$\lim_{t \to \infty} y(t) = \lim_{s \to 0} sY(s) = \lim_{s \to 0} sH(s)\frac{A}{s} = \lim_{s \to 0} H(s)A = H(0)A$$

The gain of the system is the output amplitude divided by the input amplitude, so we can compute the static gain K as

$$K = \frac{Output\ Amplitude}{Input\ Amplitude} = \frac{H(0)A}{A} = H(0)$$

So we have the result (for asymptotically stable systems) that $K = H(0)$ and $\lim_{t \to \infty} y(t) = KA$ for a step input of amplitude A.

Example 7.5.1. Determine the static gain and steady state value of the output for the system

$$H(s) = \frac{s+2}{s^2 + 0.4s + 3}$$

for a step input of amplitude 5. Here $K = \frac{2}{3}$ and $\lim_{t \to \infty} y(t) = KA = \frac{2}{3} \times 5 = \frac{10}{3}$. The response of this system is displayed in Figure 7.3.

Example 7.5.2. Determine the static gain and steady state value of the output for the system

$$H(s) = \frac{2}{s^2 + s + 1}$$

for a step input of amplitude 5. Here $K = 2$ and $\lim_{t \to \infty} y(t) = KA = 2 \times 5 = 10$ The response of this system is displayed in Figure 7.4.
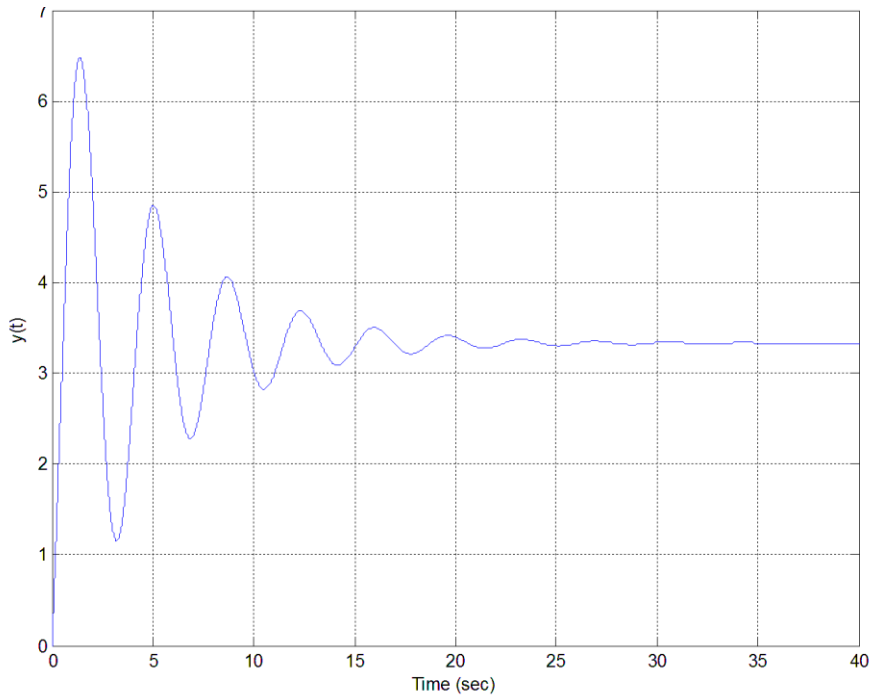
Figure 7.3. Response of system for Example 7.5.1 to a step of amplitude 5. The static gain of this system is 2/3 and the final value of the output is then $10/3 = 3.3$, as the figure shows.
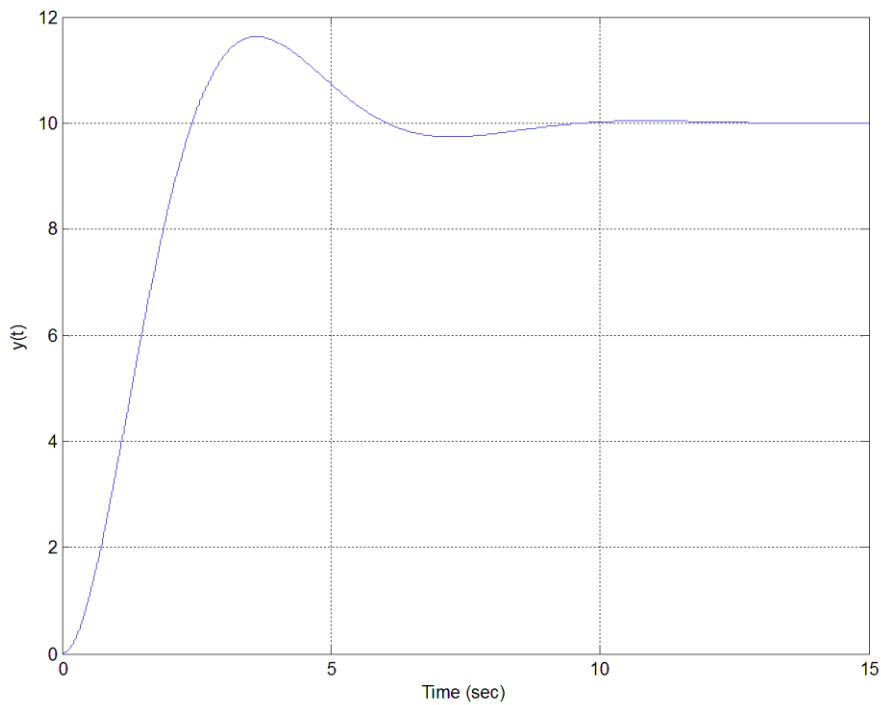
Figure 7.4. Response of system for Example 7.5.2 to a step of amplitude 5. The static gain of this system is 2 and the final value of the output is then 10, as the figure shows.

7.6 Ideal Second Order Systems (Again)

Recall that the differential equation that describes an ideal second order system is of the form

$$\frac{d^2 y(t)}{dt^2} + 2\zeta\omega_n \frac{dy(t)}{dt} + \omega_n^2 y(t) = K\omega_n^2 x(t)$$

where ζ is the damping ratio, ω_n is the natural frequency, and K is the static gain of the system. To determine the transfer function we assume all of the initial conditions are zero and take the Laplace transform of each term,

$$\mathcal{L}\left\{\frac{d^2 y(t)}{dt^2}\right\} = s^2 Y(s)$$

$$\mathcal{L}\left\{2\zeta\omega_n \frac{dy(t)}{dt}\right\} = 2\zeta\omega_n s Y(s)$$

$$\mathcal{L}\left\{\omega_n^2 y(t)\right\} = \omega_n^2 Y(s)$$

$$\mathcal{L}\left\{K\omega_n^2 x(t)\right\} = K\omega_n^2 X(s)$$

Combining these we get,

$$\left[s^2 + 2\zeta\omega_n s + \omega_n^2\right] Y(s) = \left[K\omega_n^2\right] X(s)$$

and transfer function

$$H(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{K}{\frac{1}{\omega_n^2}s^2 + \frac{2\zeta}{\omega_n}s + 1}$$

The characteristic equation for this system is $s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$, and for the under damped case ($0 < \zeta < 1$)the poles of the system are given by

$$s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -\zeta\omega_n \pm j\omega_d = -\sigma \pm j\omega_d$$

Figure 7.5 displays the relationship between these parameters in the complex plane. We have previously determined that the response of the system to a step input of amplitude A is given by

$$y(t) = KA + ce^{-\zeta\omega_n t}\sin(\omega_d t + \phi)$$

where the constants c and ϕ are determined by the initial conditions. What we would like to examine now are how we can specify system pole locations to achieve a desired settling time, percent overshoot, and time to peak.
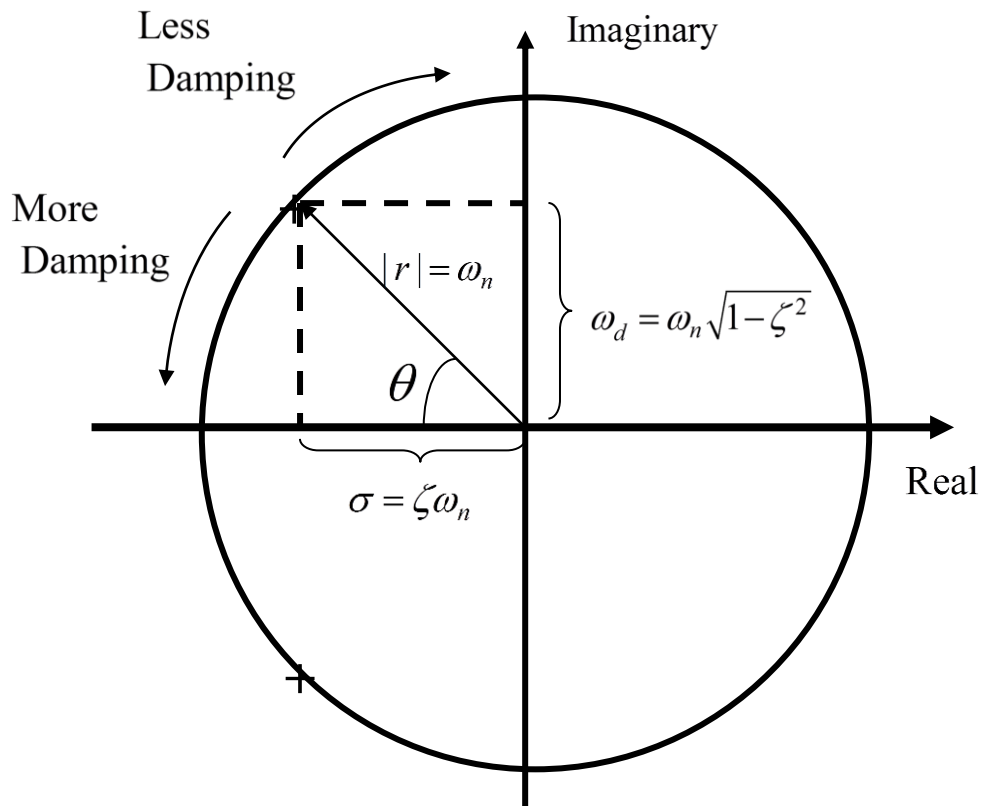
Figure 7.5. Relationship between damping ratio (ζ), natural frequency (ω_n), and damped frequency (ω_d) in the complex plane for an under damped system ($0 < \zeta < 1$). The poles of the system are located at $s = -\sigma \pm j\omega_d$.

Settling Time
We have approximated the settling time for a system with distinct poles to be the time it takes for the slowest characteristic mode to reach four time constants. The time constant for each characteristic mode is the reciprocal of the magnitude of the real part of the pole. For example, for a system with complex poles at $s = -\sigma \pm j\omega$. The corresponding characteristic modes are of the form $e^{-\sigma t}\cos(\omega t)$, $e^{-\sigma t}\sin(\omega t)$, and the time constant is $\tau = 1/\sigma$. Thus if we want our system to have a settling time of T_s^{max}, then we have

$$T_s = 4\tau = \frac{4}{\sigma} < T_s^{max}$$

or

$$\frac{4}{T_s^{max}} < \sigma$$

This means the magnitude of the real part of the pole must be greater than $\frac{4}{T_s^{max}}$, or equivalently, all

poles of the system must be to the left of $-\frac{4}{T_s^{max}}$. *Although this relationship was derived for an ideal*

second order system, it is generally a good initial approximation for any system without repeated poles.

Example 7.6.1. Determine the allowed pole locations in the *s*-plane that correspond to a settling time of less than or equal to 2.5 seconds. Here $T_s^{max} = 2.5$ and

$$\sigma = \frac{4}{T_s^{max}} = \frac{4}{2.5} = 1.6$$

Hence all poles must be to the left of -1.6. The allowable pole locations are shown as the shaded region in Figure 7.6.
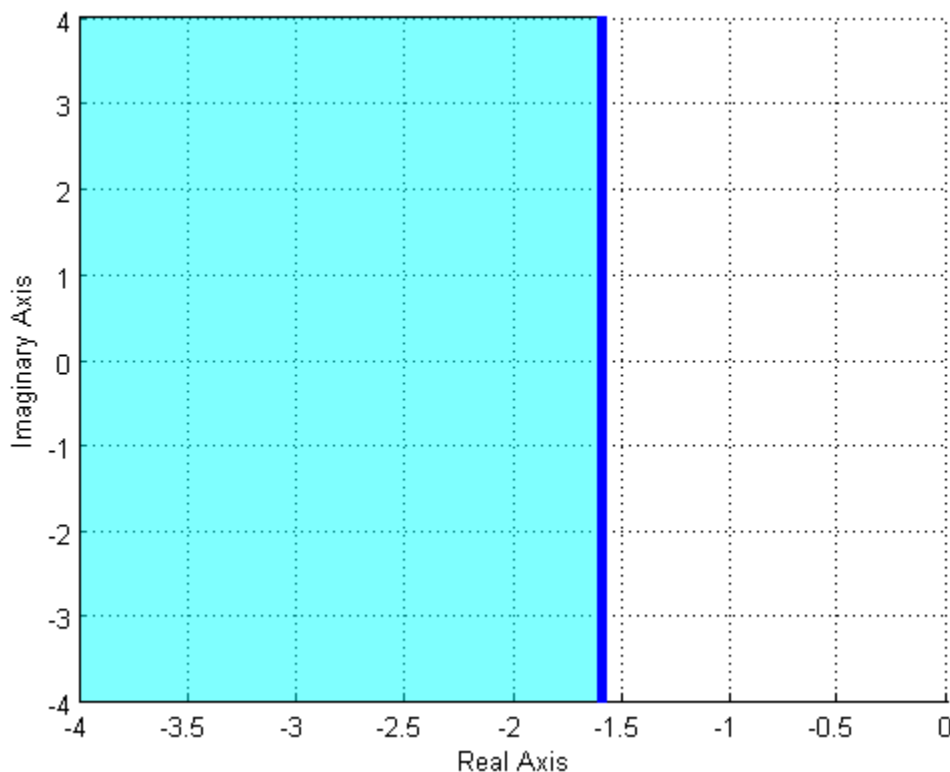
Figure 7.6. The pole locations corresponding to a settling time less that 2.5 seconds are shown in the shaded region, to the left of -1.6 on the real axis. Note that the settling time constraint only affects the real part of the pole, not the imaginary part of the poles.

Example 7.6.2. Determine the allowed pole locations in the *s*-plane that correspond to a settling time of less than or equal to 0.5 seconds. Here $T_s^{max} = 0.5$ and

$$\sigma = \frac{4}{T_s^{max}} = \frac{4}{0.5} = 8$$

Hence all poles must be to the left of -8. The allowable pole locations are shown as the shaded region in Figure 7.7.
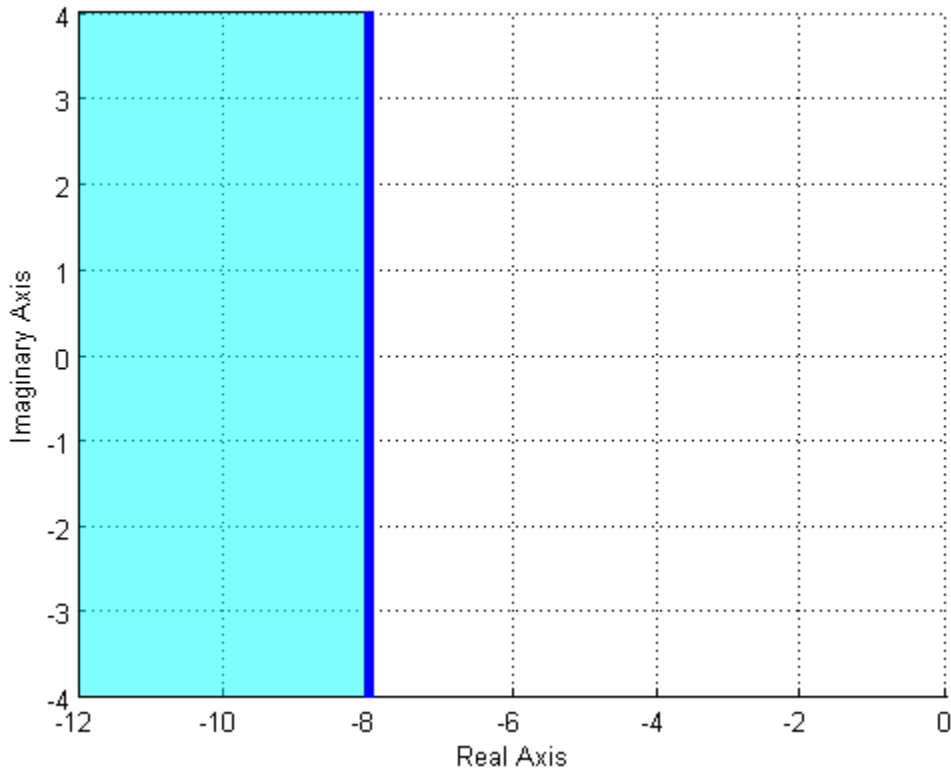
Figure 7.7. The pole locations corresponding to a settling time less that 0.5 seconds are shown in the shaded region, to the left of -8 on the real axis. Note that the settling time constraint only affects the real part of the pole, not the imaginary part of the poles.

Percent Overshoot

From our previous analysis, the percent overshoot (*PO*) is computed as

$$PO = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100\%$$

In order to determine the pole locations in the s-plane that produce an acceptable percent overshoot, we need to do some simple algebra. Let's define

$$O^{max} = \frac{PO^{max}}{100}$$

Here O^{max} is the maximum overshoot, not expressed as a percentage. To set an upper bound on the allowed percent overshoot we have

$$PO \leq PO^{max}$$

or

$$e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \leq \frac{PO^{max}}{100} = O^{max}$$

Next we need to solve for ζ,

$$e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \leq O^{max}$$

$$\frac{-\zeta\pi}{\sqrt{1-\zeta^2}} \leq \ln(O^{max})$$

$$\frac{\zeta}{\sqrt{1-\zeta^2}} \geq \frac{-\ln(O^{max})}{\pi}$$

Squaring both sides and expanding we have

$$\frac{\zeta^2}{1-\zeta^2} \geq \left[\frac{-\ln(O^{max})}{\pi}\right]^2$$

$$\zeta^2 \geq \left[\frac{-\ln(O^{max})}{\pi}\right]^2 - \zeta^2$$

$$\zeta^2 \left\{ 1 + \left[\frac{-\ln(O^{max})}{\pi}\right]^2 \right\} \geq \left[\frac{-\ln(O^{max})}{\pi}\right]^2$$

$$\zeta \geq \frac{\dfrac{-\ln(O^{max})}{\pi}}{\sqrt{1 + \left[\dfrac{-\ln(O^{max})}{\pi}\right]^2}}$$

Finally, we use the relationship depicted in Figure 7.5, $\cos(\theta) = \zeta$, or $\cos^{-1}(\zeta) = \theta$, where the angle θ is measured from the negative real axis. Note that since we usually specify a maximum allowed percent overshoot we have determined the maximum allowed value of the damping ratio, ζ. Since the damping ratio increases (and the damping decreases) as the poles get closer to the real axis, specifying the maximum allowed percent overshoot determines the maximum angle allowed. Thus the allowable pole locations to achieve a maximum percent overshoot will have the shape of a wedge. Finally, note that this relationship is only valid for ideal second order systems.

Example 7.6.3. Determine the allowable pole locations so an ideal second order system will have a maximum percent overshoot of 25%. We have

$$O^{max} = \frac{25}{100} = 0.25$$

and then

$$\zeta \geq \frac{\dfrac{-\ln(0.25)}{\pi}}{\sqrt{1 + \left[\dfrac{-\ln(0.25)}{\pi}\right]^2}} = 0.404$$

$$\cos^{-1}(\zeta) = \cos^{-1}(0.404) = 66.2^o$$

This means the maximum angle with the real axis is 66.2 degrees. The allowable pole locations are shown as the shaded region in Figure 7.8.
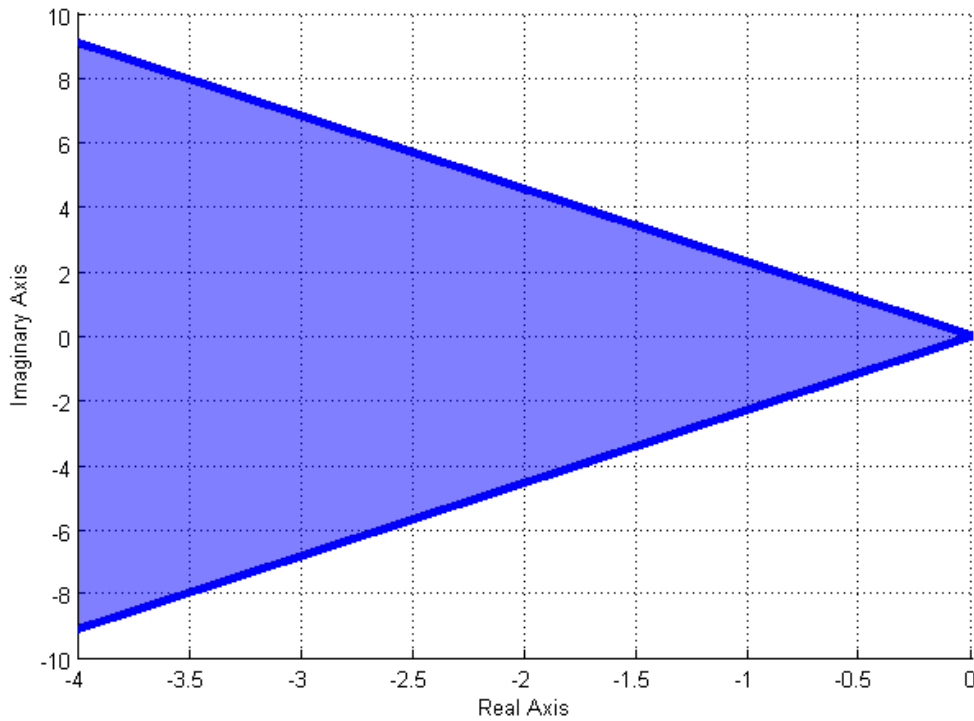
Figure 7.8. The pole locations corresponding to a maximum percent overshoot of 25% are shown in the shaded region, within the wedge.. Note that the percent overshoot affects both the real and imaginary parts of the poles.

Example 7.6.4. Determine the allowable pole locations so an ideal second order system will have a maximum percent overshoot of 10%. We have

$$O^{max} = \frac{10}{100} = 0.10$$

and then

$$\zeta \geq \frac{\dfrac{-\ln(0.10)}{\pi}}{\sqrt{1+\left[\dfrac{-\ln(0.10)}{\pi}\right]^2}} = 0.591$$

$$\cos^{-1}(\zeta) = \cos^{-1}(0.591) = 53.8^o$$

This means the maximum angle with the real axis is 53.8 degrees. The allowable pole locations are shown as the shaded region in Figure 7.9.
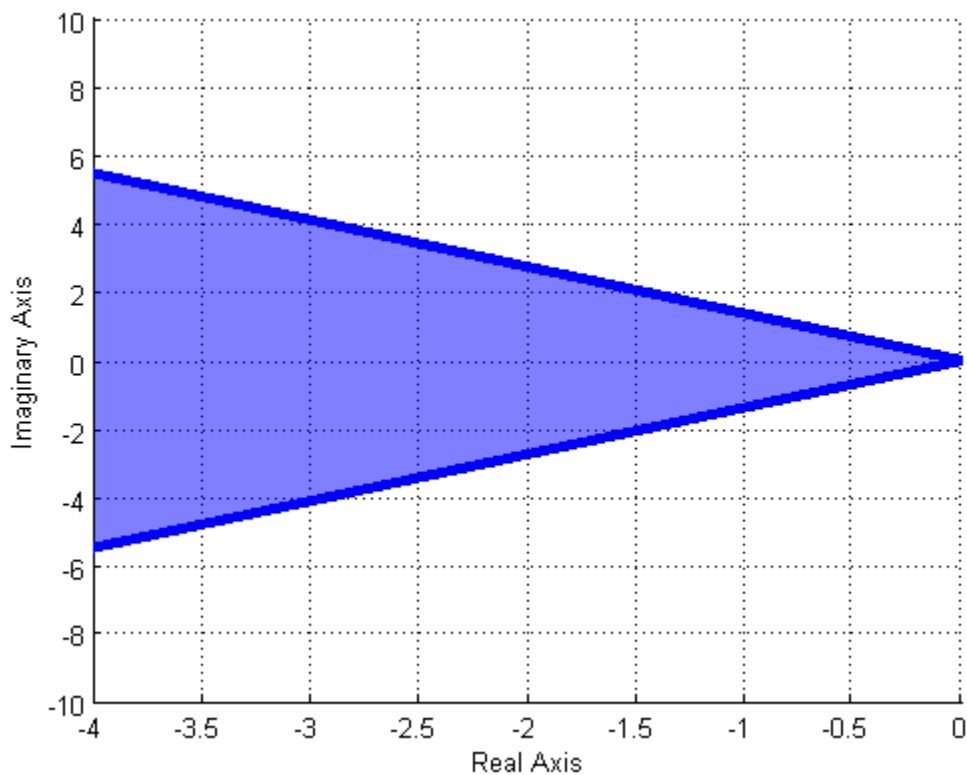
Figure 7.9. The pole locations corresponding to a maximum percent overshoot of 10% are shown in the shaded region, within the wedge. Note that the percent overshoot affects both the real and imaginary parts of the poles. Note also that the angle of this wedge is narrower than that in Figure 7.8 since the allowed percent overshoot in that case is smaller (10% compared to 25% in Figure 7.8).

Example 7.6.5. Determine the allowable pole locations for an ideal second order system so the response to a step will have a percent overshoot less than 5% and a settling time of less than 2 seconds. To solve this problem we need to determine the acceptable regions for each constraint, and then determine if there is any overlapping region so that both constraints will be satisfied. To meet the percent overshoot requirement we have

$$O^{max} = \frac{5}{100} = 0.05$$

$$\zeta \geq \frac{\dfrac{-\ln(0.05)}{\pi}}{\sqrt{1 + \left[\dfrac{-\ln(0.05)}{\pi}\right]^2}} = 0.690$$

$$\cos^{-1}(\zeta) = \cos^{-1}(0.690) = 46.4^o$$

To meet the settling time requirement we have $T_s^{max} = 2.0$ and

$$\sigma = \frac{4}{T_s^{max}} = \frac{4}{2} = 2.0$$

Hence the region of the s-plane that meets both constraints is to the left of -2 and within a wedge with an angle of 46.4 degrees. Each of these individual regions, and the overlapping region are displayed in Figure 7.10.
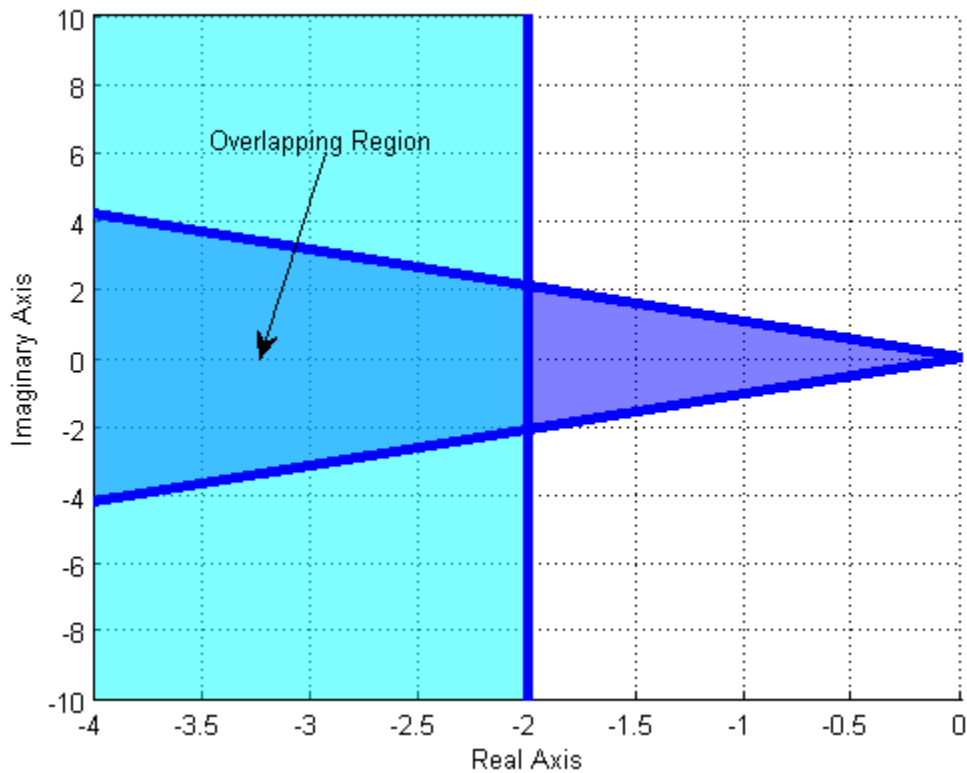
Figure 7.10. The pole locations corresponding to a maximum percent overshoot of 5% and a settling time of 2 seconds are shown in the shaded regions. The overlapping region shows the location in the *s*-plane where both conditions are met.

Time to Peak

From our previous analysis, the time to peak for an under damped ideal second order system is given by

$$T_p = \frac{\pi}{\omega_d}$$

From Figure 7.5 we can easily see that ω_d, the damped frequency, is just the imaginary part of the poles. Hence this constraint will only constrain the imaginary parts of the poles. If we define the maximum allowable time to peak as T_p^{max}, then we have

$$T_p = \frac{\pi}{\omega_d} \leq T_P^{max}$$

or

$$\frac{\pi}{T_p^{max}} \leq \omega_d$$

which indicates the imaginary part of the pole must be larger than π / T_p^{max}.

Example 7.6.6. Determine the pole locations for an ideal second order system that correspond to a time to peak of less than 0.5 seconds for an ideal second order system. We have $T_p^{max} = 0.5$ and then

$$\frac{\pi}{T_p^{max}} = \frac{\pi}{0.5} = 6.283 < \omega_d$$

This means the imaginary part of the poles must be larger than 6.283. The acceptable pole locations are shown as the shaded region in Figure 7.11.

Example 7.6.7. Determine the pole locations for an ideal second order system that correspond to a maximum time to peak of less than or equal to 1.5 seconds and a settling time of less than or equal to 1 seconds. We have $T_p^{max} = 1.5$ and then

$$\frac{\pi}{T_p^{max}} = \frac{\pi}{1.5} = 2.09 < \omega_d$$

To meet the settling time requirement we have $T_s^{max} = 1.0$ and

$$\sigma = \frac{4}{T_s^{max}} = \frac{4}{1} = 4.0$$

Hence to meet both requirements we need the real parts of the poles to the left of -4, and the imaginary parts greater than 2.09 (or less than -2.09). The acceptable pole locations are shown in Figure 7.12.
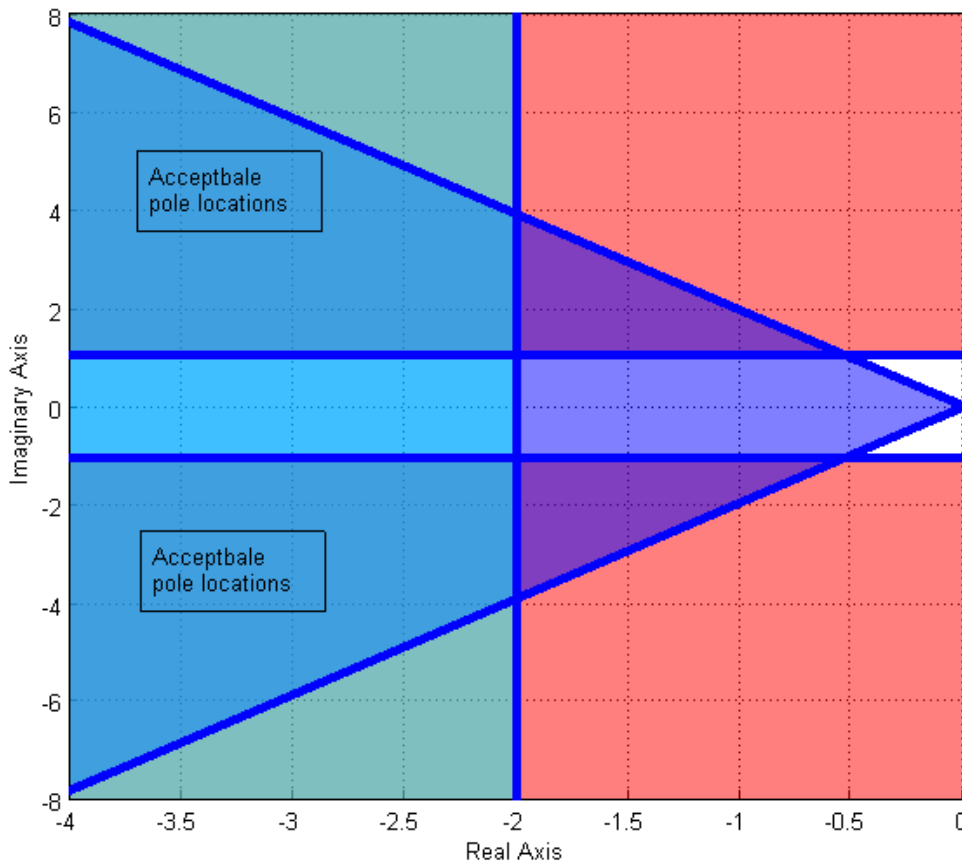
Example 7.6.7. Determine the pole locations for an ideal second order system that corresponds to a time to peak of less than or equal to 3 seconds, a settling time of less than 2 seconds, and a percent overshoot of less than 20%. We have $T_p^{max} = 3.0$
and then

$$\frac{\pi}{T_p^{max}} = \frac{\pi}{3.0} = 1.05 < \omega_d$$

To meet the settling time requirement we have $T_s^{max} = 2.0$ and

$$\sigma = \frac{4}{T_s^{max}} = \frac{4}{2} = 2.0$$

Finally to meet the percent overshoot requirement we have $O^{max} = \frac{20}{100} = 0.2$ and

$$\zeta \geq \frac{\dfrac{-\ln(0.2)}{\pi}}{\sqrt{1 + \left[\dfrac{-\ln(0.2)}{\pi}\right]^2}} = 0.456$$

$$\cos^{-1}(\zeta) = \cos^{-1}(0.456) = 62.9^o$$

Figure 7.13 displays the acceptable regions for each of the three requirements, and the overlapping region where all three requirements are met.
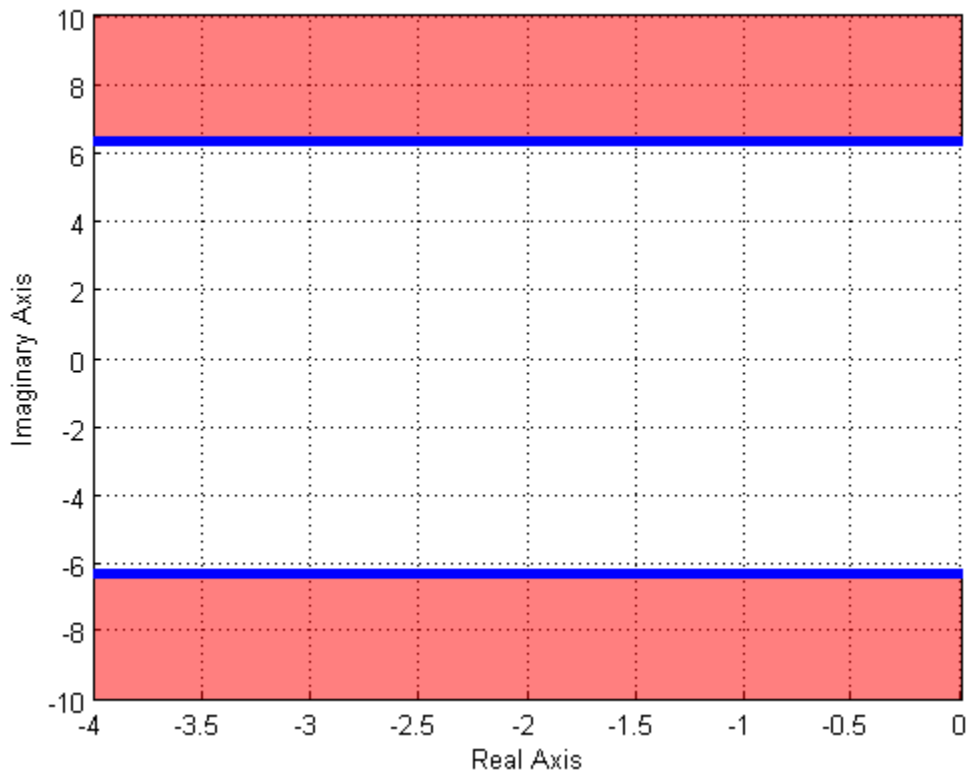
Figure 7.11. The pole locations corresponding to a maximum time to peak of 0.5 seconds. This corresponds to the imaginary part of the poles being larger than 6.28. Note that this constraint affects only the imaginary parts of the pole.
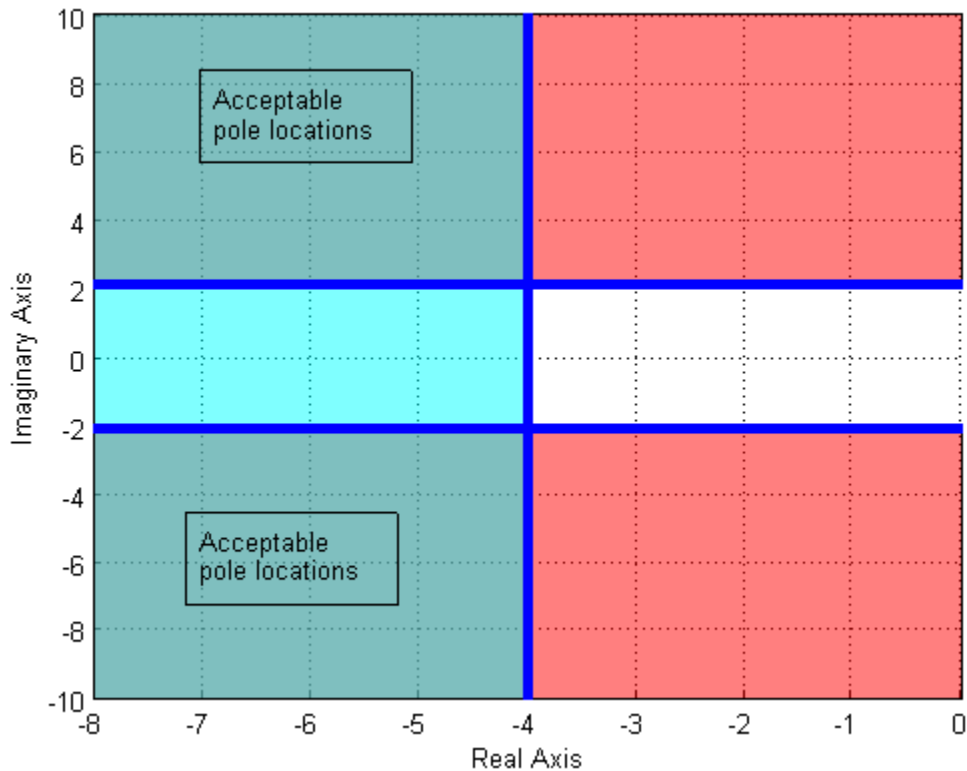
Figure 7.12. The pole locations corresponding to a maximum time to peak of 1.5 seconds and a settling time less than 1 seconds. The settling time constraint means the real part of the poles must be let than -4, and the peak time constraint means the absolute value of the imaginary part of the pole must be greater than 2.09. The pole locations that meet both of these constraints is the overlapping regions, labeled as "Acceptable pole locations".

Figure 7.13. The acceptable pole locations for an ideal second order system that corresponds to a time to peak of less than or equal to 3 seconds, a settling time of less than 2 seconds, and a percent overshoot of less than 20%. The pole locations that meet all of these constraints is the overlapping regions, labeled as "Acceptable pole locations".

7.7 Block Diagrams

We often need to analyze and design interconnected systems. When we introduced convolution for LTI systems, we demonstrated some simple interconnected systems. However, using convolution techniques for these systems is often difficult. Instead we utilize transfer functions relating the output of one system to the input of another system. We then use the fact that the time domain relationship

$$y(t) = h_1(t) \star h_2(t) \star h_3(t) \star x(t)$$

is equivalent to the *s*-domain algebraic relationship

$$Y(s) = H_1(s)H_2(s)H_3(s)X(s)$$

Example 7.7.1. Consider the op-amp circuit shown in Figure 7.14. The input to the circuit is $v_{in}(t)$, the output is $v_{out}(t)$, and $v_m(t)$ is the voltage depicted in the figure as the output of the first op amp.
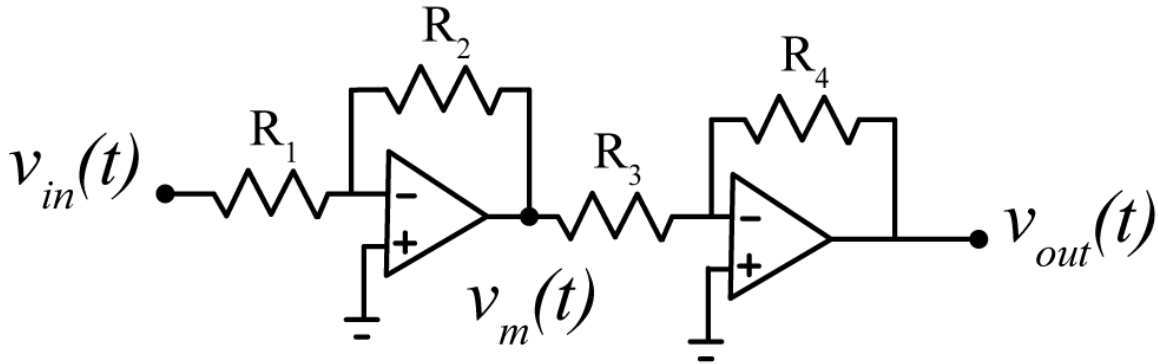
Figure 7.14. Proportional gain circuit for Example 7.7.1.

At the negative terminal of the first op amp we have

$$\frac{V_{in}(s)}{R_1} + \frac{V_m(s)}{R_2} = 0$$

or

$$V_m(s) = -\frac{R_2}{R_1} V_{in}(s)$$

So the transfer function between input $V_{in}(s)$ and output $V_m(s)$ is $H_1(s) = -\frac{R_2}{R_1}$

Similarly we have

$$V_{out}(s) = -\frac{R_4}{R_3} V_m(s)$$

So the transfer function between input $V_m(s)$ and output $V_{out}(s)$ is $H_2(s) = -\frac{R_4}{R_3}$. We can depict these relationships graphically as
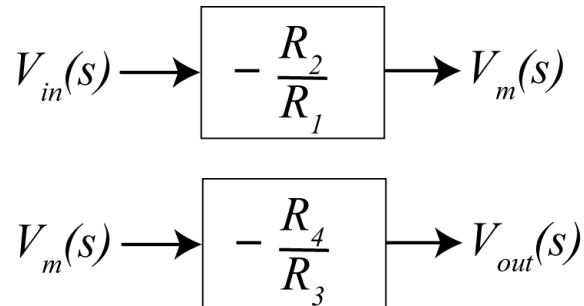
We can then combine these blocks as follows:

$$V_{in}(s) \longrightarrow \boxed{-\frac{R_2}{R_1}} \xrightarrow{V_m(s)} \boxed{-\frac{R_4}{R_3}} \longrightarrow V_{out}(s)$$
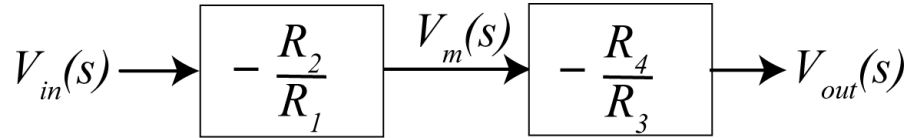
This block diagram indicates graphically that

$$V_m(s) = -\frac{R_2}{R_1}V_{in}(s), \quad V_{out}(s) = -\frac{R_4}{R_3}V_m(s), \quad \text{and} \quad V_{out}(s) = \frac{R_2 R_4}{R_1 R_3}V_{in}(s)$$

Finally, we can write this as

$$V_{out}(s) = k_p V_{in}(s)$$

where k_p is a proportionality constant.

Example 7.7.2. Consider the op-amp system shown in Figure 7.15.
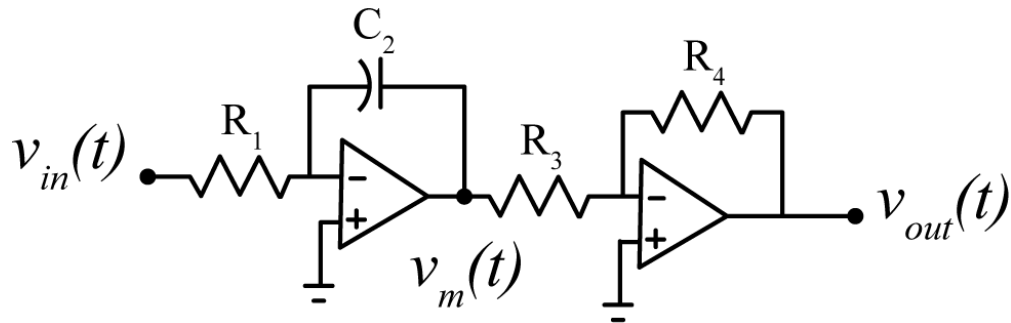
Figure 7.15. Integral and gain op amp circuit for Example 7.7.2.

At the negative node of the first op amp we have

$$\frac{V_{in}(s)}{R_1} + \frac{V_m(s)}{\dfrac{1}{C_2 s}} = 0 , \text{ or } V_m(s) = -\frac{1}{R_1 C_2 s}V_{in}(s)$$

At the negative node of the second op amp we again have

$$V_{out}(s) = -\frac{R_4}{R_3}V_m(s)$$

We can depict these relationships graphically as

$$V_{in}(s) \longrightarrow \boxed{-\dfrac{1}{R_1 C_2 s}} \longrightarrow V_m(s)$$

$$V_m(s) \longrightarrow \boxed{-\dfrac{R_4}{R_3}} \longrightarrow V_{out}(s)$$
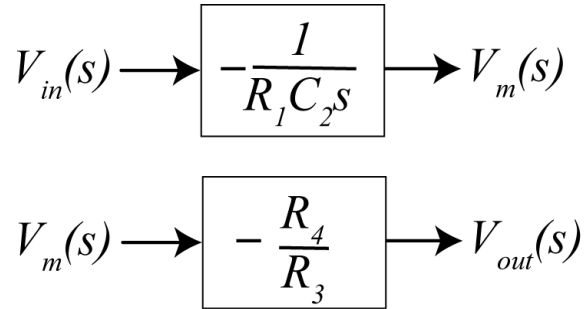
Again, we can combine these as

$$V_{in}(s) \longrightarrow \boxed{-\dfrac{1}{R_1 C_2 s}} \xrightarrow{\ V_m(s)\ } \boxed{-\dfrac{R_4}{R_3}} \longrightarrow V_{out}(s)$$
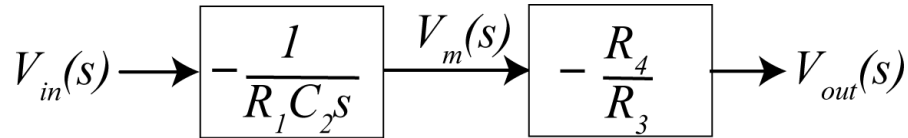
This block diagram indicates graphically that

$$V_{out}(s) = \left(-\frac{1}{R_1 C_2 s}\right)\left(-\frac{R_4}{R_3}\right)V_{in}(s) = \left(\frac{R_4}{R_1 R_3 C_2 s}\right)V_{in}(s) = \frac{k_i}{s}V_{in}(s)$$

Here we have written the proportionality constant as

$$k_i = \frac{R_4}{R_1 R_3 C_2}$$

Example 7.7.3. Consider the *differential amplifier* circuit shown in Figure 7.16. We assume $v_{out}(t) = v_a(t) - v_b(t)$. At the negative input terminal we have

$$\frac{V_f(s) - V^-(s)}{R} + \frac{V_a(s) - V^-(s)}{R_g} = 0$$

and at the positive terminal we have

$$\frac{V_r(s) - V^+(s)}{R} + \frac{V_b(s) - V^+(s)}{R_g} = 0$$

Since under the ideal op amp assumption $V^+(s) = V^-(s)$, we can rearrange these as

$$\frac{V_f(s)}{R} + \frac{V_a(s)}{R_g} = V^-(s)\left[\frac{1}{R} + \frac{1}{R_g}\right] = V^+(s)\left[\frac{1}{R} + \frac{1}{R_g}\right] = \frac{V_r(s)}{R} + \frac{V_b(s)}{R_g}$$

Simplifying this we get

$$V_a(s) - V_b(s) = V_{out}(s) = \frac{R_g}{R}\left[V_r(s) - V_f(s)\right]$$

or

$$V_{out}(s) = k\left[V_r(s) - V_f(s)\right]$$

This relationship is depicted graphically in Figure 7.17 and is commonly used in feedback systems. Note that if $R_g = R$ then $k = 1$.
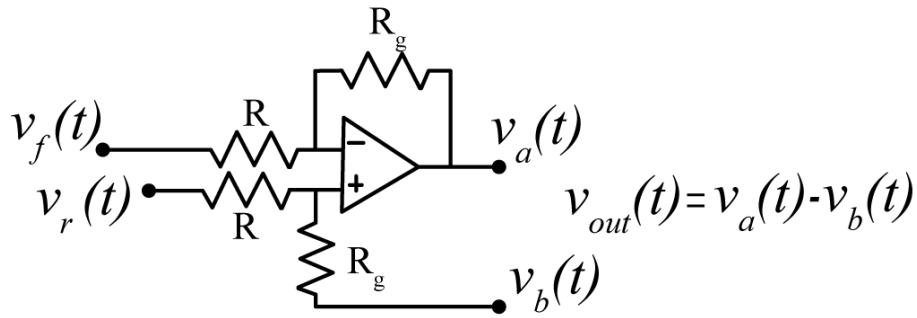
Figure 7.16. Differential amplifier circuit used in Example 7.7.3.
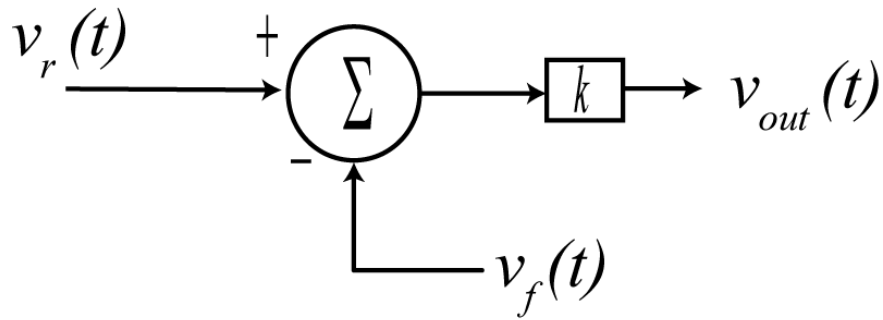
Figure 7.17. Block diagram for the differential amplifier circuit. This configuration is the basis for feedback.

Example 7.7.4. Consider the model of an armature controlled DC motor shown in Figure 7.18. Te armature (the part that does the work) is located on the rotor, while the field (the part that creates the magnetic field) is located on the stator. The field source is constant and hence the strength of the field does not vary, this means the constants K_e and K_t do not vary. The system input is the applied voltage $v_a(t)$. The developed motor torque, $T_m(t)$, is proportional to the current flowing in the loop, $i_a(t)$, so $T_m(t) = K_t i_a(t)$. The motor also develops a "back emf", $e_b(t)$, which is proportional to the speed of the motor, $e_b(t) = K_e \omega(t)$. The motor is used to drive a load with moment of inertia J, damping B, and a load torque $T_l(t)$.
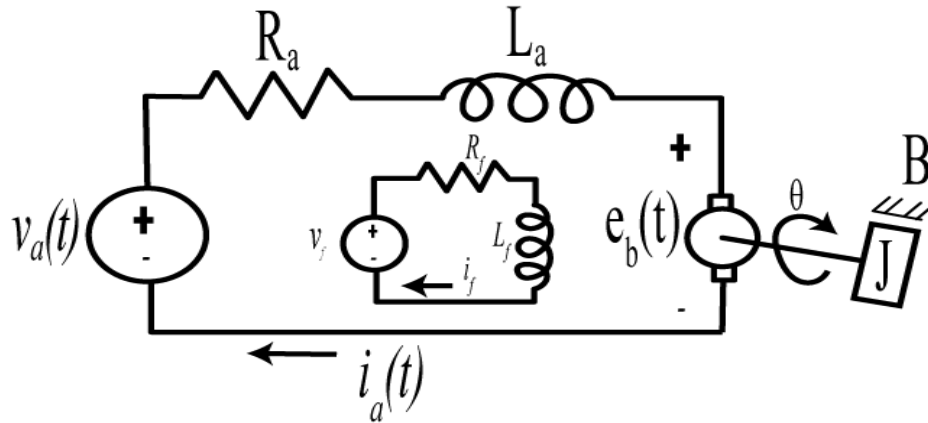
Figure 7.18. Model of an armature controlled DC motor.

To determine the current we have

$$V_a(s) - I_a(s)[R_a + L_a s] = E_b(s)$$

or

$$I_a(s) = \frac{V_a(s) - E_b(s)}{R_a + L_a s}$$

The torque developed by the motor is then

$$T_m(s) = K_t I_a(s)$$

This torque is then used to spin the load and overcome frictional forces and any external applied loads. The free body diagram for the load is shown below.
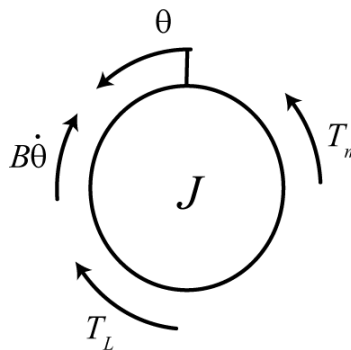
Conservation of angular momentum gives

$$J\ddot{\theta} = T_m - B\dot{\theta} - T_L$$

In the Laplace domain we get

$$s^2 J\Theta(s) + Bs\Theta(s) = T_m(s) - T_L(s)$$

or

$$s\Theta(s) = \Omega(s) = \frac{T_m(s) - T_L(s)}{sJ + B}$$

Finally, we have

$$E_b(s) = K_e \Omega(s)$$

The block diagram for this system is shown in Figure 7.19. Note that this system includes feedback.
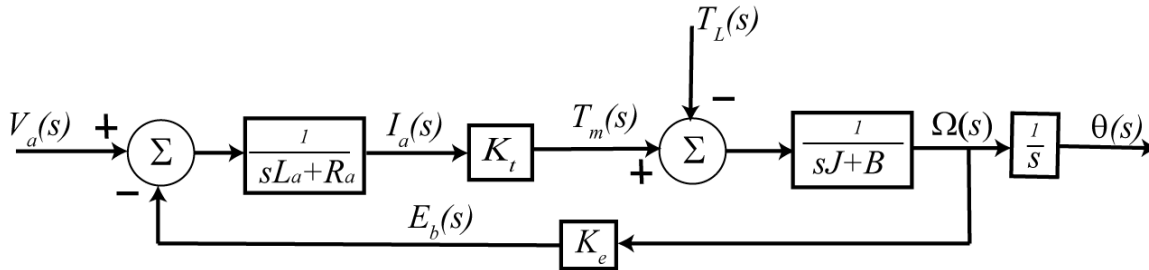
Figure 7.19. Block diagram representation for the armature controlled DC motor from Example 7.7.4.

Although block diagrams and transfer functions are widely used in various engineering disciplines for showing the interconnection of subsystems, one needs to be careful to avoid *loading effects* when representing a system with a block diagram. Consider the simple resistive circuit shown in Figure 7.20 with input $v_{in}(t)$ and output $v_{out}(t)$.
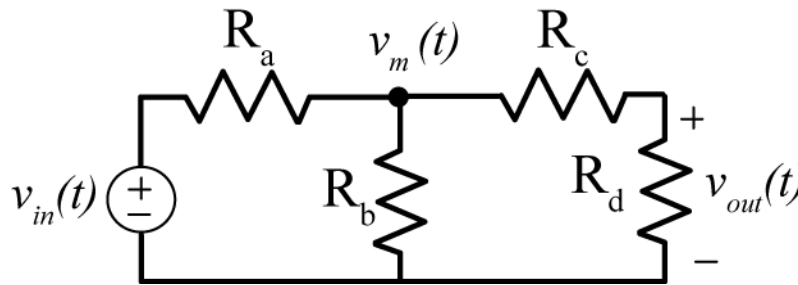
Figure 7.20. Circuit used to demonstrate loading.

At the node with voltage $v_m(t)$ we can write

$$\frac{V_{in}(s) - V_m(s)}{R_a} = \frac{V_m(s)}{R_b} + \frac{V_m(s)}{R_c + R_d}$$

This can be simplified to be

$$V_m(s) = \left[\frac{R_b(R_c + R_d)}{(R_a + R_b)(R_c + R_d) + R_a R_b} \right] V_{in}(s)$$

and finally we get

$$V_{out}(s) = \left[\frac{R_b R_d}{(R_a + R_b)(R_c + R_d) + R_a R_b} \right] V_{in}(s)$$

Now suppose we decided to write the circuit as two interconnected subsystems, as shown in Figure 7.21.
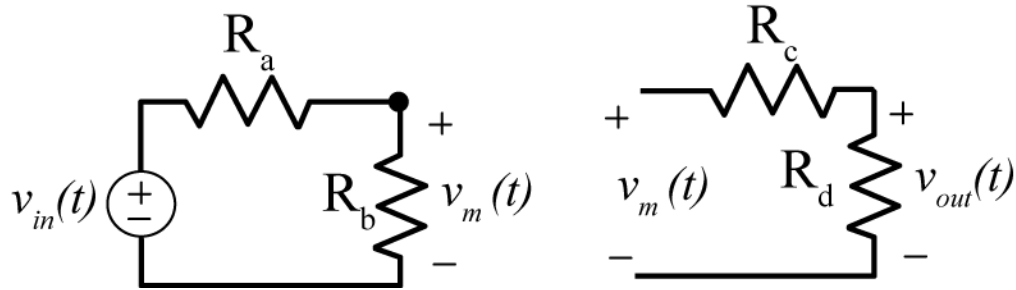
Figure 7.21. Circuit from Figure 7.20 written (incorrectly) as two subsystems.

The transfer function for the first system is clearly

$$\frac{V_m(s)}{V_{in}(s)} = \frac{R_b}{R_a + R_b}$$

and the transfer function for the second subsystem is

$$\frac{V_{out}(s)}{V_m(s)} = \frac{R_d}{R_c + R_d}$$

From this we can determine the system transfer function to be

$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{R_b R_d}{(R_a + R_b)(R_c + R_d)}$$

and the output is

$$V_{out}(s) = \left[\frac{R_b R_d}{(R_a + R_b)(R_c + R_d)}\right]V_{in}(s)$$

This is clearly the wrong answer, so what when wrong? In general, if a transfer function changes after a system is connected to it, the connections is said to have a *loading effect.* For our circuit the relationship between the system input $v_{in}(t)$ and the output $v_m(t)$ changes when the second half of the circuit is added to the system. For electronic system we can often insert an isolating amplifier to remove loading between subsystems. However, loading can occur for non-electrical systems also, so you need to be aware of it.

7.8 Feedback Systems

Assume we have a transfer function, $G_p(s)$, that represents a system, and we want to make this system behave in a certain manner. Typically we call any system we are trying to control a *plant*. For example, assume we have the mass-spring-damper system shown in Figure 7.22. In this system, the input is the voltage applied to a motor (which is modeled as a simple gain) and the motor output is a force applied to the cart. The system output is the displacement of the cart from equilibrium.
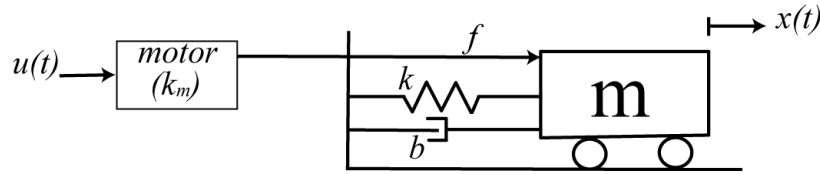
Figure 7.22. Spring-mass-damper system. The motor is modeled as a simple gain.

The input to our plant is the control signal $u(t)$ and the output is the displacement of the cart $x(t)$. Note that in this context $u(t)$ is not necessarily a unit step. It can be any allowable input. However, it is conventional in control systems to label the input to a plant as $u(t)$. A free body diagram of our system is as follows:
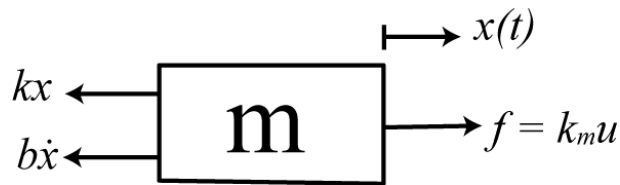
Applying conservation of linear momentum we get the equation of motion,

$$m\ddot{x}(t) = k_m u(t) - kx(t) - b\dot{x}(t)$$

which gives us the transfer function for the plant as

$$G_p(s) = \frac{k_m}{s^2 m + sb + k}$$

There are two general methods for trying to control the behavior of a plant, *open loop control* and *closed loop control*. These two methods are displayed in Figure 7.23. For the open loop control system, we have the system transfer function

$$G_o(s) = \frac{Y(s)}{R(s)} = G_c(s)G_p(s)$$

For the closed loop system we need to do a bit more work. In analyzing a closed loop system we usually look at an intermediate signal that relates input and output and then try to eliminate any intermediate signals. So for this system we have

$$E(s) = R(s) - Y(s)H(s)$$

We can then write the output in terms of the error signal as

$$Y(s) = E(s)G_c(s)G_p(s)$$
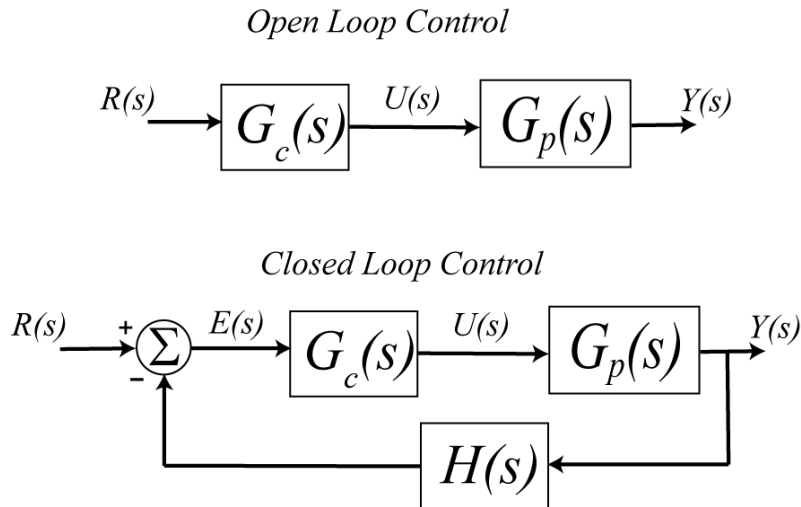
Figure 7.23. Open loop and closed loop control of a plant. Here $r(t)$ is the reference input, $u(t)$ is the control effort or control signal, $y(t)$ is the system output, and $e(t)$ is the error signal. Often the transfer function in the feedback loop, $H(s)$, is some type of transducer which converts the output to the same form at the input.

Finally we need to remove the error signal from these two equations,

$$Y(s) = [R(s) - Y(s)H(s)]G_c(s)G_p(s)$$

Rearranging these we get the closed loop transfer function.

$$G_o(s) = \frac{Y(s)}{R(s)} = \frac{G_c(s)G_p(s)}{1 + H(s)G_c(s)G_p(s)}$$

At this point, it should be obvious to you that using this transfer function to determine properties of the system is much easier than the equivalent time-domain convolution based representation,

$$y(t) \star [\delta(t) + h(t) \star g_c(t) \star g_p(t)] = r(t) \star g_c(t) \star g_p(t)$$

Most control systems are closed loop control systems. As you will see, a closed loop system has the ability to correct for errors in modeling the plant, or if the plant changes over time as components age.

7.9 Steady State Errors

Often we design a control system to track, or follow, the reference input. How well the system tracks the reference input is usually then divided into two parts: the *transient (time-varying) response*, and the *steady state response*. The most common reference input is a step input, and we can use our previously defined measures of settling time, percent overshoot, and rise time to measure how well our system tracks the step input during the transient time. Once the system has reached steady state we often want to use the steady state error as a measure of how well our system tracks the input.

The steady state error, e_{ss}, is usually defined as the difference between the reference input , $r(t)$, and output of the system, $y(t)$, in steady state, or

$$e_{ss} = \lim_{t \to \infty} [r(t) - y(t)]$$

While we can use partial fractions and inverse Laplace transforms to compute this, it is often easier to do this computation in the *s*-domain using the final value Theorem. If we assume the system is asymptotically stable, then we have

$$e_{ss} = \lim_{t \to \infty} [r(t) - y(t)] = \lim_{s \to 0} s[R(s) - Y(s)]$$

If we assume our system has transfer function $G_0(s)$, then we have

$$e_{ss} = \lim_{s \to 0} s[R(s) - Y(s)] = \lim_{s \to 0} s[R(s) - G_0(s)R(s)] = \lim_{s \to 0} sR(s)[1 - G_0(s)]$$

Finally, if we assume our input is a step of amplitude A, $r(t) = Au(t)$, then

$$R(s) = \frac{A}{s}$$

and

$$e_{ss} = \lim_{s \to 0} A[1 - G_0(s)]$$

Clearly for a steady state error of zero, we want $G_0(0) = 1$. Note also that this means we want the *static gain* of the system to be one.

Example 7.9.1. Assume we have the system transfer function

$$G_0(s) = \frac{1}{(s+1)(s+2)}$$

and the input to the system is a step of amplitude 3. Determine the steady state error in the time and *s*-domain. In the time-domain we can use partial fractions,

$$Y(s) = G_0(s)R(s) = \frac{1}{(s+1)(s+2)} \times \frac{3}{s} = \frac{3}{2}\frac{1}{s} - 3\frac{1}{s+1} + \frac{3}{2}\frac{1}{s+2}$$

or

$$y(t) = \frac{3}{2}u(t) - 3e^{-t}u(t) + \frac{3}{2}e^{-2t}u(t)$$

Then

$$e_{ss} = \lim_{t \to \infty}[3u(t) - y(t)] = \lim_{t \to \infty}[3u(t) - \frac{3}{2}u(t) + 3e^{-t}u(t) - \frac{3}{2}e^{-2t}u(t)] = \frac{3}{2}$$

In the *s*-domain we have

$$e_{ss} = \lim_{s \to 0} A[1 - G_0(s)] = 3[1 - G_0(0)] = \frac{3}{2}$$

Example 7.9.2. For the system depicted in Figure 7.14, determine the value of the *prefilter gain* , G_{pf} , so the steady state error for a step is zero. For this system we have the closed loop transfer function

$$G_0(s) = \frac{\dfrac{1}{s+2}\dfrac{3}{s+1}G_{pf}}{1 + 5\dfrac{1}{s+2}\dfrac{3}{s+1}} = \frac{3G_{pf}}{(s+2)(s+1)+15}$$

For zero steady state error we need

$$G_0(0) = 1 = \frac{3G_{pf}}{17}$$

So we need $G_{pf} = 17/3$. Note that in this case we do not really need to simplify the transfer function, we can directly evaluate the transfer function at $s = 0$,

$$G_0(s) = \frac{\dfrac{1}{2}\dfrac{3}{1}G_{pf}}{1 + 5\dfrac{1}{2}\dfrac{3}{1}} = \frac{\dfrac{3}{2}G_{pf}}{1 + \dfrac{15}{2}} = \frac{3G_{pf}}{17}$$

Clearly in this example we want the prefilter gain to be 17/3.
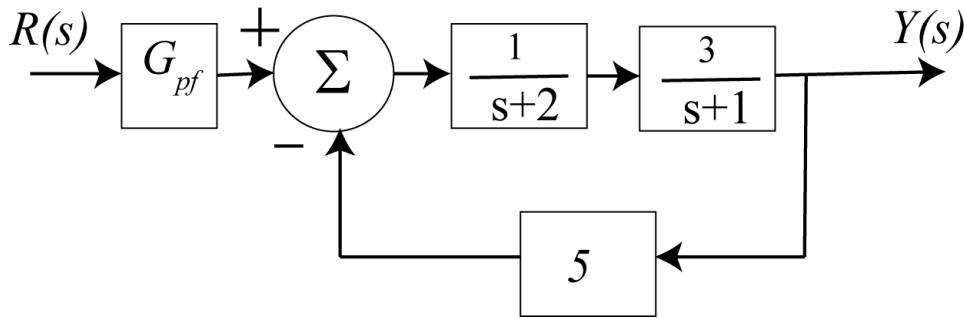
Figure 7.14. Block diagram for Example 7.9.2. For a unit step input, the prefilter should be 17/3 to produce zero steady state error.

Example 7.9.3. For the system depicted in Figure 7.15, determine the value of the prefilter so the steady state error for a step is zero. For this system we have the closed loop transfer function

$$G_0(s) = \frac{\dfrac{1}{s}\dfrac{2}{s^2+2s+1}G_{pf}}{1 + \dfrac{1}{s}\dfrac{2}{s^2+2s+1}}$$

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Note that we cannot immediately set $s = 0$ in this form. We could multiply the transfer function out, but it is easier to just multiply the top and bottom by s,

$$G_0(s) = \frac{\dfrac{2}{s^2 + 2s + 1} G_{pf}}{s + \dfrac{2}{s^2 + 2s + 1}}$$

Now we can set $s = 0$ to get $G_0(0) = G_{pf}$, so for a zero steady state error we need the prefilter to be 1.
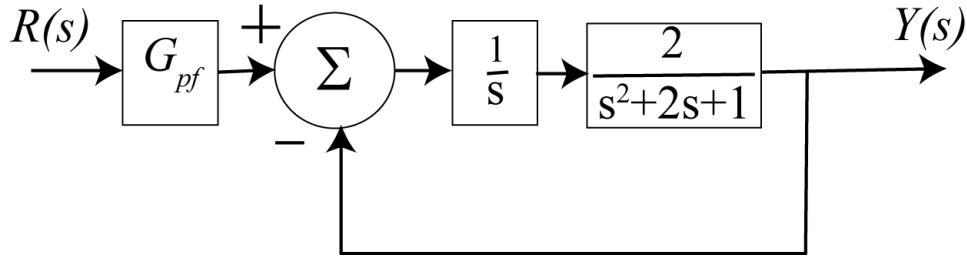

Figure 7.15. Block diagram for Example 7.9.3. For a unit step input, the prefilter should be one to produce zero steady state error.

Example 7.9.4. For the system depicted in Figure 7.16, determine the value of the parameter k so the steady state error for a unit step is less than or equal to 0.1, $e_{ss} \le 0.1$. For this system we have the closed loop transfer function

$$G_0(s) = \frac{k \dfrac{3s + 2}{4s^2 + 5s + 1}}{1 + k \dfrac{3s + 2}{4s^2 + 5s + 1}}$$

and

$$G_0(0) = \frac{2k}{1 + 2k}$$

Then we want

$$e_{ss} = 1 - G_0(0) = 1 - \frac{2k}{1 + 2k} = \frac{1}{1 + 2k} \le \frac{1}{10}$$

or

$$1 + 2k \ge 10$$

This means we need $k \ge 4.5$.
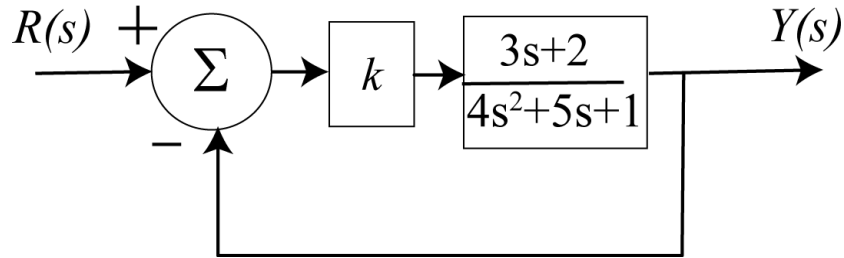
Figure 7.16. Block diagram for Example 7.9.4. For a unit step input, if we want the steady state error less than or equal to 0.1, we need $k > 4.5$.

7.10 Initial Control Effort

Although it is often straightforward to design a control system to produce a given steady state error or an acceptable transient response, sometimes these controllers require a control effort that is not possible to produce. In many, though not all, instances, the initial control effort is the largest control effort when the input is a step. In order to quickly determine the initial control effort we use the initial value Theorem. Recall that the initial value Theorem stated that if $x(t) \leftrightarrow X(s)$ and $X(s)$ is asymptotically stable, then

$$\lim_{t \to 0^+} x(t) = \lim_{s \to \infty} sX(s)$$

For our standard closed loop system in Figure 7.17, the control effort is denoted by $U(s)$. We can solve for this as follows:

$$E(s) = R(s) - H(s)Y(s)$$
$$Y(s) = E(s)G_c(s)G_p(s)$$

and

$$U(s) = E(s)G_c(s)$$

Hence

$$Y(s) = U(s)G_p(s)$$
$$E(s) = \frac{U(s)}{G_c(s)}$$

Combining these we have

$$\frac{U(s)}{G_c(s)} = R(s) - H(s)G_p(s)U(s)$$
$$U(s) = R(s)G_c(s) - H(s)G_c(s)G_p(s)U(s)$$

This yields the following expression for the control effort

$$U(s) = \frac{G_c(s)R(s)}{1 + H(s)G_c(s)G_p(s)}$$

Finally, to determine the initial control effort, we have

$$u(0^+) = \lim_{s \to \infty} sU(s) = \lim_{s \to \infty} \frac{sG_c(s)R(s)}{1 + H(s)G_c(s)G_p(s)}$$

If we assume the input is a step of amplitude A then we have

$$u(0^+) = \lim_{s \to \infty} \frac{G_c(s)A}{1 + H(s)G_c(s)G_p(s)}$$

As an example, let's consider a plant with the transfer function

$$G_p(s) = \frac{5}{s^2 + 2s + 2}$$

We will assume the closed loop control configuration shown in Figure 7.17, and look at the results using three different controllers. We will assume our reference input is a unit step for all three examples.
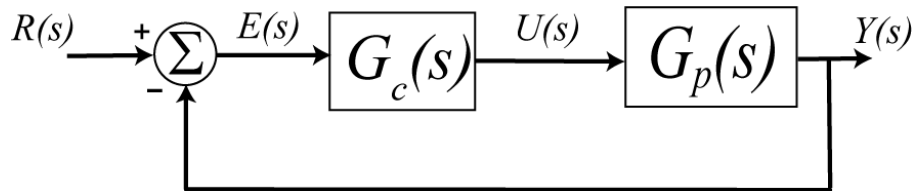
Figure 7.17. Block diagram used for controlling the plant $G_p(s) = \dfrac{5}{s^2 + 2s + 2}$.

In the first case, we will assume we have a *proportional (P) controller,* where the control effort is proportional to the error signal. Here we will have

$$G_c(s) = k_p, \, k_p = 20$$

where we have assigned $k_p = 20$. The steady state error for this system can easily be determined to be $e_{ss} = 0.02$. The closed loop poles are at $-1 \pm j10.05$ which gives an approximate settling time of 4 seconds.

In the second case, we will assume we have a *proportional plus integral (PI) controller.* Here the control effort is made up of two components, one is proportional to the error signal, and one is proportional to the integral of the error signal. Here we will have

$$G_c(s) = k_p + \frac{k_i}{s}, \, k_p = 0.04, k_i = 0.2$$

where we have assigned $k_p = 0.04$ and $k_i = 0.2$. The steady state error for this system is zero, $e_{ss} = 0$. The closed loop poles are at (approximately) $-0.60 \pm j0.94$ and -0.81, which gives an approximate settling time of 6.7 seconds.

In the third case, we will assume we have a *proportional plus derivative (PD) controller*. Here the control effort is again made up of two components, one is again proportional to the error signal and the other is proportional to the derivative of the error signal. This controller will have the form

$$G_c(s) = k_p + k_d s, \; k_p = 8, k_d = 0.4$$

where we have assumed $k_p = 8$ and $k_d = 0.4$. The steady state error for this system is approximately 0.05. The closed loop poles are at (approximately) $-2.0 \pm j6.16$ which gives an approximate settling time of 2 seconds.

The response of the plant to each of these controllers is shown in Figure 7.18.

Next let's look at the initial control effort for each of these controllers. For the proportional controller we have

$$u(0^+) = \lim_{s \to \infty} \frac{k_p}{1 + (1)(k_p)\left(\dfrac{5}{s^2 + 2s + 1}\right)} = k_p = 20$$

For the proportional plus integral controller we have

$$u(0^+) = \lim_{s \to \infty} \frac{k_p + \dfrac{k_i}{s}}{1 + (1)\left(k_p + \dfrac{k_i}{s}\right)\left(\dfrac{5}{s^2 + 2s + 1}\right)} = k_p = 0.04$$

Finally, for the proportional plus derivative controller we have

$$u(0^+) = \lim_{s \to \infty} \frac{k_p + k_d s}{1 + (1)(k_p + k_d s)\left(\dfrac{5}{s^2 + 2s + 1}\right)} = \infty$$

For these three controllers, the PI controller requires the least initial control effort. While the control effort for the P controller is finite, it may be more difficult to implement this controller using op amps with fixed voltage sources. Finally, the initial control effort is infinite for the PD controller. However, often this means the source will just saturate and not reach infinity. However, it is something you will need to be aware of.
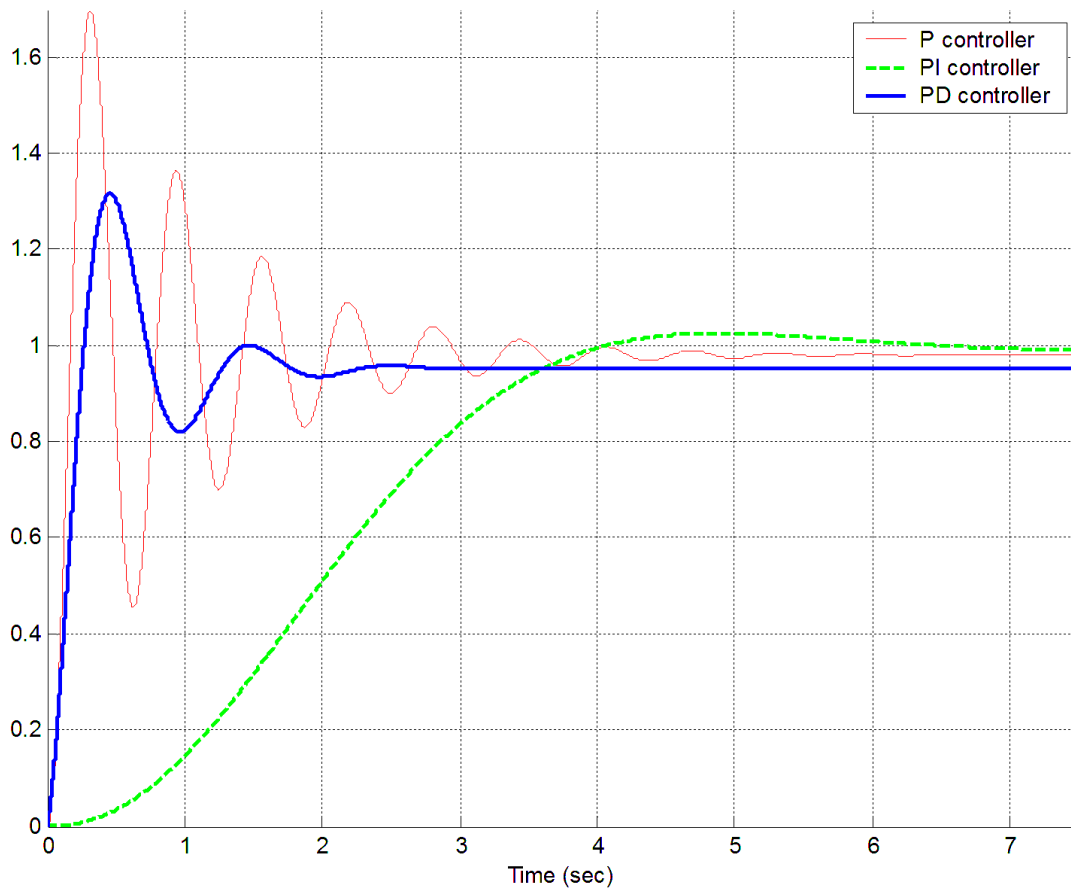
Figure 7.18. The response of the plant $G_p(s) = \dfrac{5}{s^2 + 2s + 2}$ to the proportional (P) controller $G_c(s) = 20$,

the proportional plus integral (PI) controller $G_c(s) = 0.04 + \dfrac{0.2}{s}$, and the proportional plus derivative

controller $G_c(s) = 8 + 0.4s$. The input was a unit step.

Chapter 7 Problems

1) For the following transfer functions, determine
- the characteristic polynomial
- the characteristic modes
- if the system is (asymptotically) stable, unstable, or marginally stable
-

a) $H(s) = \dfrac{s-1}{s(s+2)(s+10)}$

b) $H(s) = \dfrac{s(s-1)}{(s+1)^2(s^2+s+1)}$

c) $H(s) = \dfrac{1}{s^2(s+1)}$

d) $H(s) = \dfrac{s^2-1}{(s-1)(s+2)(s^2+1)}$

e) $H(s) = \dfrac{1}{(s^2+2)(s+1)}$

Partial Answer: 1 stable, 2 unstable, 2 marginally stable

2) For a system with the following pole locations, estimate the settling time and determine the dominant poles

a) -1,-2,-4,-5

b) -4, -6, -7, -8

c) -1+j, -1-j, -2, -3

d) -3-2j, -3+2j, -4+j, -4-j

Scrambled Answers: 4/3, 4, 4, 1

3) Determine the static gain for the systems represented by the following transfer functions, and then the steady state output for an input step of amplitude 3:

$$H(s) = \frac{s+2}{s^2+s+1}, \quad H(s) = \frac{1}{s^2+4s+4}, \quad H(s) = \frac{s-4}{s^2+s+1}$$

Answers: 2, 0.25, -4, 6, 0.75, -12 (this should be very easy)

4) Consider the following simple feedback control block diagram. The plant, the thing we want to control, has the transfer function $G_p(s) = \dfrac{3}{s+2}$ and the controller is a proportional controller, so $G_c(s) = k_p$.
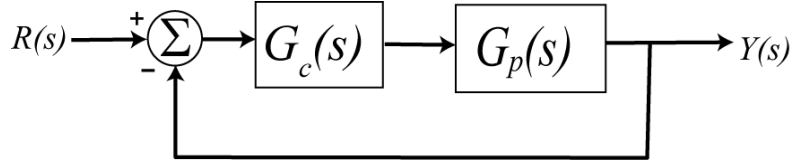

a) Determine the settling time of the plant alone (assuming there is no feedback)

b) Determine the closed loop transfer function, $G_0(s)$

c) Determine the value of k_p so the settling time of the system is 0.5 seconds.

d) If the input to the system is a unit step, determine the output of the system.

e) The steady state error is the difference between the input and the output as $t \to \infty$. Determine the steady state error for this system.

Partial Answer: $y(t) = \dfrac{3}{4}\left[1 - e^{-8t}\right] u(t), \quad e_{ss} = 0.25$

5) Show that the following circuit can be used to implement the PI controller

$$G_c(s) = \frac{U(s)}{E(s)} = k_p + k_i \frac{1}{s} = \frac{R_4 R_2}{R_3 R_1} + \frac{R_4}{R_3 R_1 C_2}\frac{1}{s}$$
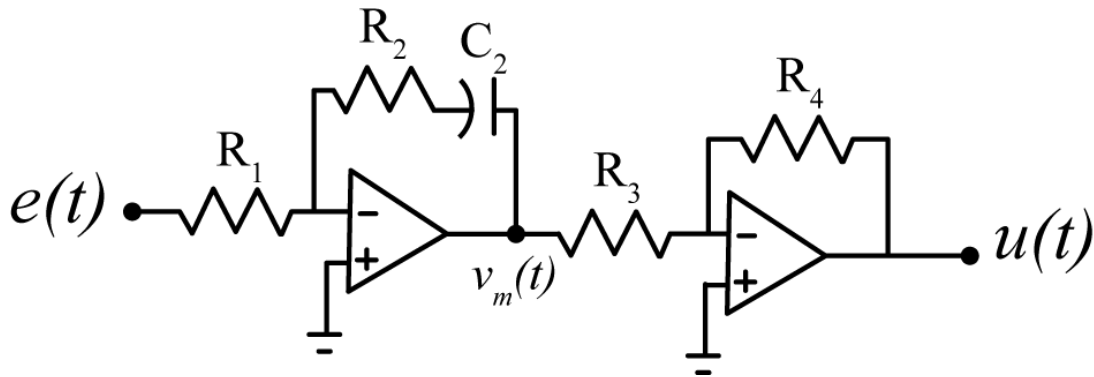
6) Consider the following simple feedback control block diagram. The plant, the thing we want to control, has the transfer function $G_p(s) = \dfrac{3}{s+2}$ and the controller is an integral controller, so $G_c(s) = \dfrac{k_i}{s}$.

R(s) ——→(+−Σ)——→|$G_c(s)$|——→|$G_p(s)$|——→ Y(s)
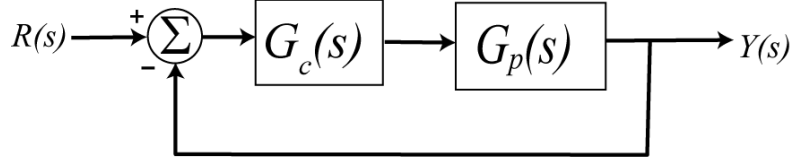
a) Determine the closed loop transfer function, $G_0(s)$

b) Determine the poles of value of $G_0(s)$ and show they are only real if $0 < k_i < \dfrac{1}{3}$. Note that the best possible setting time is 4 seconds. Use $k_i = \dfrac{1}{3}$ for the remainder of this problem.

c) If the input to the system is a unit step, determine the output of the system.

d) The steady state error is the difference between the input and the output as $t \to \infty$. Determine the steady state error for this system.

Partial Answers: $y(t) = \left[1 - e^{-t} - te^{-t}\right]u(t), \quad e_{ss} = 0$

7) Consider the following simple feedback control block diagram. The plant is $G_p(s) = \dfrac{2}{s+4}$. The input is a unit step.

R(s) ——→(+−Σ)——→|$G_c(s)$|——→|$G_p(s)$|——→ Y(s)
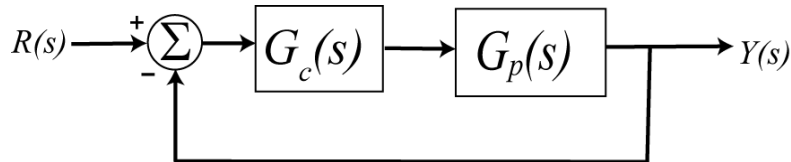
a) Determine the settling time and steady state error of the plant alone (assuming there is no feedback)

b) Assuming a proportional controller, $G_c(s) = k_p$, determine the closed loop transfer function, $G_0(s)$

c) Assuming a proportional controller, $G_c(s) = k_p$, determine the value of k_p so the steady state error for a unit step is 0.1, and the corresponding settling time for the system.

d) Assuming a proportional controller, $G_c(s) = k_p$, determine the value of k_p so the settling time is 0.5 seconds, and the corresponding steady state error.

e) Assuming an integral controller, $G_c(s) = k_i / s$, determine closed loop transfer function, $G_0(s)$

f) Assuming an integral controller, $G_c(s) = k_i / s$, determine the value of k_i so the steady state error for a unit step is less than 0.1 and the system is stable.

Partial Answers: $T_s = 1, e_{ss} = 0.5, k_p = 18, k_p = 2, T_s = 0.1, e_{ss} = 0.5, k_i > 0$

8) (Model Matching) Consider the following closed loop system, with plant $G_p(s)$ and controller $G_c(s)$.

R(s) → Σ (+, −) → $G_c(s)$ → $G_p(s)$ → Y(s)
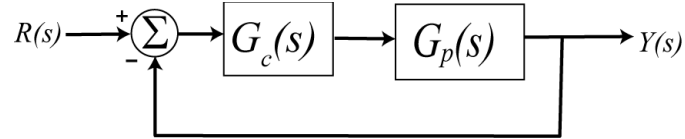
One way to choose the controller is to try and make your closed loop system match a transfer function that you choose (hence the name model matching). Let's assume that our **desired** closed loop transfer function, $G_o(s)$, our plant can be written in terms of numerators and denominators as

$$G_o(s) = \frac{N_o(s)}{D_o(s)} \quad G_p(s) = \frac{N_p(s)}{D_p(s)}$$

Show that our controller is then $G_c(s) = \dfrac{N_o(s)D_p(s)}{N_p(s)[D_o(s) - N_o(s)]}$

Note that there are some restrictions here, in that for implementation purposes the controller must be stable, and it must be proper.

9) For the following system, with plant $G_p(s) = \dfrac{1}{s+1}$, and controller $G_c(s)$

R(s) → Σ (+, −) → $G_c(s)$ → $G_p(s)$ → Y(s)
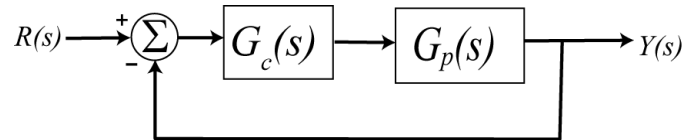
a) Using the results from problem 8, determine the controller so that the closed loop system matches a second order ITAE (**I**ntegral of **T**ime and **A**bsolute **E**rror) optimal system, i.e., so that the closed loop transfer function is

$$G_0(s) = \frac{\omega_0^2}{s^2 + 1.4\omega_0 s + \omega_0^2}$$

Anwes. $G_c(s) = \dfrac{\omega_0^2(s+1)}{s(s+1.4\omega_0)}$, *note that there is a pole/zero cancellation between the controller and the plant and there is a pole at zero in the controller.*

b) Show that the damping ratio for this system is 0.7, the closed loop poles of this system are at $-0.7\omega_0 \pm j0.714\omega_0$. For faster response should ω_0 be large or small?

**c)** What is the steady state error for this system if the input is a unit step?

d) Determine the controller so that the closed loop system matches a third order **deadbeat** system, i.e., so that the closed loop transfer function is

$$G_0(s) = \frac{\omega_0^3}{s^3 + 1.90\omega_0 s^2 + 2.20\omega_0^2 s + \omega_0^3}$$

Ans. $G_c(s) = \dfrac{\omega_0^3(s+1)}{s(s^2 + 1.9\omega_0 s + 2.20\omega_0^2)}$, *note that there is a pole/zero cancellation between the controller and the plant and there is a pole at zero in the controller.*

e) What is the steady state error of this system for a unit step input?

10) When implementing a control system, we need to be careful the control effort does not get too large. Sometimes this is overcome by changing the feedback structure. Consider the following two feedback control structures,
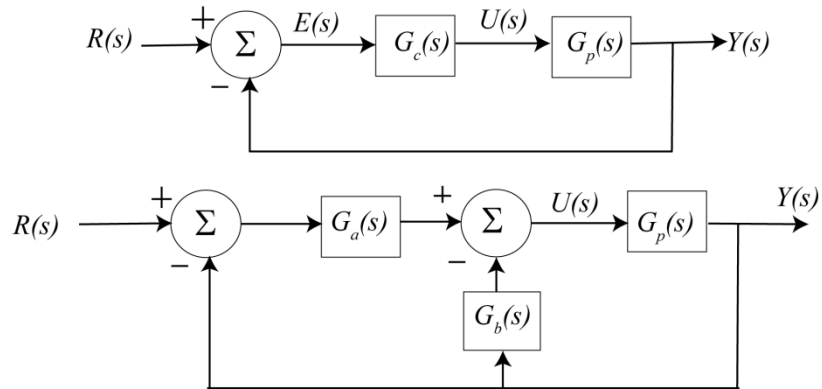
For the first structure, the initial control effort is given by

$$u(0^+) = \lim_{s\to\infty} sU(s) = \lim_{s\to\infty} \frac{sR(s)G_c(s)}{1+G_c(s)G_p(s)}$$

For the second structure the initial control effort is given by

$$u(0^+) = \lim_{s\to\infty} sU(s) = \lim_{s\to\infty} \frac{sR(s)G_a(s)}{1+\left[G_a(s)+G_b(s)\right]G_p(s)}$$

Let's assume we can break up the controller so that we can write the transfer functions as

$$G_p(s) = \frac{N_p(s)}{D_p(s)}, G_c(s) = \frac{N_c(s)}{D_c(s)}, G_a(s) = \frac{N_a(s)}{D_a(s)}, G_b(s) = \frac{N_b(s)}{D_b(s)}$$

a) Show that if we can partition the controller so that

$$G_a(s) + G_b(s) = G_c(s)$$

Then we can write the closed loop transfer functions as

$$G_o(s) = \frac{N_c(s)N_p(s)}{D_c(s)D_p(s)+N_c(s)N_p(s)}, G_o(s) = \frac{D_c(s)N_a(s)N_p(s)}{D_a(s)\left[D_c(s)D_p(s)+N_c(s)N_p(s)\right]}$$

Hence, if $D_a(s)$ is a constant, the two structures produce the same closed loop poles.

b) Assume we have the plant $G_p(s) = \dfrac{3}{s+2}$, and assume we are trying to use a PD controller. So
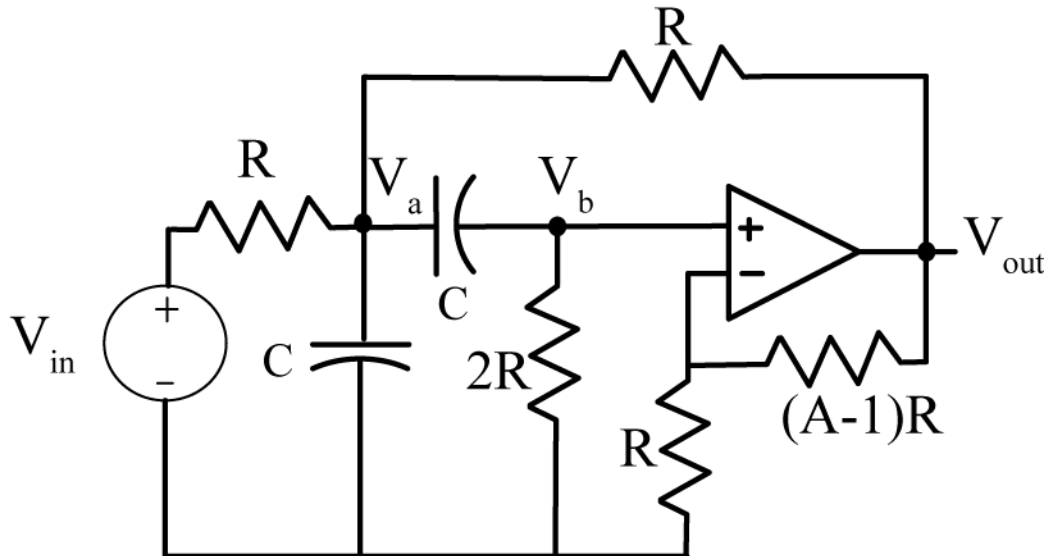
$$G_c(s) = k_p + k_d s, \quad G_a(s) = k_p, \quad G_b(s) = k_d s$$

Show that if the input is a step of amplitude A, then for the two control structures we have

$$u(0^+) = \infty, \ e_{ss} = \frac{A2}{2+3k_p}, \quad u(0^+) = \frac{Ak_p}{1+3k_d}, e_{ss} = \frac{A2}{2+3k_p}$$

This second control structure is often used with PID controllers, as I-PD and PI-D controllers.

11) (Matlab Problem) Consider the following Sallen-Key filter, shown in the figure below:

We are going to solve for the output of this circuit for a range of frequencies using Matlab.

a) Download the m-file **homework9.m** from the class website. You will need to modify this code in this problem.

b) Write node equations for this system in terms of the nodes **Va**, **Vb**, and **Vout** and the input voltage **Vin**. Note that you will not be able to write a node equation at the output of the op amp, but will have to relate **Vb** and **Vout** using the ideal op-amp assumption. The impedance for a capacitor is $Z(j\omega) = \dfrac{1}{j\omega C}$
. In Matlab, we can use **j** as the square root of -1 (though you may have to multiply by **j** explicitly). The capacitor is 1 microfarad, the resistors are 1000 ohms, and the parameter A is 2.5. These parameters have already been entered in the program. The frequency will be changing however.

c) In the Matlab program, write the node equations in terms of the matrix M, and the input vector b. Assume we write the matrix equation as $Mx = b$, or

$$[M] \begin{bmatrix} V_a \\ V_b \\ V_{out} \end{bmatrix} = \begin{bmatrix} V_{in} \\ 0 \\ 0 \end{bmatrix}$$

d) Use Matlab to solve the matrix equation for the nodal voltages (x). In this problem we only care about the output voltage, the third element of x. Assume the input voltage is 1 volt with a phase of zero degrees.

e) Use the Matlab commands **abs** and **angle** to get the magnitude and phase of the output signal. The command angle gives the phase in radians, and you will need to convert this to degrees.

Your final plot should look like that in Figure 1. Turn in your code and your plot.

:
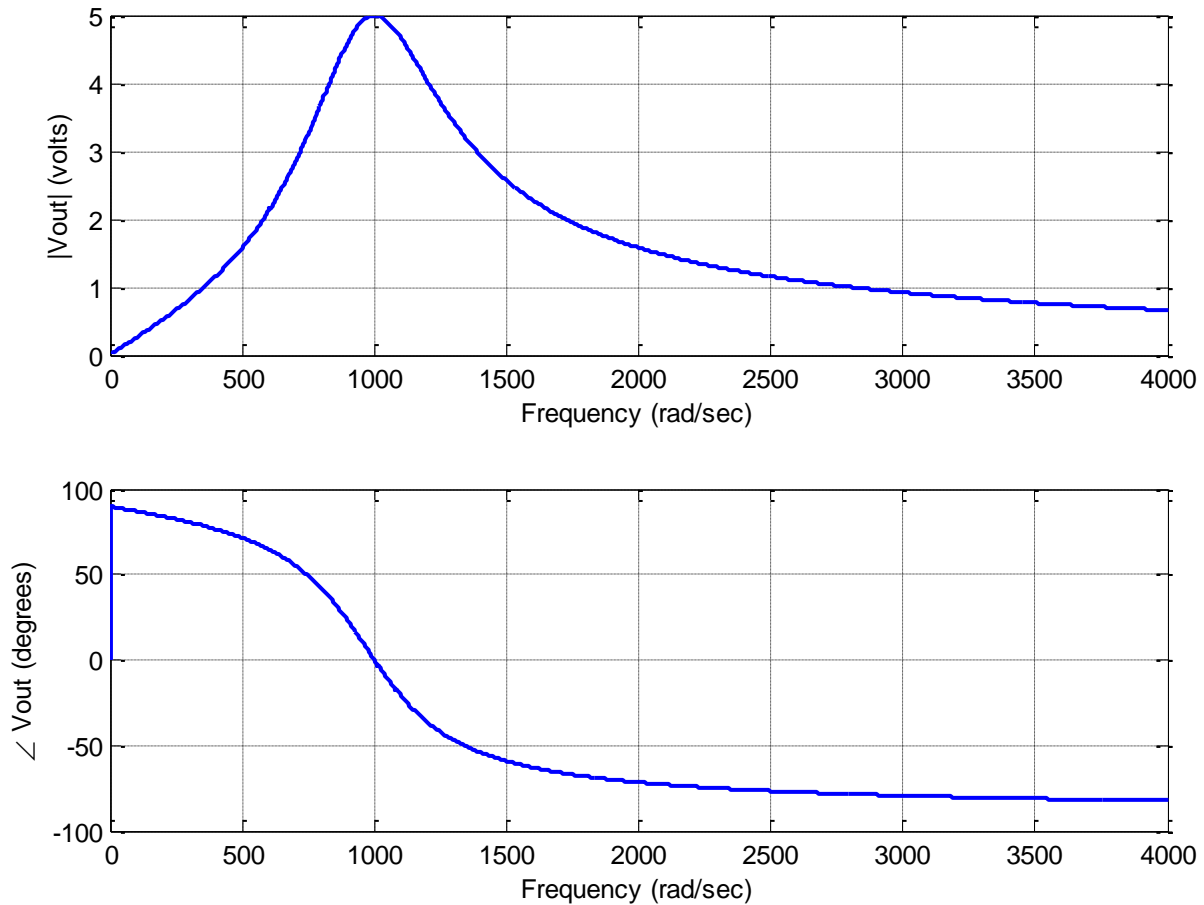
Figure 1. Frequency response of the Sallen-Key filter from problem 11.

8.0 Steady State Frequency Response

Consider the response of two LTI system with transfer functions

$$H_1(s) = \frac{5}{s+1}$$

$$H_2(s) = \frac{80}{s^2 + 1.2s + 10}$$

to the inputs $x(t) = \sin(\omega_0 t)u(t)$ for $\omega_0 = 5$, 10, and 15 radians/sec. Figure 8.1 displays the response of the first system to the three input sinusoids, while Figure 8.2 displays the response of the second system to the three input sinusoids. In both figures, the input sinusoid is displayed as a dashed line and the output is a continuous line. Both figures include heavy solid lines that bound the amplitude of the output sinusoid in steady state. As the figures demonstrate, both of the systems go through some initial transients, and then reach a steady state response. The first system has a pole at -1 and the second system has its poles at approximately $-0.6 \pm 3.1j$, so the 2% setting times for the two systems are estimated to be 4 and 6.7 seconds which corresponds pretty well with the results in the figures. Once the systems come into steady state, the output of the system has a constant amplitude, and there is a constant relationship between the input signal and the output signal. It is important to note that *the settling time for the system is not a function of the frequency of the input, but is a property of the system!*

At this point we know how to quickly estimate the settling time of a system based on the poles of the system, and we next want to be able to quickly estimate the steady state output of an asymptotically stable system with a sinusoidal input. In order to do this we first need to review Euler's identities and write them in a different form than you are used to seeing.

8.1 Euler's Identity and Other Useful Relationships

The usual form of Euler's identity is

$$e^{j\omega_0 t} = \cos(\omega_0 t) + j\sin(\omega_0 t)$$

We can also write this as

$$e^{-j\omega_0 t} = \cos(\omega_0 t) - j\sin(\omega_0 t)$$

If we add and subtract these two expressions we get

$$e^{j\omega_0 t} + e^{-j\omega_0 t} = 2\cos(\omega_0 t)$$

$$e^{j\omega_0 t} - e^{-j\omega_0 t} = 2j\sin(\omega_0 t)$$

Finally, we get our alternate forms of Euler's identity

$$\cos(\omega_0 t) = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$$

$$\sin(\omega_0 t) = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}$$

This form of Euler's identity is very useful in both this and in later courses.
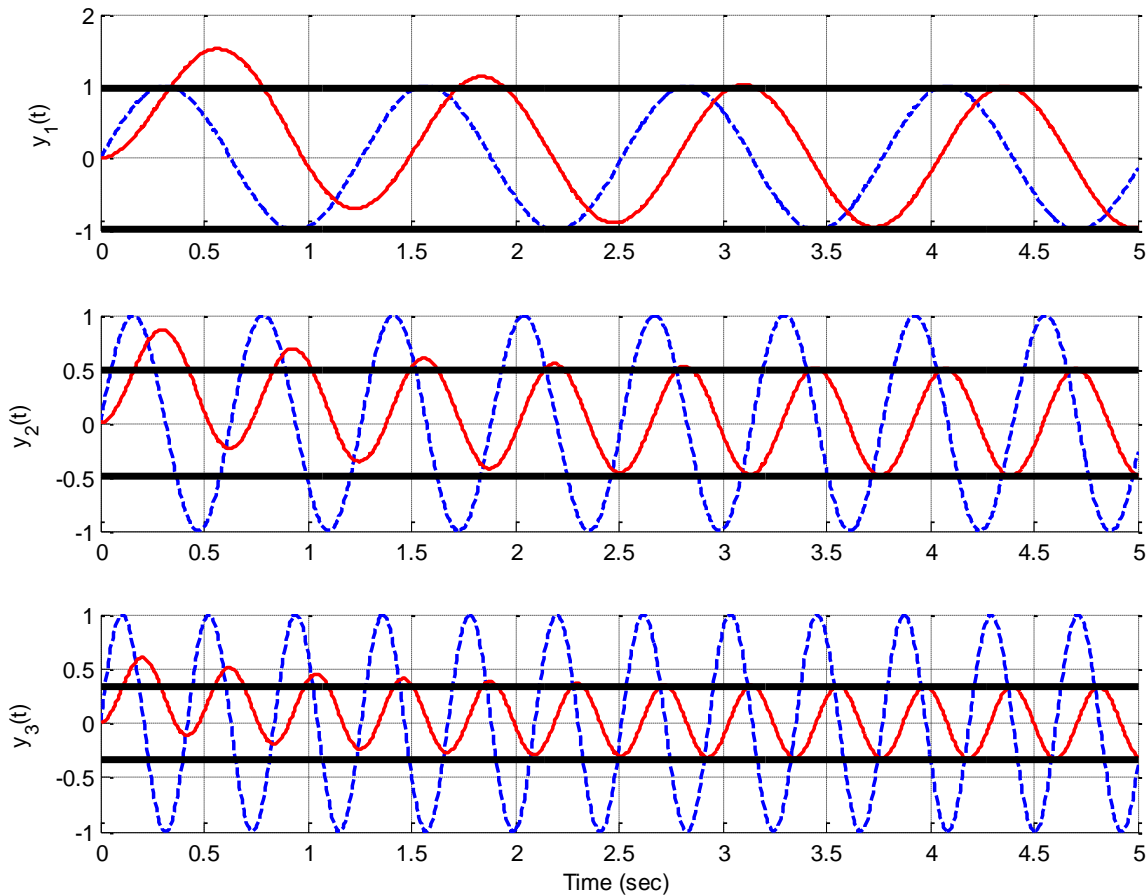
Figure 8.1. Response of the first system to sinusoids of 5, 10, and 15 radians/sec. The estimated settling time of the system is 4 seconds. The input signal is the dashed line and the output signal is the solid line. The thick solid line bounds the steady state amplitude.

Next, let's assume our usual case of a proper transfer function that is the ratio of two polynomials,

$$H(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

Since this is proper we have $m \leq n$. Let's also assume that we have a real valued system, so that if the input is a real valued function the output will be a real valued function. This means that all of the coefficients must be real values. To understand this, remember that this transfer function means the input and output are related by the differential equation

$$\frac{d^n y(t)}{dt^n} + a_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + \cdots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = b_m \frac{d^m x(t)}{dt^m} + b_{m-1} \frac{d^{m-1} x(t)}{dt^{m-1}} + \cdots + b_1 \frac{dx(t)}{dt} + b_0 x(t)$$

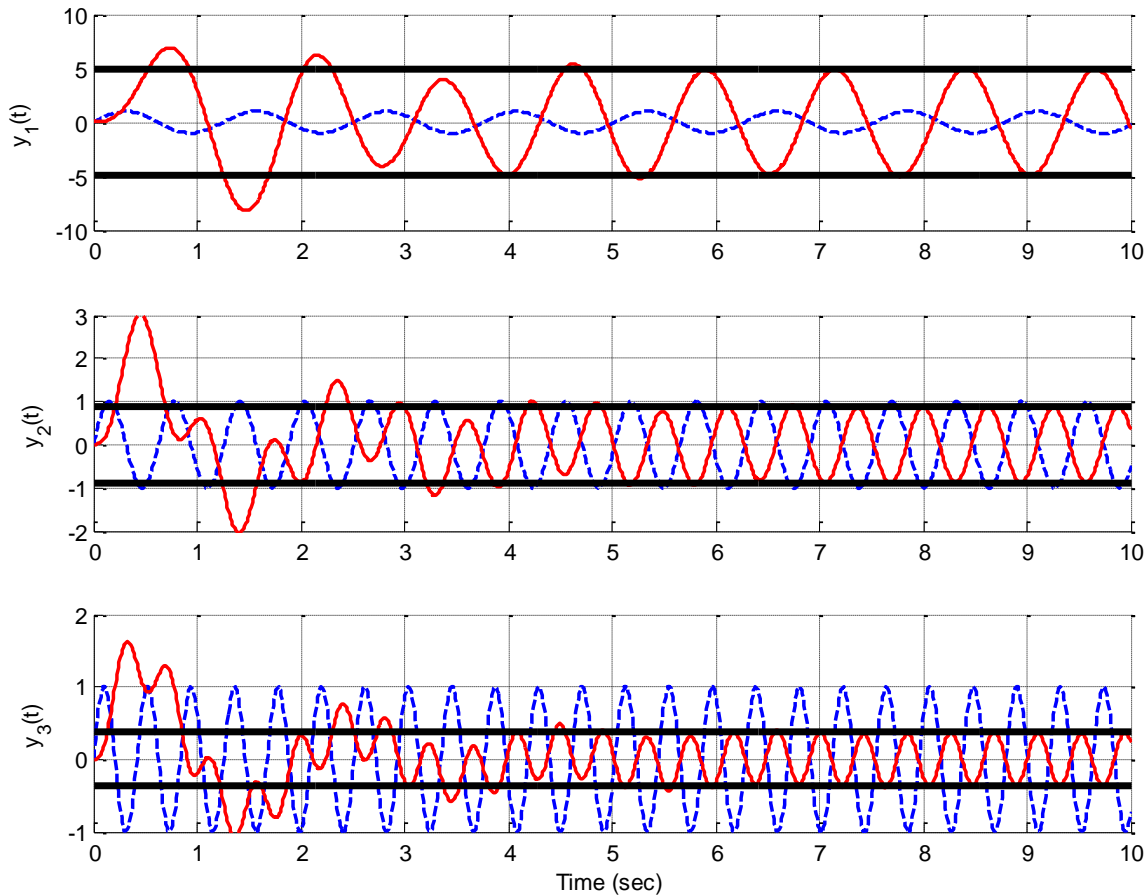Hence, if the input is real and we want the output to be real all of the coefficients must also be real.

Figure 8.2. Response of the second system to sinusoids of 5, 10, and 15 radians/sec. The estimated settling time of the system is 6.7 seconds. The input signal is the dashed line and the output signal is the solid line. The thick solid line bounds the steady state amplitude.

Next, let's assume we want to evaluate the transfer function at $s = j\omega_0$ and also at $s = -j\omega_0$ and if we can relate the two. Remember that as we raise j to increasing powers we cycle through the same four values, i.e., $j^1 = j$, $j^2 = -1$, $j^3 = -j$, $j^4 = 1$, and $j^5 = j$ so we are right back where we started. To determine what is going on with transfer functions we will look at low order systems and build are way up. Finally, remember that to determine the complex conjugate of a number, you replace j with $-j$.

First Order System:

$$H(s) = \frac{b_1 s + b_0}{s + a_0}, H(j\omega_0) = \frac{jb_1\omega_0 + b_0}{j\omega_0 + a_0}, H(-j\omega_0) = \frac{-jb_1\omega_0 + b_0}{-j\omega_0 + a_0} = H^*(j\omega_0)$$

Second Order System:

$$H(s) = \frac{b_2 s^2 + b_1 s + b_0}{s^2 + a_1 s + a_0}, H(j\omega_0) = \frac{-b_2\omega_0^2 + jb_1\omega_0 + b_0}{-\omega_0^2 + ja_1\omega_0 + a_0},$$

$$H(-j\omega_0) = \frac{-b_2\omega_0^2 - jb_1\omega_0 + b_0}{-\omega_0^2 - ja_1\omega_0 + a_0} = H^*(j\omega_0)$$

Third Order System:

$$H(s) = \frac{b_3 s^3 + b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0}, H(j\omega_0) = \frac{-jb_3\omega_0^3 - b_2\omega_0^2 + jb_1\omega_0 + b_0}{-j\omega_0^3 - a_2\omega_0^2 + ja_1\omega_0 + a_0},$$

$$H(-j\omega_0) = \frac{jb_3\omega_0^3 - b_2\omega_0^2 - jb_1\omega_0 + b_0}{j\omega_0^3 - a_2\omega_0^2 - ja_1\omega_0 + a_0} = H^*(j\omega_0)$$

Fourth Order System:

$$H(s) = \frac{b_4 s^4 + b_3 s^3 + b_2 s^2 + b_1 s + b_0}{s^4 + a_4 s^3 + a_2 s^2 + a_1 s + a_0}, H(j\omega_0) = \frac{b_4\omega_0^4 - jb_3\omega_0^3 - b_2\omega_0^2 + jb_1\omega_0 + b_0}{\omega_0^4 - ja_3\omega_0^3 - a_2\omega_0^2 + ja_1\omega_0 + a_0},$$

$$H(-j\omega_0) = \frac{b_4\omega_0^4 + jb_3\omega_0^3 - b_2\omega_0^2 - jb_1\omega_0 + b_0}{\omega_0^4 + ja_3\omega_0^3 - a_2\omega_0^2 - ja_1\omega_0 + a_0} = H^*(j\omega_0)$$

At this point it is pretty straightforward to generalize the relationship $H^*(j\omega) = H(-j\omega)$.

The last thing we need to do is look at representing the transfer function in polar form to determine two important properties of transfer functions for real-valued systems. Assume that we have the complex function written in rectangular form as

$$z(\omega) = a(\omega) + jb(\omega)$$

where $a(\omega)$ and $b(\omega)$ are real valued functions. We can write this in polar form as

$$z(\omega) = c(\omega)e^{jd(\omega)} = |z(\omega)| e^{j\angle z(\omega)}$$

where

$$c(\omega) = \sqrt{a(\omega)^2 + b(\omega)^2} = |z(\omega)|$$

$$d(\omega) = \tan^{-1}\left(\frac{b(\omega)}{a(\omega)}\right) = \angle z(\omega)$$

Now let's assume we have a complex valued transfer function $H(j\omega)$. We can write this in polar form as

$$H(j\omega) = |H(j\omega)| e^{j\angle H(j\omega)}$$

Next, let's look at $H^*(j\omega)$ and $H(-j\omega)$ to derive some useful relationships.

$$H^*(j\omega) = |H^*(j\omega)| e^{-j\angle H(j\omega)} = |H(j\omega)| e^{-j\angle H(j\omega)}$$
$$H(-j\omega) = |H(-j\omega)| e^{j\angle H(-j\omega)} = |H(-j\omega)| e^{j\angle H(-j\omega)}$$

Now since we know that for a real valued system $H^*(j\omega) = H(-j\omega)$ we can conclude the following useful information:

- $|H(j\omega)| = |H(-j\omega)|$ (the magnitude is an **even** function of frequency)
- $-\angle H(j\omega) = \angle H(-j\omega)$ (the phase is an **odd** function of frequency)

Although we have only shown this to be true for transfer functions that are ratios of polynomials, it is true in general for any transfer function that corresponds to a real valued system. It may also seem a bit odd to talk about *negative* frequencies, but this is necessary for the mathematics to work out. Often when we plot the frequency response of a system we will plot both negative and positive frequencies, even if we can only measure the positive frequencies.

With this background, we can now solve our problem.

8.2 Steady State Response to a Periodic Input

Let's assume we have *strictly proper* rational transfer function for a real valued asymptotically stable system which has n distinct poles. Then we can represent the transfer function as

$$H(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{(z - p_1)(z - p_2) \cdots (z - p_n)}$$

Next, assume the input to our system is a cosine with amplitude A, $x(t) = A\cos(\omega_0 t) u(t)$. We can then use Laplace transforms to determine the output of the system,

$$Y(s) = H(s) X(s) = H(s) \frac{As}{s^2 + \omega_0^2} = H(s) \frac{As}{(s - j\omega_0)(s + j\omega_0)}$$

Using partial fraction expansion we will have

$$Y(s) = H(s) \frac{As}{(s - j\omega_0)(s + j\omega_0)} = \frac{B_1}{s - p_1} + \frac{B_2}{s - p_2} + \cdots + \frac{B_n}{s + p_n} + \frac{C_1}{s - j\omega_0} + \frac{C_2}{s + j\omega_0}$$

In the time domain we can represent this as

$$y(t) = B_1 \phi_1(t) + B_2 \phi_2(t) + \cdots + B_n \phi_n(t) + C_1 e^{j\omega_0 t} + C_2 e^{-j\omega_0 t}$$

where $\phi_i(t)$ is the ith characteristic mode. Now, since we have assumed our system is asymptotically stable, in steady state, as $t \to \infty$, all of the characteristic modes will go to zero and we will have

$$y_{ss}(t) = C_1 e^{j\omega_0 t} + C_2 e^{-j\omega_0 t}$$

Using the partial fraction expansion we have

$$C_1 = AH(j\omega_0)\frac{j\omega_0}{j\omega_0 + j\omega_0} = \frac{A}{2}H(j\omega_0)$$

$$C_2 = AH(-j\omega_0)\frac{-j\omega_0}{-j\omega_0 - j\omega_0} = \frac{A}{2}H(-j\omega_0)$$

so we have

$$y_{ss}(t) = \frac{A}{2}H(j\omega_0)e^{j\omega_0 t} + \frac{A}{2}H(-j\omega_0)e^{-j\omega_0 t}$$

Now we need the relationships we derived in the last section. We can write

$$H(j\omega) = |H(j\omega)|e^{j\angle H(j\omega)}$$

$$H(-j\omega) = |H(j\omega)|e^{-j\angle H(j\omega)}$$

Inserting this into our steady state response, and combining exponents and factoring out common terms we have

$$y_{ss}(t) = A|H(j\omega_0)|\frac{e^{j(\omega_0 t + \angle H(j\omega_0))} + e^{-j(\omega_0 t + \angle H(j\omega_0))}}{2}$$

Using Euler's identity we finally have

$$y_{ss}(t) = A|H(j\omega_0)|\cos(\omega_0 t + \angle H(j\omega_0))$$

This is a very important result that you will use repeatedly in the future. Although we assumed our system had distinct poles, it should be obvious that that is not necessary. Also, although we assumed the input contained no phase, that was to keep the mathematics to a simpler level. If the input has a phase, it is just added to the output.

In summary if we have a proper, rational transfer function for a real valued system that is asymptotically stable, then we can compute the steady state output of the system as follows

$$x(t) = A\cos(\omega_0 t + \theta) \rightarrow \boxed{H(s)} \rightarrow y_{ss}(t) = A|H(j\omega_0)|\cos(\omega_0 t + \angle H(j\omega_0) + \theta)$$

Note that *this result is only valid after the transients have died* out, usually after approximately four time constants (i.e., the system has passed the settling time).

Example 8.2.1. Determine the steady state output for the system represented by the transfer function

$$H(s) = \frac{2}{s+3}$$

if the input to the system is $x(t) = 4\cos(3t + 45^o)$. First, we must recognize $\omega_0 = 3$, $\theta = 45^o$, and $A = 4$. Next we compute the magnitude and phase of the transfer function,

$$H(j3) = \frac{2}{j3+3}, |H(j3)| = 1.075, \angle H(j3) = -45^o$$

Combining these we get

$$y_{ss}(t) = (4)(1.075)\cos(3t - 45^o + 45^o) = 4.30\cos(3t)$$

Example 8.2.2. Determine the steady state output for the system represented by the transfer function

$$H(s) = \frac{s}{(s+1)(s+2)}$$

if the input to the system is $x(t) = 3\sin(2t + 30^o)$. First we must convert the sine to a cosine,

$$x(t) = 3\sin(2t + 30^o) = 3\cos(2t + 30^o - 90^o) = 3\cos(2t - 60^o)$$

Next we recognize $\omega_0 = 2$, $\theta = -60^o$, and $A = 3$. Evaluating the transfer function at the correct frequency we have

$$H(j2) = \frac{j2}{(j2+1)(j2+2)}, |H(j2)| = 0.3162, \measuredangle H(j2) = -18.4349^o$$

Finally we have

$$y_{ss}(t) = (3)(0.3162)\cos(2t - 18.4349^o - 60^o) = 0.9487\cos(2t - 68.4349^o) = 0.9487\sin(2t + 48.4349)$$

Note that we subtracted 90 degrees to convert from the cosine to the sine, and then added 90 degrees when we were done to convert back to the sine. Since the initial phase is just carried through, we really did not have to convert from the cosine to the sine to do this problem. In short, we could have used the relationship

$$x(t) = A\sin(\omega_0 t + \theta) \rightarrow \boxed{H(s)} \rightarrow y_{ss}(t) = A|H(j\omega_0)|\sin(\omega_0 t + \measuredangle H(j\omega_0) + \theta)$$

Finally, note that the notation $H(j\omega_0)$ usually refers to evaluating the Laplace transform of an asymptotically stable system at the frequency ω_0,

$$H(j\omega_0) = H(s)|_{s=j\omega_0}$$

8.3 Computation of $H(j\omega_0)$ Using The Pole-Zero Diagram

In addition to being able to compute $H(j\omega_0)$ analytically, it is sometimes useful to utilize the pole-zero diagram to determine the magnitude and phase of the transfer function. With some practice, this method allows one to get a feeling for how the magnitude and phase of the transfer function changes as the frequency ω_0 is changed.

Let's assume we have a strictly proper transfer function written in terms of poles and zeros as

$$H(s) = \frac{K(s - z_1)(s - z_2)\cdots(s - z_m)}{(s - p_1)(s - p_2)\cdots(s - p_n)}$$

We need to evaluate this at $s = j\omega_0$, so we have

$$H(j\omega_0) = \frac{K(j\omega_0 - z_1)(j\omega_0 - z_2)\cdots(j\omega_0 - z_m)}{(j\omega_0 - p_1)(j\omega_0 - p_2)\cdots(j\omega_0 - p_n)}$$

Next, we need to represent each of these terms in polar form, rather than in rectangular form. To standardize this, let's write

$$j\omega_0 - z_i = \beta_i e^{j\phi_i}, \; \beta_i = \sqrt{(Re(z_i)^2 + (\omega_0 - Im(z_i))^2}, \; \phi_i = \tan^{-1}\left(\frac{\omega_0 - Im(z_i)}{-Re(z_i)}\right)$$

$$j\omega_0 - p_i = \alpha_i e^{j\theta_i}, \; \alpha_i = \sqrt{(Re(p_i)^2 + (\omega_0 - Im(p_i))^2}, \; \theta_i = \tan^{-1}\left(\frac{\omega_0 - Im(p_i)}{-Re(p_i)}\right)$$

β_i and α_i represent the distances between zero z_i and pole p_i from the point $j\omega_0$, respectively. ϕ_i represents the angle (measured from the positive real axis) between zero z_i and $j\omega_0$, while θ_i represents the angle between pole p_i and $j\omega_0$. While these formulas look pretty messy, you will see they are actually fairly easy to understand graphically. Finally we have

$$H(j\omega_0) = \frac{K\beta_1 e^{j\phi_1}\beta_2 e^{j\phi_2}...\beta_m e^{j\phi_m}}{\alpha_1 e^{j\theta_1}\alpha_2 e^{j\theta_2}...\alpha_n e^{j\theta_n}} = \frac{K\beta_1\beta_2...\beta_m}{\alpha_1\alpha_2...\alpha_n} e^{j(\phi_1+\phi_2+\cdots+\phi_m-\theta_1-\theta_2-\cdots-\theta_n)}$$

This means

$$|H(j\omega_0)| = \frac{K\beta_1\beta_2...\beta_m}{\alpha_1\alpha_2...\alpha_n}$$

$$\angle H(j\omega_0) = \phi_1 + \phi_2 + \cdots + \phi_m - \theta_1 - \theta_2 - \cdots - \theta_n$$

To see how to use these formulas, let's do a few examples. In words, the first formula means that the magnitude is equal to K times the product of distances from the zeros to $j\omega_0$ divided by the product of distances from the poles to $j\omega_0$. The second formula indicates that the phase of the transfer function is the sum of the phases between the zeros and the point $j\omega_0$ minus the sum of the phases between the poles and the point $j\omega_0$. A few examples will help clear this up, and you will see it is quite easy to view this graphically.

Example 8.3.1 For the system with transfer function

$$H(s) = \frac{4}{s+4}$$

use the pole zero plot to determine the frequency response at frequencies ω_0 =2, 5, and 8 radians/sec. We will first look at the magnitude and then the phase of $H(j\omega_0)$ for the three frequencies. Figures 8.3-8.5 shows the corresponding pole-zero plots and the distances we need to compute.

At a frequency of 2 radians/second (Figure 8.3), we first need to compute the distance between the pole at -4 and the point $j2$, so we have

$$\alpha_1 = \sqrt{(-4-0)^2 + (0-2)^2} = \sqrt{20}$$

and then

$$|H(j2)| = \frac{K}{\alpha_1} = \frac{4}{\sqrt{20}} = 0.894$$

We next need to determine the angle between the pole at -4 and the point $j2$, which we can compute as

$$\theta_1 = \tan^{-1}\left(\frac{2}{4}\right) = 26.6^o$$

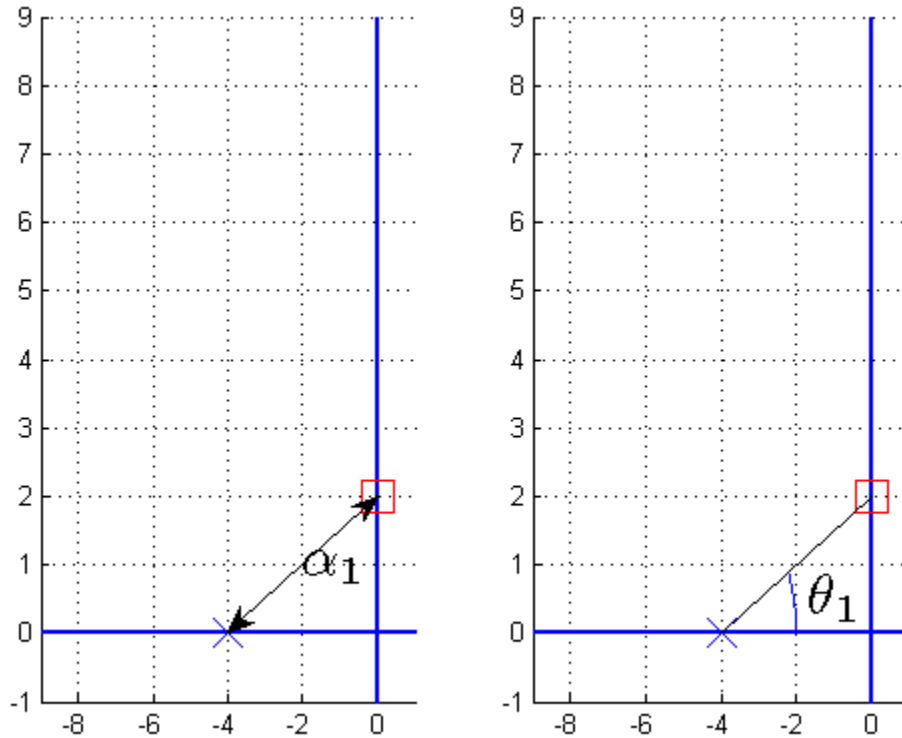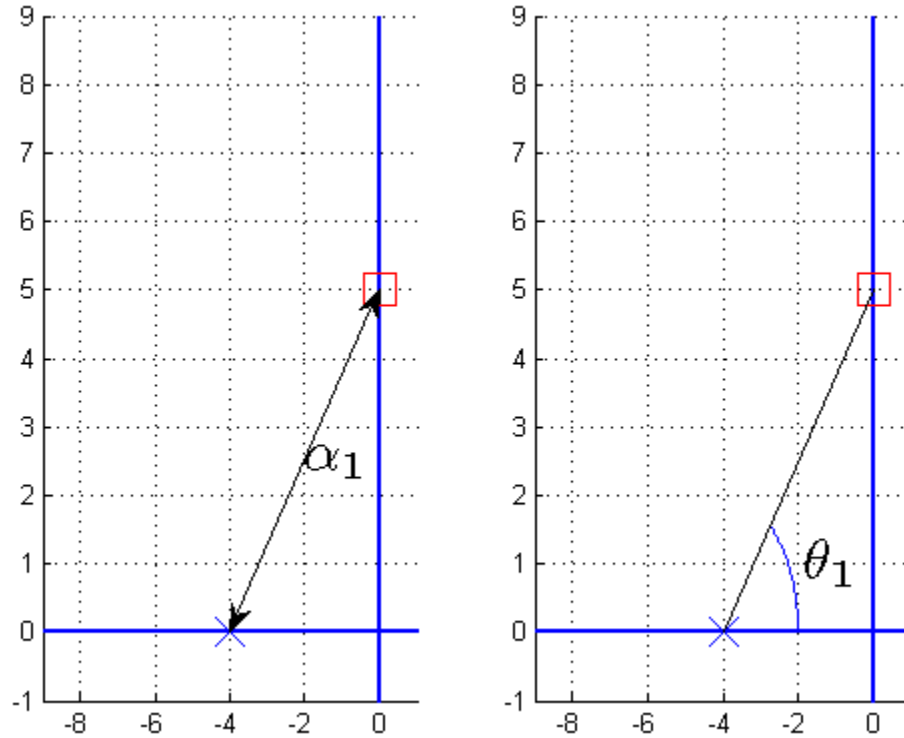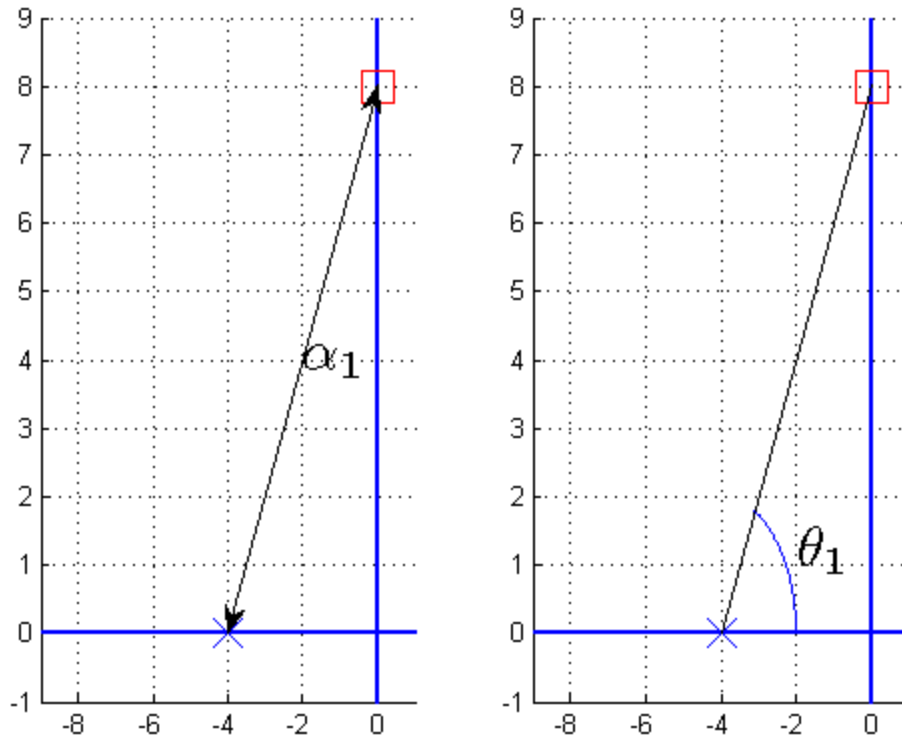Hence the phase is -26.6^o. Thus we have $H(j2) = 0.894 \angle -26.6.^o$.

Figure 8.3. Pole-zero plot of $H(s) = \dfrac{4}{s+4}$ evaluated at 2 radians/second. The distance between the pole and the point $j2$ is used in the magnitude computation, while the angle between the pole and the point $j2$ is used in the phase computation.

At a frequency of 5 radians/second (Figure 8.4), we need to compute the distance between the pole at -4 and the point $j5$ so we have

$$\alpha_1 = \sqrt{(-4-0)^2 + (0-5)^2} = \sqrt{41}$$

and then

$$|H(j5)| = \frac{K}{\alpha_1} = \frac{4}{\sqrt{41}} = 0.625$$

Similarly, to compute the phase we need the angle between the pole at -4 and the point $j5$, so we have

$$\theta_1 = \tan^{-1}\left(\frac{5}{4}\right) = 51.3^o$$

and hence $H(j5) = 0.25\angle -51.3^o$.

Figure 8.4. Pole-zero plot of $H(s) = \dfrac{4}{s+4}$ evaluated at 5 radians/second. The distance between the pole and the point $j5$ is used in the magnitude computation, while the angle between the pole and the point $j5$ is used in the phase computation.

At a frequency of 8 radians/second (Figure 8.5), we need to compute the distance between the pole at -4 and the point $j8$, so we have

$$\alpha_1 = \sqrt{(-4-0)^2 + (0-8)^2} = \sqrt{80}$$

and then

$$|H(j8)| = \frac{K}{\alpha_1} = \frac{4}{\sqrt{80}} = 0.447$$

The phase is given by

$$\theta_1 = \tan^{-1}\left(\frac{8}{4}\right) = 63.4^o$$

and we have $H(j8) = 0.447\angle -63.4^o$.

Figure 8.5. Pole-zero plot of $H(s) = \dfrac{4}{s+4}$ evaluated at 8 radians/second. The distance between the pole and the point $j8$ is used in the magnitude computation, while the angle between the pole and the point $j8$ is used in the phase computation.

Figure 8.6 displays the magnitude and phase as a continuous function of frequency, with the points we just calculated shown with a star. Note that the *magnitude is an even function of the frequency*, and the *phase is an odd function of the frequency*.
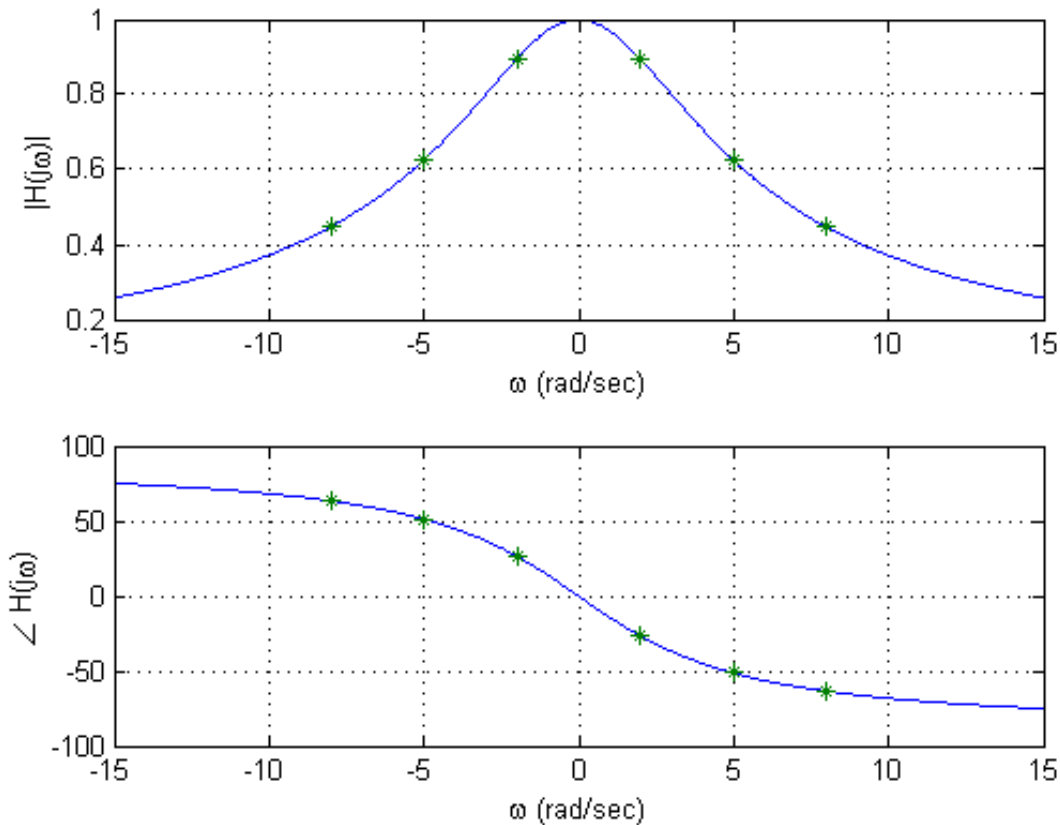
Figure 8.6. Magnitude and phase plots of $H(s) = \dfrac{4}{s+4}$. The stars (*) are located at the discrete points we evaluated in Example 8.3.1.

Example 8.3.2. Consider the system with transfer function

$$H(s) = \frac{s^2 + 20s + 200}{(s^2 + 16s + 80)(s + 10)} = \frac{(s + 10 + 10j)(s + 10 - 10j)}{(s + 8 + 4j)(s + 8 - 4j)(s + 10)}$$

Use the pole-zero diagram to determine $H(j2)$. The appropriate pole-zero diagram is shown in Figure 8.7. Computing the distances and angles between the zeros and the point $j2$ we have

$$\beta_1 = \sqrt{(-10-0)^2 + (10-2)^2} = \sqrt{164} = 12.806, \quad \phi_1 = \tan^{-1}\left(\frac{-8}{10}\right) = -38.66^o$$

$$\beta_2 = \sqrt{(-10-0)^2 + (-10-2)^2} = \sqrt{244} = 15.621, \quad \phi_2 = \tan^{-1}\left(\frac{12}{10}\right) = 50.19^o$$

Computing the distances and angles between the poles and the point $j2$ we have

$$\alpha_1 = \sqrt{(-10-0)^2+(0-2)^2} = \sqrt{104} = 10.198, \quad \theta_1 = \tan^{-1}\left(\frac{2}{10}\right) = 11.31^o$$

$$\alpha_2 = \sqrt{(-8-0)^2+(4-2)^2} = \sqrt{68} = 8.246, \quad \theta_2 = \tan^{-1}\left(\frac{-2}{8}\right) = -14.04^o$$

$$\alpha_3 = \sqrt{(-8-0)^2+(-4-2)^2} = \sqrt{100} = 10.000, \quad \theta_3 = \tan^{-1}\left(\frac{6}{8}\right) = 36.87^o$$

We can then compute

$$|H(j2)| = \frac{\beta_1\beta_2}{\alpha_1\alpha_2\alpha_3} = \frac{(12.806)(15.621)}{(10.198)(8.246)(10)} \approx 0.238$$

$$\angle H(j2) = \phi_1 + \phi_2 - \theta_1 - \theta_2 - \theta_3 = -38.66^o + 50.19^o - 11.31^o + 14.04^o - 36.87^o \approx -22.6^o$$

A plot of the magnitude and phase of this transfer function as a continuous function of frequency is shown in Figure 8.8. In this figure, the point on the plot we just computed is identified with a star (*).

While computing the magnitude and phase of the transfer function this way can become quite tedious if we need to compute it for many frequencies, it does help to visualize the frequency response. For example, if a system has a single real pole at $-p$, then the smallest the distance between this point and any point on the $j\omega$ axis occurs when $\omega = 0$. This means the maximum of $|H(j\omega)|$ occurs when $\omega = 0$ and will be monotonically decreasing in magnitude as the frequency increase. If a system has two complex conjugate poles, such as at $-\sigma \pm j\omega_0$, then at the frequency $\omega = \omega_0$ the denominator tends to have its smallest value, and the magnitude of the transfer function its largest value. This is not always true, but is a general trend especially when the imaginary part of the pole is large compared to the real part of the pole ($\omega_0 \gg \sigma$). Figure 8-9 demonstrates this type of behavior. In this figure, the imaginary part of the pole is fixed at ± 16 as the real part varies. The magnitude of each plot is scaled to a maximum of one. As the figure shows, as the magnitude of the imaginary part of the pole becomes larger than the magnitude of the real part of the pole, we see peaks or resonances at the frequency corresponding to the imaginary part of the poles.
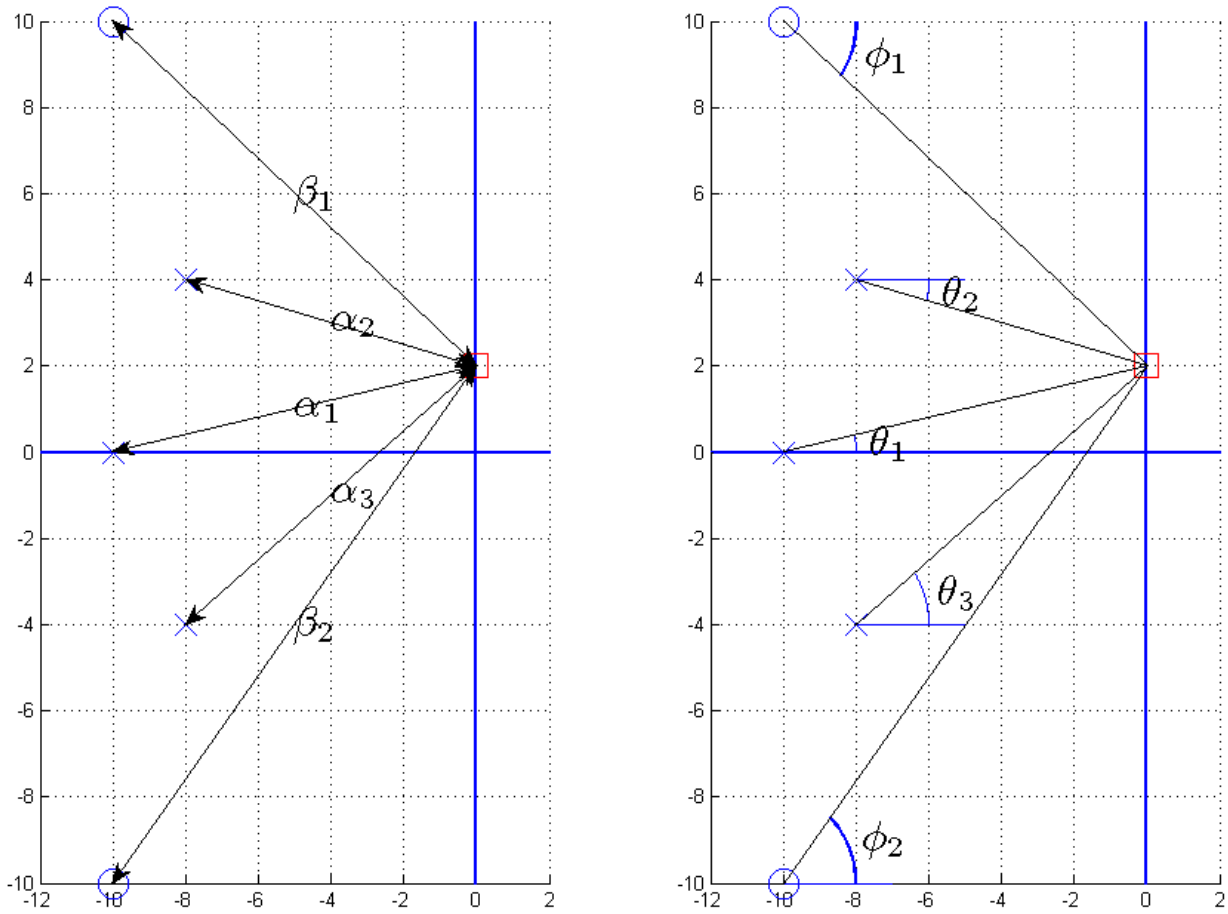
Figure 8.7. Pole-zero diagram for Example 8.3.2. We are evaluating the transfer function

$$H(s) = \frac{(s+10+10j)(s+10-10j)}{(s+8+4j)(s+8-4j)(s+10)}$$ at the frequency 2 radians/sec.
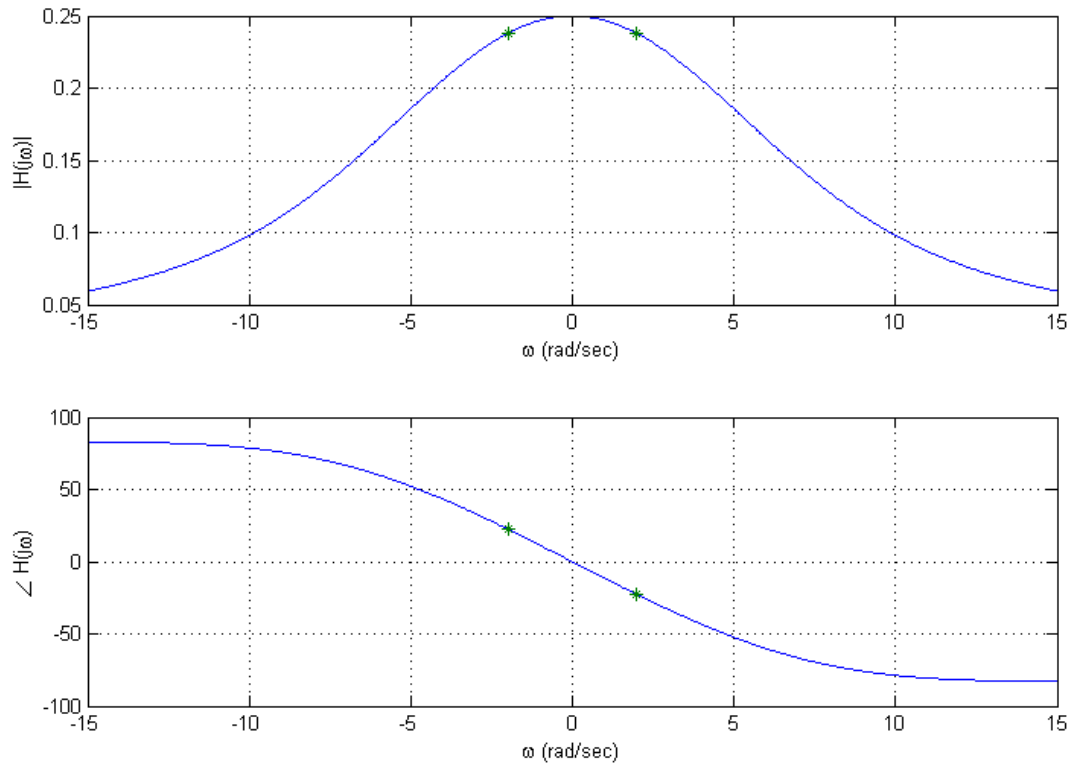
Figure 8.8. Plot of $H(s) = \dfrac{(s+10+10j)(s+10-10j)}{(s+8+4j)(s+8-4j)(s+10)}$ as a continuous function of frequency. This is the transfer function from Example 8.3.2, and the stars (*) denote the points we computed in that example.
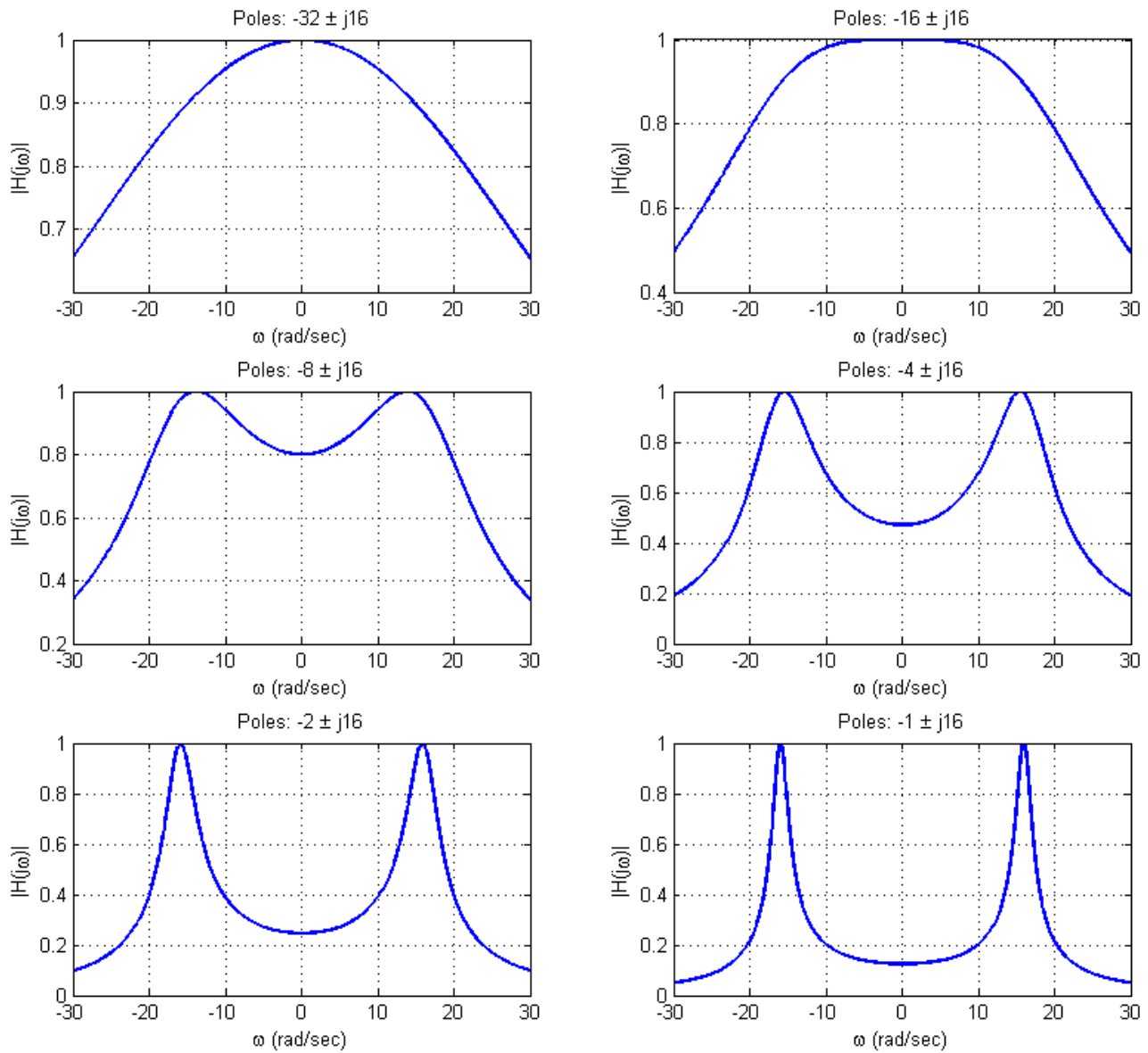
Figure 8.9. This figure shows the magnitude of the frequency response of various systems that have two complex conjugate poles. All of these systems have poles of the form $\sigma \pm 16$ where the real part of the pole varies from -32 to -1. The magnitudes for these systems have been scaled to have a maximum value of one.

8.4 Using Decibels

Before the advent of calculators and computers, people needed an easy way to sketch frequency response plots, and it was determined that the use of a logarithmic scale was the best choice. This is primarily because of the following relationships

$$\log(ab) = \log(a) + \log(b)$$
$$\log(a/b) = \log(a) - \log(b)$$
$$\log(a^N) = N\log(a)$$

In many engineering systems, we are interested in the relationship between the input and the output. One very useful measure for many applications is to measure the ratio of the output power to the input power, so this idea is also included in our measurement scale. We then define the power gain G in decibels (or dBs) as

$$G_{dB} = 10\log_{10}\left(\frac{P_{out}}{P_{in}}\right)$$

Note that we are using a base ten logarithmic scale. If we think about a circuit, we often want to measure the ratio of the output voltage (or current) to the input voltage (or current). It is customary to assume we measure the voltage across (or through) a common resistance (typically we assume 1 ohm), so we have

$$G_{dB} = 10\log_{10}\left[\frac{|V_{out}^2|/R}{|V_{in}^2|/R}\right] = 10\log_{10}\left[\frac{|V_{out}^2|}{|V_{in}^2|}\right] = 20\log_{10}\left[\frac{|V_{out}|}{|V_{in}|}\right]$$

Now note that if we have a transfer function relating the input and output,

$$V_{out}(s) = H(s)V_{in}(s)$$

then we have

$$\frac{|V_{out}(s)|}{|V_{in}(s)|} = |H(s)|$$

and

$$G_{dB} = 20\log_{10}\left[\frac{|V_{out}|}{|V_{in}|}\right] = 20\log_{10}|H|$$

We can also measure the power with respect to a known reference power. The reference power is measured as the rms value of the signal across (through) a 1 ohm resistor. However, since we are assuming the common one ohm resistor, we just reference it to an rms voltage. For example, to measure a signal in dBV, we reference the signal to a 1 volt rms signal,

$$G_{dBV} = 20\log_{10}\left[\frac{V_{rms}}{1\,volt\,(rms)}\right]$$

To measure power in dBmV, we reference our signal to a 1 mV rms signal, and compute it as

$$G_{dBmV} = 20\log_{10}\left[\frac{V_{rms}}{1\,millivolt\,(rms)}\right]$$

In what follows, we will just use the convention above, that

$$G_{dB} = 20\log_{10}|H(j\omega)|$$

8.5 Bode Plots

A very common method of representing the frequency response of a transfer function is with a Bode plot. This form of representation was very important before the advent of computers. However, it is still very useful to be able to quickly sketch Bode plots to estimate the frequency response of a system, and also to be able to determine how the frequency response may change as poles and zeros are added to a transfer function.

The Bode plot is based on knowledge of how to construct the Bode plot for basic *building blocks* that make up our transfer functions and then combining the Bode plots for these basic building blocks into the Bode plot for the entire transfer function. The ability to easily combine the Bode plots of the building blocks is based on the properties of the polar form of writing complex functions of frequency and properties of the logarithm,.

The building blocks that make up the transfer functions we have been studying are the following:

- Constant gains, K
- Integrators or differentiators, s^n
- Simple poles or zeros, $(\tau s + 1)^n$

- Complex conjugate poles or zeros, $\left(\dfrac{1}{\omega_n^2}s^2 + \dfrac{2\zeta}{\omega_n}s + 1\right)^n$

Basically, any transfer function we have studied (except for those with delays) can be written as a combination of these building blocks.

Next, let's assume we have two complex functions, $A(j\omega)$ and $B(j\omega)$. We can represent these in polar

form as

$$A(j\omega) = |A(j\omega)|e^{j\angle A(j\omega)}$$
$$B(j\omega) = |B(j\omega)|e^{j\angle B(j\omega)}$$

Let's now look at two new functions of these basic functions,

$$Z_1(j\omega) = A(j\omega)B(j\omega)$$
$$Z_2(j\omega) = \frac{A(j\omega)}{B(j\omega)}$$

If we write these new functions in polar form we will have

$$Z_1(j\omega) = |A(j\omega)|e^{j\angle A(j\omega)}|B(j\omega)|e^{j\angle B(j\omega)} = |A(j\omega)||B(j\omega)|e^{j(\angle A(j\omega)+\angle B(j\omega))}$$

$$Z_2(j\omega) = \frac{|A(j\omega)|e^{j\angle A(j\omega)}}{|B(j\omega)|e^{j\angle B(j\omega)}} = \frac{|A(j\omega)|}{|B(j\omega)|}e^{j(\angle A(j\omega)-\angle B(j\omega))}$$

Then we can write the magnitudes and phases as,

$$|Z_1(j\omega)| = |A(j\omega)||B(j\omega)| \quad \angle Z_1(j\omega) = \angle A(j\omega)+\angle B(j\omega)$$

$$|Z_2(j\omega)| = \frac{|A(j\omega)|}{|B(j\omega)|} \qquad \angle Z_2(j\omega) = \angle A(j\omega)-\angle B(j\omega)$$

Represent the magnitudes in terms of dBs, then we have

$$|Z_1(j\omega)|_{dB} = 20\log_{10}\big[|A(j\omega)||B(j\omega)|\big] = 20\log_{10}|A(j\omega)|+20\log_{10}|B(j\omega)| = |A(j\omega)|_{db}+|B(j\omega)|_{dB}$$

$$|Z_2(j\omega)|_{dB} = 20\log_{10}\left[\frac{|A(j\omega)|}{|B(j\omega)|}\right] = 20\log_{10}|A(j\omega)|-20\log_{10}|B(j\omega)| = |A(j\omega)|_{dB}-|B(j\omega)|_{dB}$$

Finally, representing the magnitude in dB and the phase in either degrees or radians we have

$$|Z_1(j\omega)|_{dB} = |A(j\omega)|_{db}+|B(j\omega)|_{dB} \quad \angle Z_1(j\omega) = \angle A(j\omega)+\angle B(j\omega)$$

$$|Z_2(j\omega)|_{dB} = |A(j\omega)|_{dB}-|B(j\omega)|_{dB} \quad \angle Z_1(j\omega) = \angle A(j\omega)+\angle B(j\omega)$$

It is these simple relationships that form the basis of constructing the Bode plot. We basically need to figure out how to plot the magnitude or phase of a building block function, and then add the responses. Note that a Bode plot really refers to two different plots, a magnitude plot and a phase plot. Finally, it will be convenient to plot the Bode plots as semi-log plots, where the frequency axis is plotted on a logarithmic scale and the magnitude (in dB) or phase is plotted on a linear scale.

8.5.1. Constant Terms

The first building block is the constant term in the transfer function, $H(j\omega) = K$. We can represent this in terms of magnitude and phase as $H(j\omega) = |K|e^{j\angle K}$. It is important to note that the magnitude must be zero or positive. Hence if the gain is negative, then this will be included in the phase. For example, we can write $-8 = 8e^{j180^\circ}$. Next, when we look at the magnitude we have

$$|H(j\omega)|_{dB} = 20\log_{10}|K|$$

We can break this into three regions

$$0 < |K| < 1 \quad 20\log_{10}|K| < 0$$
$$|K| = 1 \quad 20\log_{10}|K| = 0$$
$$|K| > 1 \quad 20\log_{10}|K| > 0$$

In summary, the magnitude plot of a constant will be a flat line, and the phase plot will also be a flat line.

8.5.2. Integrators and Differentiators

The transfer function for these building blocks are of the form $H(s) = s^n$ where n can be positive (differentiators) or negative (integrators). To determine the frequency response we look at

$$H(j\omega) = (j\omega)^n = \left(\omega e^{j90^o}\right)^n = \omega^n e^{jn90^o}$$

The phase of this building block is then easy to determine as $n90^o$, which is just a constant. To determine the magnitude part, we have

$$20\log_{10} \omega^n = n20\log_{10} \omega$$

At this point, it is important to note that when $\omega = 1$ we have $20\log_{10} \omega = 20log_{10} 1 = 0$, so the point $(1,0)$ will be on the Bode plot for this building block. Next, let's see how much the magnitude changes as the frequency changes by a factor of 10 (a *decade)*. To do this, let's assume that at some frequency ω_1 we have $\omega_1 = 10^a$, and then we have another frequency ω_2 a decade later, $\omega_2 = 10^{a+1}$. Looking at the magnitudes for these frequencies we have

$$20\log_{10} \omega_1^n = n20\log_{10} \omega_1 = n20\log_{10} 10^a = n20a$$
$$20\log_{10} \omega_2^n = n20\log_{10} \omega_2 = n20\log_{10} 10^{a+1} = n20(a+1)$$

Thus in a decade (factor of 10) change in frequency, the magnitude changes by $20n$ dB. This is corresponds to a slope of *20n* dB/decade.

Example 8.5.1. Sketch the Bode plot for the transfer function $G(s) = \dfrac{1}{s}$. The magnitude of this transfer function goes through the point $(1\,\text{rad}/\text{sec}, 0\,\text{dB})$ and has a slope of -20 dB/decade. The phase is a constant -90°. The Bode plot for this system is shown in Figure 8.10.

Example 8.5.2. Sketch the Bode plot for the transfer function $G(s) = s^2$. The magnitude of this transfer function goes through the point $(1\,\text{rad}/\text{sec}, 0\,\text{dB})$ and has a slope of $+40$ dB/decade. The phase is a constant -180°. The Bode plot for this system is shown in Figure 8.11.

Example 8.5.3. Sketch the Bode plot for the transfer function $G(s) = \dfrac{100}{s}$. Note that there are two building blocks for this example, $G(s) = 100 \times \dfrac{1}{s} = G_1(s) \times G_2(s)$. We will construct the Bode plot for the individual components, and then add them. For $G_1(s) = 100$ we have $20\log_{10} 100 = 40\text{dB}$ and the phase is 0°. For $G_2(s) = \dfrac{1}{s}$ the magnitude goes through the point $(1\,\text{rad}/\text{sec}, 0\,\text{dB})$ with a slope of -20 dB/decade, and the phase is a constant -90°. The Bode plots for G_1 and G_2 are shown in Figure 8.12 as

dashed lines. The Bode plot for $G(s)$ is then determined by adding these components, and is displayed as a solid line in Figure 8.12.
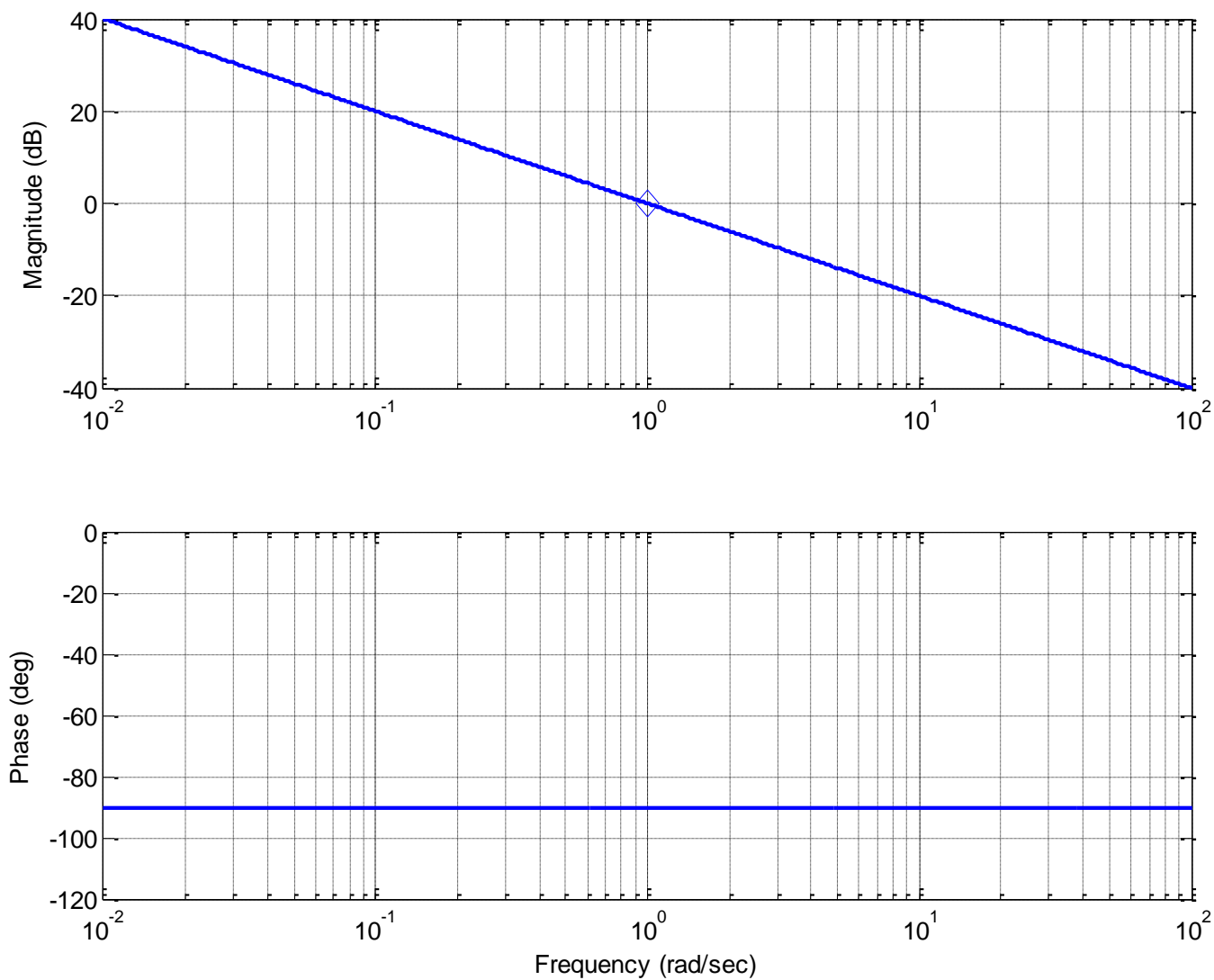

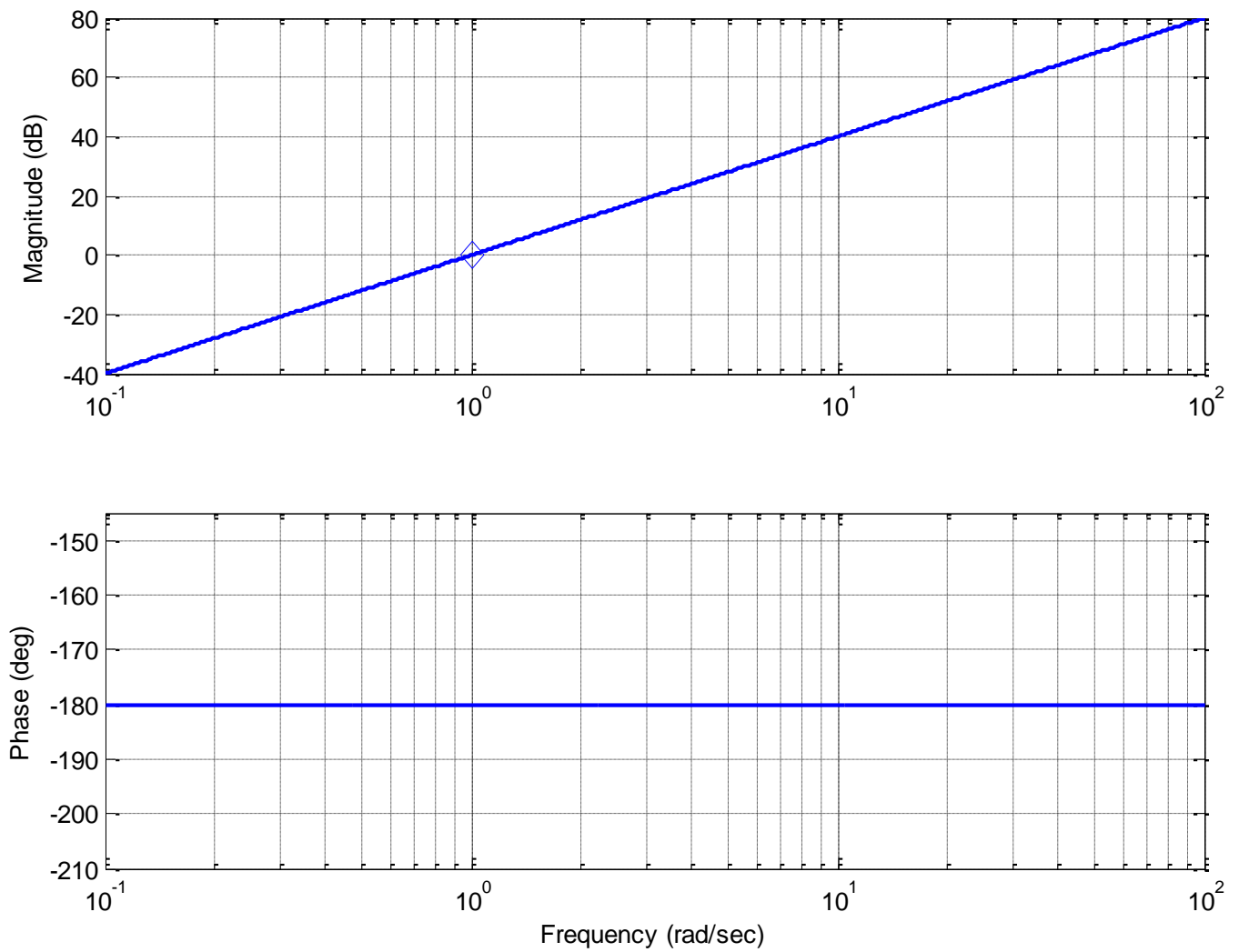**Figure 8.10.** Bode plot of the transfer function $G(s) = s^2$. Note that that a diamond delineates the point $(1 \, \text{rad} / \text{sec}, 0 \, \text{dB})$.

Figure 8.11. Bode plot of the transfer function $G(s) = \dfrac{1}{s}$. Note that that a diamond delineates the point $(1\,\text{rad}/\sec, 0\,\text{dB})$.
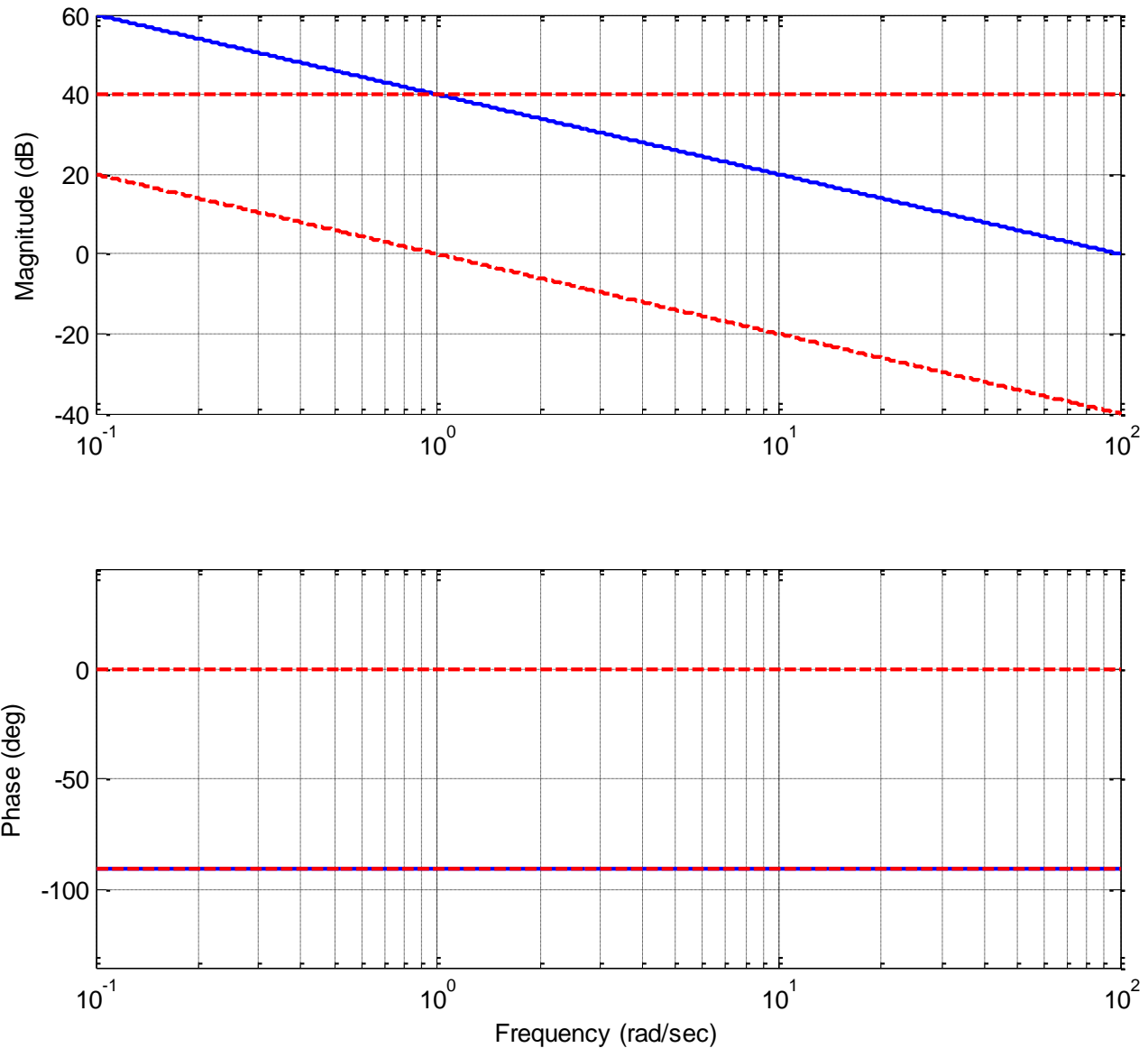
Figure 8.12. Bode plot of the transfer function $G(s) = \dfrac{100}{s}$. The Bode plots of the two subcomponents are shown as dashed lines, the final Bode plot is shown as a solid line. In the phase diagram, the subcomponent at -90 degrees overlaps with the final phase of -90 degrees (since the other subcomponent's phase is zero). Note that the sum of any two values (at a given frequency) on the dashed lines equals the value on the solid line at the same frequency.

8.5.3. Simple Poles and Zeros

This building block has the form $H(s) = (\tau s + 1)^n$ where again n is negative (for a pole) and positive (for a zero). To get the frequency response, we look at

$$H(j\omega) = (\tau j\omega + 1)^n = \left[\sqrt{(\tau\omega)^2 + 1} e^{j\tan^{-1}\left(\frac{\tau\omega}{1}\right)} \right]^n = \left[(\tau\omega)^2 + 1 \right]^{\frac{n}{2}} e^{jn\tan^{-1}\left(\frac{\tau\omega}{1}\right)}$$

The phase response is

$$\measuredangle H(j\omega) = n\tan^{-1}\left(\frac{\omega\tau}{1}\right)$$

Rather than trying to exactly determine what this looks like, let's just look at a few points. For small frequencies, $\omega \approx 0$, we have $\measuredangle H(j\omega) \approx n\tan^{-1}\left(\frac{0}{1}\right) \approx 0$. For very large frequencies, $\omega \approx \infty$, we have

$\measuredangle H(j\omega) \approx n\tan^{-1}\left(\frac{\infty}{1}\right) \approx n90^o$. Finally, when $\omega = \frac{1}{\tau}$ we have $\measuredangle H(j\omega) \approx n\tan^{-1}\left(\frac{1}{1}\right) \approx n45^o$.

A more precise formula is as follows:

$$\measuredangle H(j\omega) \approx 0^o \qquad \omega = \frac{0.1}{\tau} \qquad \left(\text{one decade before } \frac{1}{\tau} \right)$$

$$\measuredangle H(j\omega) \approx n45^o \qquad \omega = \frac{1}{\tau}$$

$$\measuredangle H(j\omega) \approx n90^o \qquad \omega = \frac{10}{\tau} \qquad \left(\text{one decade after } \frac{1}{\tau} \right)$$

When we sketch the phase as follows: for $\omega < \frac{0.1}{\tau}$ the phase is zero, and for $\omega > \frac{10}{\tau}$ the phase is $n90^o$. We then connect these points with a straight line.

The magnitude response is $|H(j\omega)| = \left[(\tau\omega)^2 + 1 \right]^{\frac{n}{2}}$, and again we will look at three points. For small frequencies, $\omega \approx 0$, we have $|H(j\omega)|_{dB} \approx 20log_{10}(1) \approx 0$. For large very frequencies, $\omega \approx \infty$, we have

$$|H(j\omega)|_{dB} \approx 20\log_{10}\left[(\tau\omega)^2 \right]^{\frac{n}{2}} = n20\log_{10}(\tau\omega) = n20\log_{10}\tau + n20\log_{10}\omega \approx n20log_{10}\omega$$

This means that for large frequencies we will have a slope of *20n* dB/decade. Finally, when $\omega = \frac{1}{\tau}$ we

have $|H(j\omega)|_{dB} = 20\log_{10}\left[2\right]^{\frac{n}{2}} = n10\log_{10}(2) = 3n$. Summarizing these results we have

	Magnitude	*Phase*

$$\omega \approx 0 \qquad\qquad 0\ dB \qquad\qquad\qquad \angle H(j\omega) \approx 0^o\ at\ \omega = \frac{0.1}{\tau}\left(one\ decade\ before\ \frac{1}{\tau}\right)$$

$$\omega = \frac{1}{\tau} \qquad\qquad 3n\ dB \qquad\qquad\qquad \angle H(j\omega) \approx n45^o$$

$$\omega \approx \infty \quad slope \approx 20n\ dB\ /\ decade \quad \angle H(j\omega) \approx n90^o\ at\ \omega = \frac{10}{\tau}\left(one\ decade\ after\ \frac{1}{\tau}\right)$$

Example 8.5.5. Sketch the Bode plot for the transfer function $G(s) = \dfrac{1}{0.1s+1}$. Here we have $\dfrac{1}{\tau} = 10$,

and this is referred to as the *break* or *corner* frequency. The magnitude is then zero until $\omega = 10 = \dfrac{1}{\tau}$. At

this point the magnitude decreases linearly with a slope of -20 dB/decade. Both the approximate (straight line) approximation and true magnitude portion of the Bode plot is shown in the top panel of Figure 8.13. The approximate magnitude is shown in the dashed line while the true magnitude is shown as a solid line. Note that at the corner frequency there is a difference of approximately 3 dB between the estimate and true magnitude plot. The phase plot is zero until $\dfrac{0.1}{\tau} = 0.1 \times 10 = 1$ and is -90° for

frequencies above $\dfrac{10}{\tau} = 10 \times 10 = 100$. Between these points we draw a straight line. The estimated phase

plot (dashed line) and true phase plots are shown in the lower panel of Figure 8.13.

Example 8.5.6 Sketch the Bode plot for the transfer function $G(s) = \dfrac{1}{100}\left(\dfrac{1}{100}s+1\right)^2$. This transfer

function is again made up of two building blocks, $G_1(s) = \dfrac{1}{100}$ and $G_2(s) = \left(\dfrac{1}{100}s+1\right)^2$. The magnitude

of the first building block is just the constant $20\log_{10}(0.01) = -40\ dB$ and the phase is zero. The corner frequency for the second building block is 100 rad/sec. The magnitude of the estimate will be zero before this corner point, and will have a positive slope of 40 dB/decade for frequencies above this. The phase of this component will be zero until 10 rad/sec, and will be 180° for frequencies above 1000 rad/sec. The estimated phase will be linear between these points. The estimated magnitude and phase plots, as well as the correct magnitude and phase plots, are shown in Figure 8.14. In this plot, the building block estimates are dashed lines, the sum of the building blocks are dotted lines, and the correct plots are shown as solid lines.

Example 8.5.7. Sketch the Bode plot approximation for the transfer function $G(s) = \dfrac{100}{s(s+10)}$. First we

need to put this into the correct form,

$$G(s) = \frac{100}{s(s+10)} = \frac{100}{s \times 10 \times (0.1s+1)} = \frac{10}{s(0.1s+1)}$$

We will then have three components, or building blocks,

$$G_1(s) = 10$$

$$G_2(s) = \frac{1}{s}$$

$$G_3(s) = \left(\frac{1}{10}s + 1\right)^{-1}$$

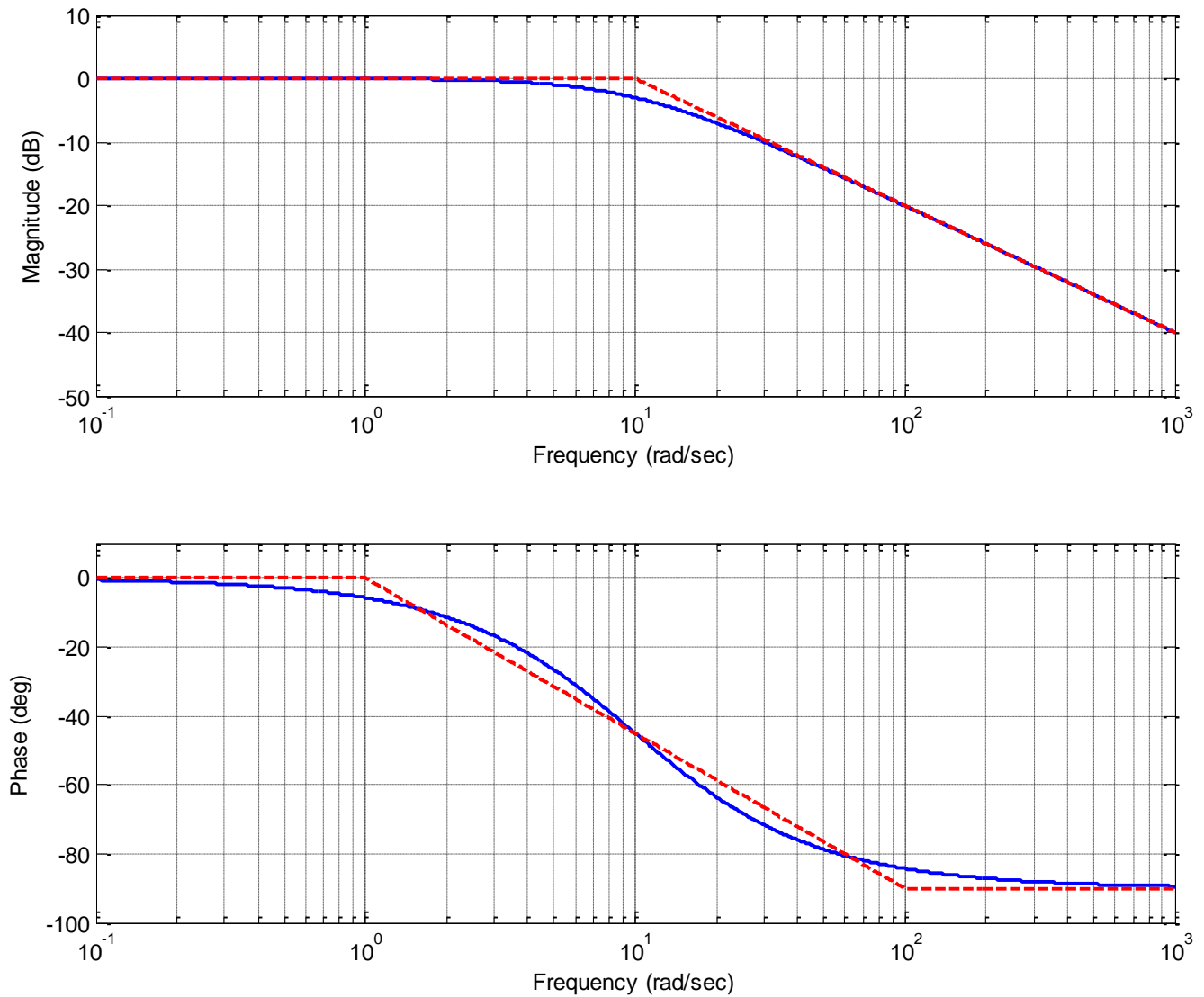The straight line approximations for the building blocks, and the final Bode plot are shown in Figure 8.15.

Example 8.5.8. Sketch the Bode plot for the transfer function $G(s) = \dfrac{s+1}{(s+10)(s+100)}$. Again, we first need to put this into the correct form,

$$G(s) = \frac{s+1}{(s+10)(s+100)} = \frac{s+1}{10 \times (0.1s+1) \times 100 \times (0.01s+1)} = \frac{10^{-3}(s+1)}{(0.1s+1)(0.01s+1)}$$

The building blocks are then

$$G_1(s) = 10^{-3}$$
$$G_2(s) = (s+1)$$
$$G_3(s) = (0.1s+1)^{-1}$$
$$G_4(s) = (0.01s+1)^{-1}$$

The straight line approximation for the building blocks, and the final Bode plot are shown in Figure 8.16.

Example 8.5.9. Sketch the Bode plot for the transfer function $G(s) = \dfrac{10s}{(s+100)^2}$. Putting this transfer function into the correct form and identifying the building blocks we have

$$G(s) = \frac{10s}{(s+100)^2} = \frac{10s}{\left(100 \times (0.01s+1)\right)^2} = \frac{10^{-3}s}{(0.01s+1)^2}$$
$$G_1(s) = 10^{-3}$$
$$G_2(s) = s$$
$$G_3(s) = (0.01s+1)^{-2}$$

The straight line approximation for the building blocks, and the final Bode plot are shown in Figure 8.17.

Figure 8.13. Bode plot for the transfer function $G(s) = \dfrac{1}{0.1s+1}$. Note the 3 dB discrepancy at the corner frequency in the magnitude plot. The Bode plot of the straight line approximation is shown as dashed lines while the exact Bode plot is shown as a solid line.

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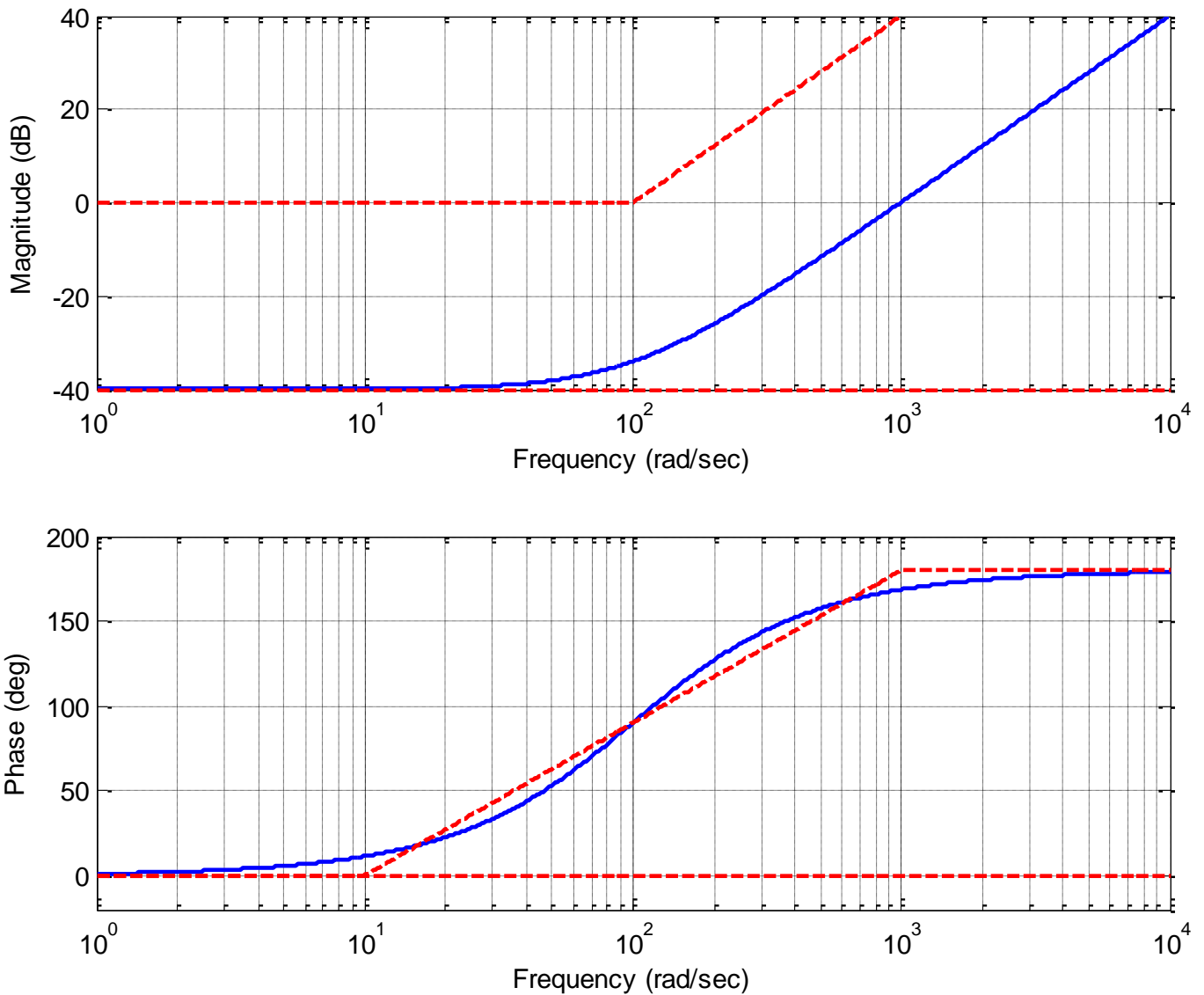
Figure 8.14. Bode plot for the transfer function $G(s) = \dfrac{1}{100}\left(\dfrac{1}{100}s+1\right)^2$. Note the 6 dB discrepancy at the corner frequency in the magnitude plot. The straight line approximations of the individual components are shown as dashed lines, the actual Bode plot is shown as a continuous line. Note that for any given frequency, if you sum the values of the straight line approximations you get reasonable approximations to the true (continuous curve) Bode plot.
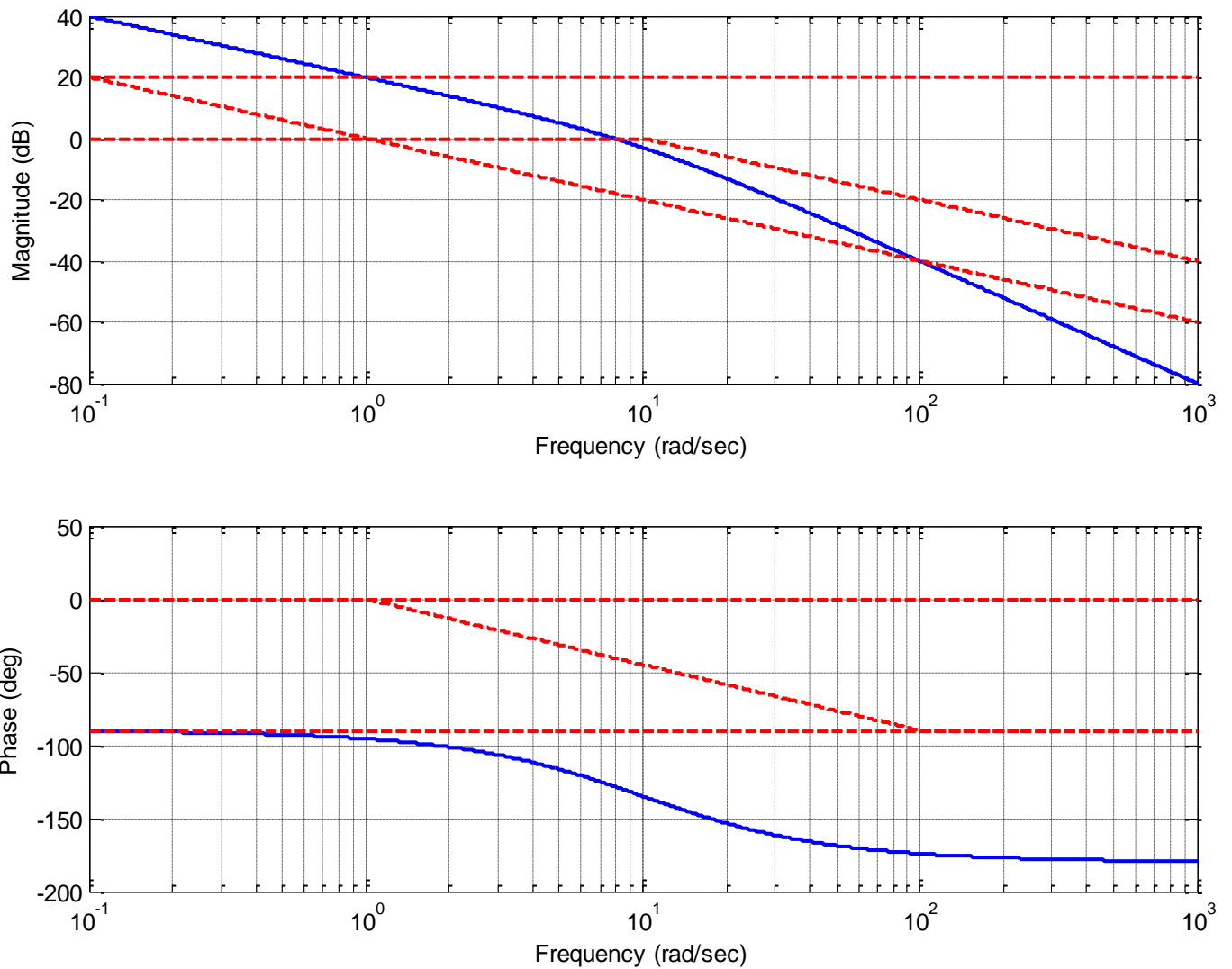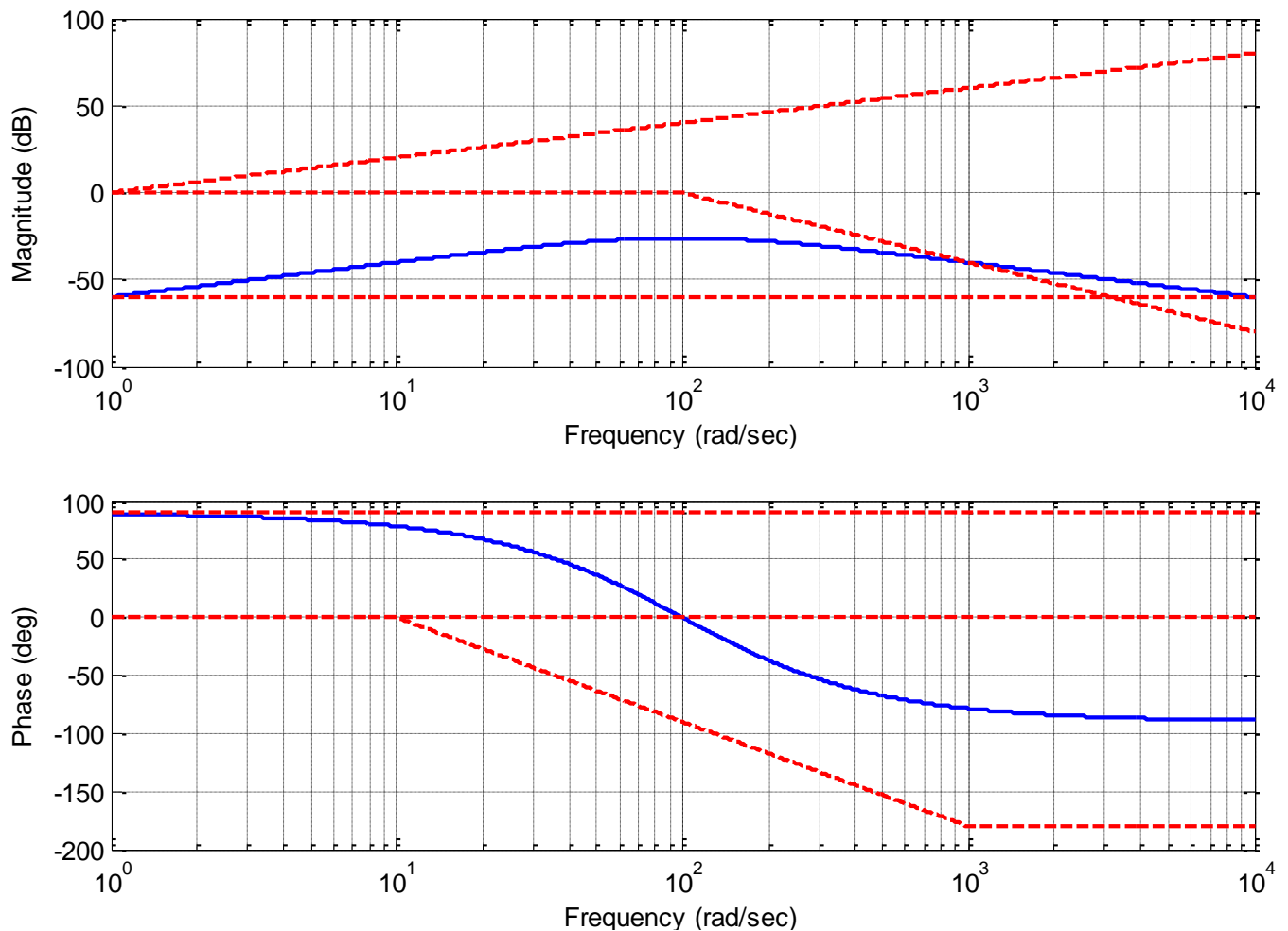
Figure 8.15. Bode plot for the transfer function $G(s) = \dfrac{100}{s(s+10)} = \dfrac{10}{s(0.1s+1)}$. The straight line

approximations of the individual components are shown as dashed lines, the actual Bode plot is shown as a continuous line. Note that for any given frequency, if you sum the values of the straight line approximations you get reasonable approximations to the true (continuous curve) Bode plot.
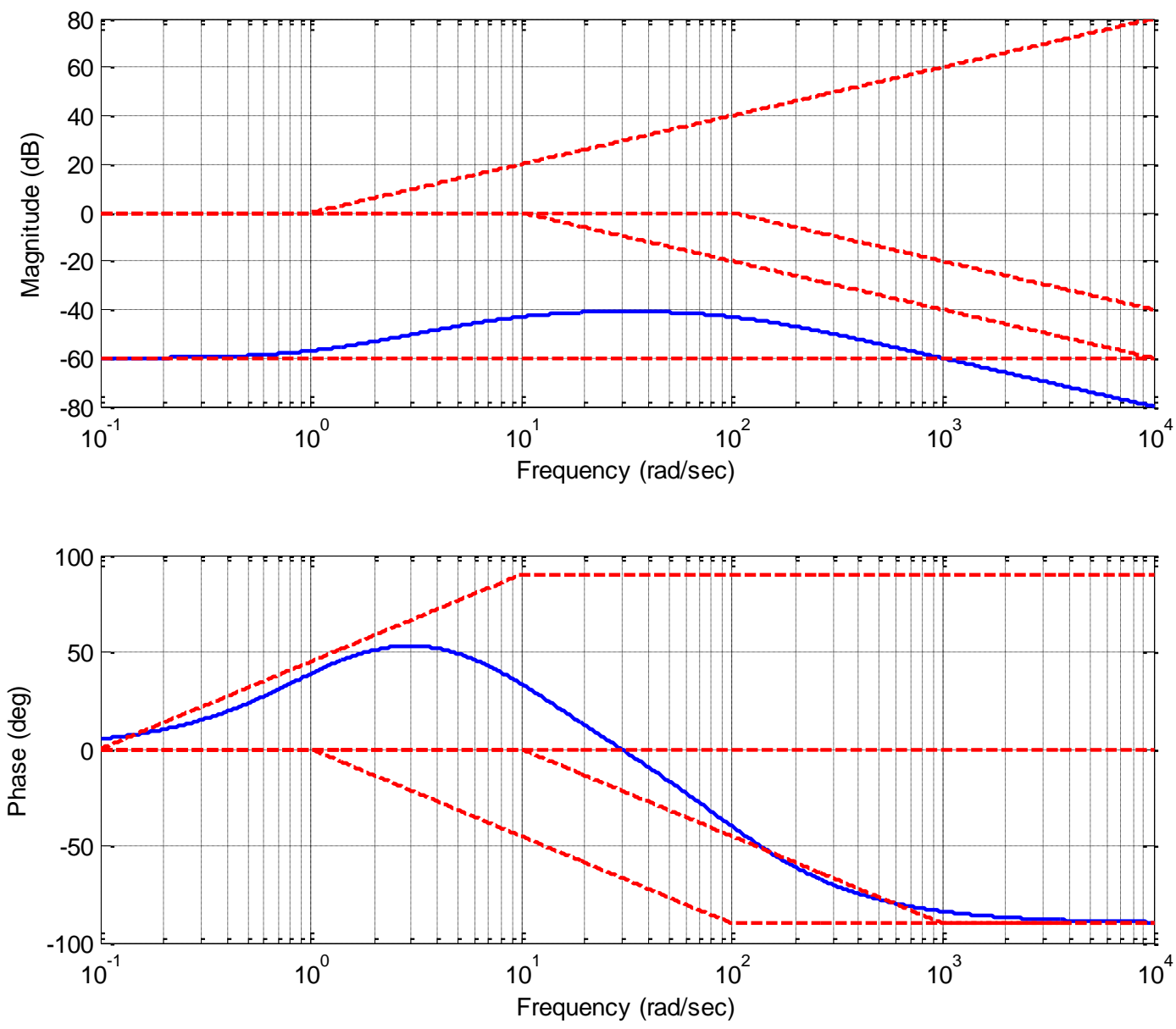
Figure 8.16. Bode plot for the transfer function $G(s) = \dfrac{10^{-3}(s+1)}{(0.1s+1)(0.01s+1)}$. The straight line approximations of the individual components are shown as dashed lines, the actual Bode plot is shown as a continuous line. Note that for any given frequency, if you sum the values of the straight line approximations you get reasonable approximations to the true (continuous curve) Bode plot.

Figure 8.17. Bode plot for the transfer function. $G(s) = \dfrac{10^{-3}s}{(0.01s+1)^2}$ The straight line approximations of the individual components are shown as dashed lines, the actual Bode plot is shown as a continuous line. Note that for any given frequency, if you sum the values of the straight line approximations you get reasonable approximations to the true (continuous curve) Bode plot.

8.5.3. Complex Conjugate Poles and Zeros

The basic building block for a system with complex conjugate poles or zeros can be put into the standard form

$$G(s) = \left(\frac{1}{\omega_n^2} s^2 + \frac{2\zeta}{\omega_n} s + 1 \right)^n$$

Evaluating this at $s = j\omega$ we get

$$G(j\omega) = \left(1 - \left[\frac{\omega}{\omega_n}\right]^2 + j2\zeta\frac{\omega}{\omega_n}\right)^n = \left(1 - u^2 + j2\zeta u\right)^n$$

where we have defined $u = \frac{\omega}{\omega_n}$. Computing the magnitude and phase we have

$$|G(j\omega)| = \left[(1 - u^2)^2 + (2\zeta u)^2\right]^n, \quad \angle G(j\omega) = n\tan^{-1}\left(\frac{2\zeta u}{1 - u^2}\right)$$

Let's first look at the magnitude in decibels,

$$|G(j\omega)|_{dB} = 20\log_{10}[(1 - u^2)^2 + (2\zeta u)^2]^{\frac{n}{2}} = 10n\log_{10}[(1 - u^2)^2 + (2\zeta u)^2]$$

For $\omega \approx 0$ we have $u \approx 0$, and $|G(j\omega)| \approx 1$, so $|G(j\omega)|_{dB} \approx 0$.

For larger frequencies, $\omega \gg \omega_n$, we have $|G(j\omega)| \approx u^{4n} = \left(\frac{\omega}{\omega_n}\right)^{4n}$, so in decibels we have

$$|G(j\omega)|_{dB} \approx 10n\log_{10}[u^4] = 40n\log_{10} u = 40n\log_{10}\omega - 40n\log_{10}\omega_n \approx 40n\log_{10}\omega$$

So for large frequencies we have a slope of 40n dB/decade.

For $\omega \approx \omega_n$ we have $u \approx 1$. Then we have $|G(j\omega)| \approx (2\zeta)^n$, and in decibels,

$$|G(j\omega)|_{dB} \approx 20n\log_{10}(2\zeta)$$

Now if we have two regions to consider,

$$0 < \zeta < 0.5 \quad \log_{10}(2\zeta) < 0$$
$$0.5 < \zeta < 1 \quad \log_{10}(2\zeta) > 0$$

Figure 8.18 displays the magnitude plot of a the second order system

$$G(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

for $K = 10, \omega_n = 20$, and $\zeta = 0.01, 0.1, 0.25 0.5, 0.75, 0.99$. Note that there is a sharp peak at the *resonant frequency*, which is given by $\omega_r = \omega_n\sqrt{1 - 2\zeta^2}$. Not that for $\zeta > \sqrt{0.5} = 0.707$ there is no resonance. As this figure shows, the smaller the damping ratio, the higher the peak frequency will be.

Now we need to look at the phase of the response of this system. For low frequencies, $u \approx 0$ we have $\angle G(j\omega) \approx n\tan^{-1}(2\zeta u) = 0^o$. For large frequencies, $\omega \gg \omega_n$ we have $\angle G(j\omega) \approx n\tan^{-1}\frac{2\zeta}{-\infty} \approx n180^o$.

Finally, for $\omega = \omega_n$ $(u = 1)$ we have $\angle G(j\omega) = n\tan^{-1}\left(\frac{2\zeta}{0}\right) = n90^o$. Figure 8.19 shows the phase plot of

the second order system $G(s) = \dfrac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$ for $K = 10, \omega_n = 20,$ and

$\zeta = 0.01, 0.1, 0.250.5, 0.75, 0.99$.

Notice that as the damping ratio is smaller, the phase transition is much sharper. Also notice that all of the phase plots pass through -90° when $\omega = \omega_n = 20\,\text{rad/sec}$.

We can then summarize our rules as

<div align="center">

Magnitude   *Phase*

</div>

$\omega \approx 0$ $0\,dB$ $\measuredangle H(j\omega) \approx 0^o\,at\,\omega = \dfrac{\omega_n}{10}\,\big(one\,decade\,before\,\omega_n\big)$

$\omega = \omega_n$ *depends on* ζ $\measuredangle H(j\omega) \approx n45^o$

$\omega \approx \infty$ $slope \approx 40n\,dB/decade$ $\measuredangle H(j\omega) \approx n180^o\,at\,\omega = 10\omega_n\,\big(one\,decade\,after\,\omega_n\big)$

8.6 Bandwidth, Filter Types, and Quality Factor

How a system affects an input signal is often used to classify the system type in terms of its *filtering* characteristics. Sometimes, when we are designing a control system for example, we are primarily concerned with making the output match the input with a reasonable transient response, and are not primarily concerned with the frequency response of our system. In other situations, such as designing a circuit for a music system, we are designing a system to enhance or remove frequencies from the input signal. In these cases we are interested in talking about these filtering characteristics that are most relevant in those applications. Three of the most useful system characteristics in filtering applications are the bandwidth of the filter, the filter type, and the quality of the filter.
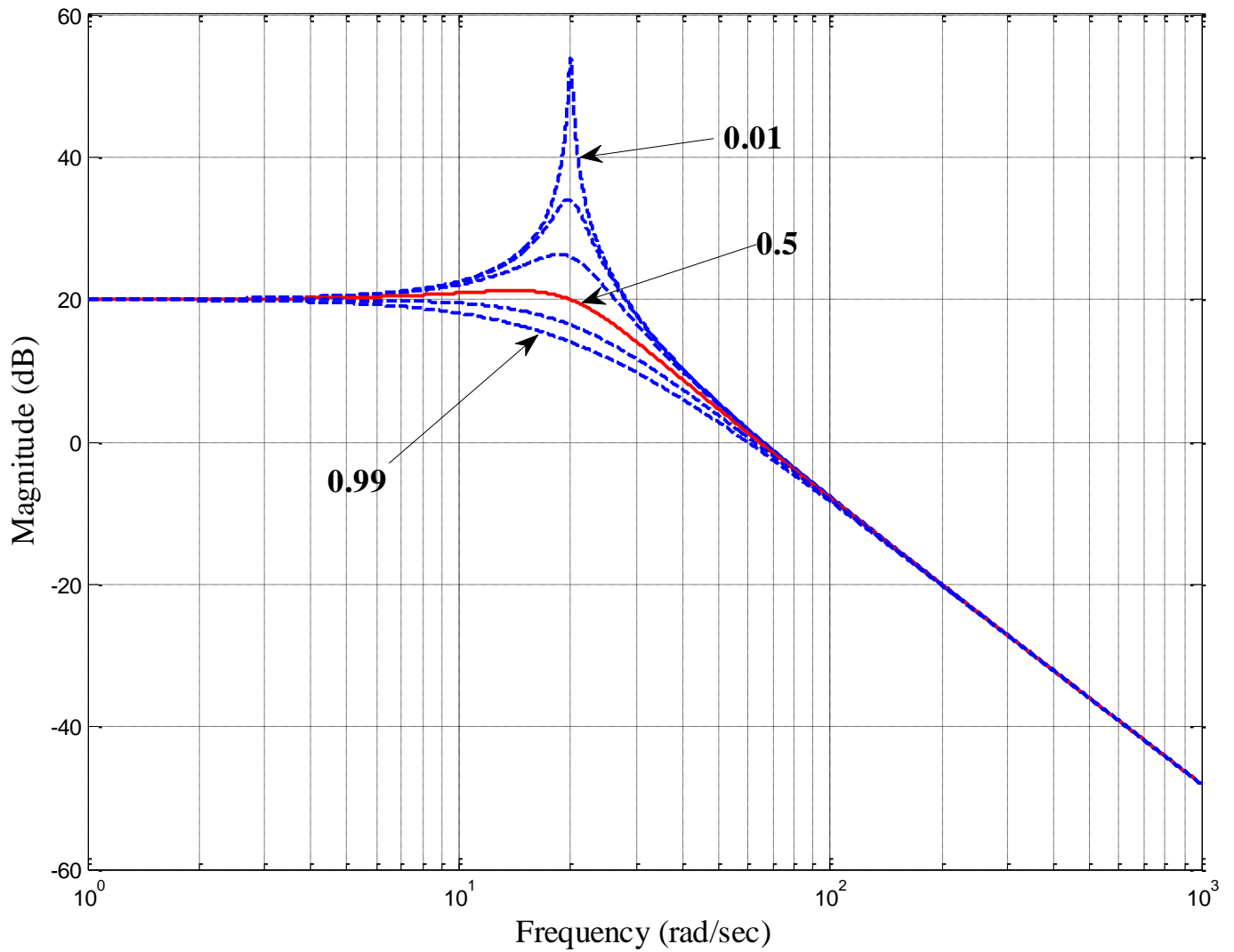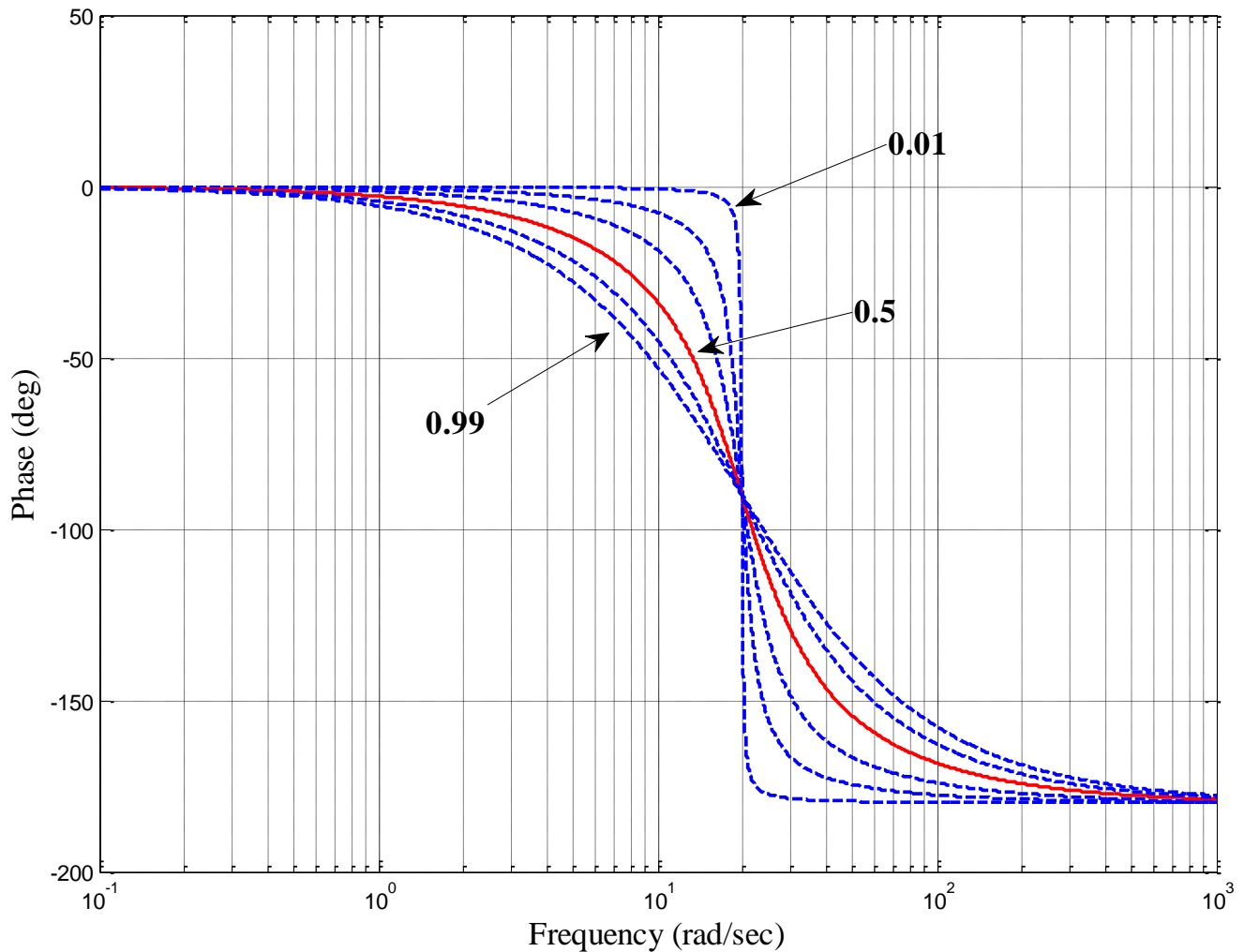
Figure 8.18. Magnitude of the frequency response for the transfer function $G(s) = \dfrac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$ for

$K = 10, \omega_n = 20,$ and $\zeta = 0.01, 0.1, 0.25 \, 0.5, 0.75, 0.99$

Figure 8.19. Magnitude of the frequency response for the transfer function $G(s) = \dfrac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$ for

$K = 10, \omega_n = 20,$ and $\zeta = 0.01, 0.1, 0.25, 0.5, 0.75, 0.99$

8.6.1 System Bandwidth

The system bandwidth is usually defined as the frequency range for which the power of the output is within one half of the peak output power. Thus we determine the bandwidth as the difference between the minimum and maximum frequencies as which

$$P_{out}(\omega) = \frac{1}{2} P_{out}^{max}$$

In terms of dB's we have

$$10\log_{10}\left[P_{out}(\omega)\right]=10\log_{10}\left[\frac{1}{2}P_{out}^{max}\right]=10\log_{10}\left[\frac{1}{2}\right]+10\log_{10}\left[P_{out}^{max}\right]$$

Then we have

$$10\log_{10}\left[\frac{1}{2}\right]\approx-3\,\text{dB}$$

Hence we determine the bandwidth as

$$Bandwidth=\omega_{high}-\omega_{low}$$

where

$$10\log_{10}[P_{out}(\omega)]_{\omega_{low}\leq\omega\leq\omega_{max}}\geq10\log_{10}[P_{out}^{max}]-3\,\text{dB}$$

This form is useful for reading information off of Bode plots. Sometimes we don't want to have to convert to Bode plots. Then we can just use

$$P_{out}(\omega)_{\omega_{low}\leq\omega\leq\omega_{max}}\geq\frac{1}{\sqrt{2}}P_{out}^{max}$$

In terms of transfer functions, we have

$$|H(j\omega)|_{\omega_{low}\leq\omega\leq\omega_{max}}\geq\frac{1}{\sqrt{2}}|H^{max}(j\omega)|$$

or

$$10\log_{10}\left[|H(j\omega)|\right]_{\omega_{low}\leq\omega\leq\omega_{max}}\geq10\log_{10}|H^{max}(j\omega)|-3\,\text{dB}$$

Example 8.6.1. Consider the RC circuit shown in Figure ??, The transfer function for this circuit is clearly

$$H(s)=\frac{1}{RCs+1}$$

In terms of frequency response we have

$$H(j\omega)=\frac{1}{j\omega RC+1}$$

The magnitude of the transfer function is then

$$|H(j\omega)|=\frac{1}{\sqrt{(RC\omega)^2+1}}$$

Clearly the maximum of this transfer function occurs when $\omega=0$, so for this system

$$|H^{max}(j\omega)|=1$$

We next need to find the frequency at which

$$|H(j\omega)|\leq\frac{1}{\sqrt{2}}|H^{max}(j\omega)|=\frac{1}{\sqrt{2}}$$

Since our transfer function is a decreasing function of frequency, we can just find the largest frequency for which this is true,

$$|H(j\omega)| = \frac{1}{\sqrt{(RC\omega)^2 + 1}} = \frac{1}{\sqrt{2}}$$

which leads to $RC\omega = 1$, or $\omega = \frac{1}{RC}$. Thus the bandwidth for this system is

$$\omega_{high} - \omega_{low} = \frac{1}{RC} - 0 = \frac{1}{RC}$$

A Bode plot of the magnitude of this transfer function for $R = 1\,k\Omega$, $C = 1\,\mu f$ is shown in Figure 8.21.

We expect the bandwidth to be $\frac{1}{RC} = 1000$ rad/sec. The maximum value of the transfer function is 1 (or 0 dB), and the bandwidth is then the frequency at which the magnitude has dropped 3 dB. This Bode plot has a dashed line at the -3 dB point, and you can see it intersects the magnitude of the transfer function at 1000 rad/sec.
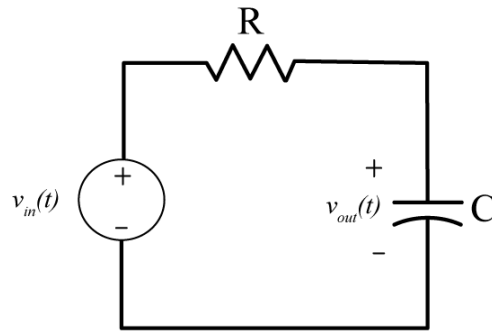
Figure 8.20. RC circuit for Example 8.6.1.

Figure 8.22 displays three different bandpass systems with bandwidths of 400, 100, and 50 radians/sec. In these figures, the -3 dB line is shown as a dashed line, and is measured from the peak amplitude.
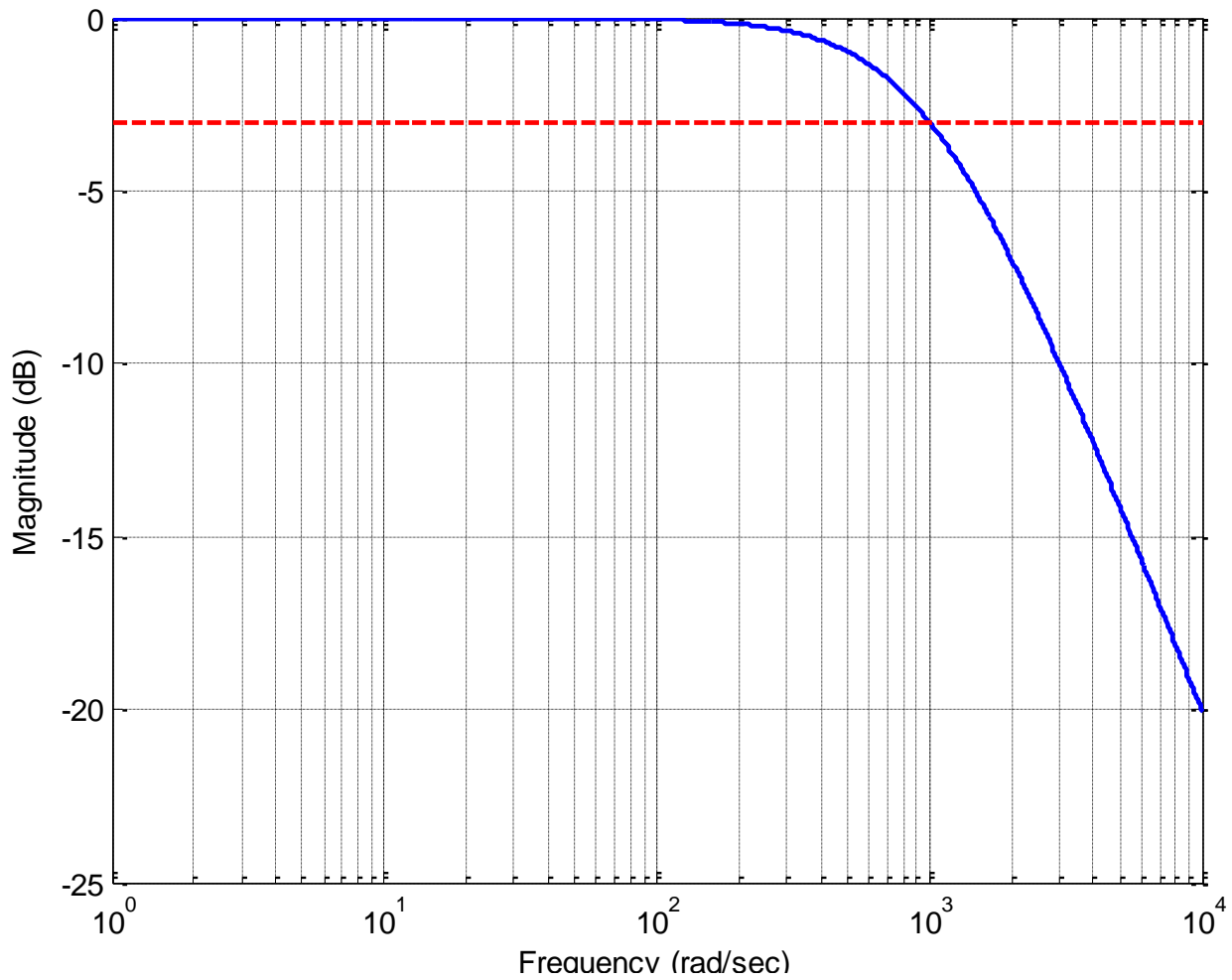
Figure 8.21. Bode plot of the magnitude of the transfer function for Example 8.6.1. The maximum value of the transfer function is 1 (0 dB), so the bandwidth will be determined by the point where the magnitude falls to – 3dB. This occurs at 1000 rad/sec.
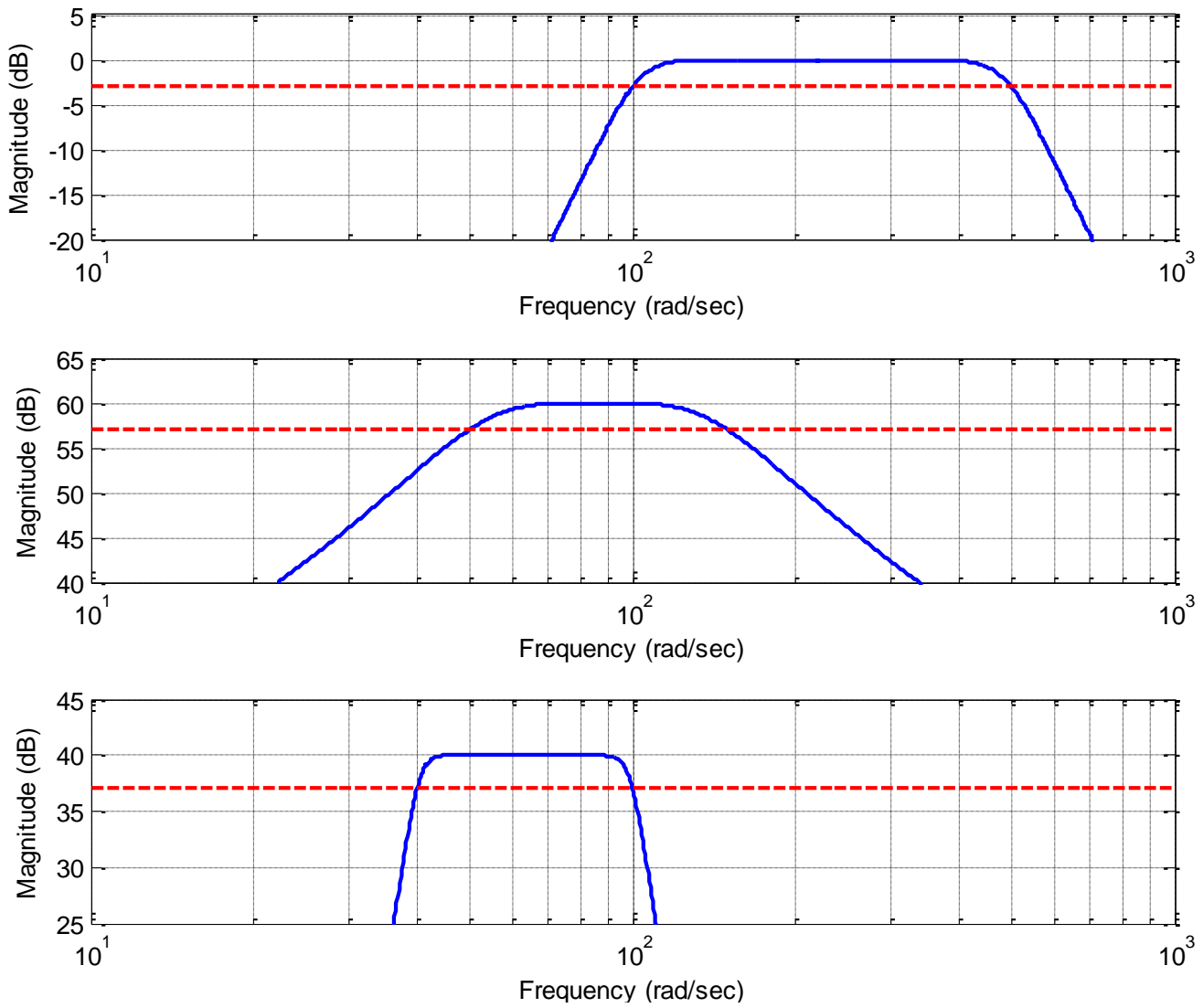
Figure 8.22. Three different bandpass systems with bandwidths of 400, 100, and 50 radians/sec. The -3 dB line is shown as a dashed line, and is measured from the peak amplitude.

8.6.2. Filter Types

In discussing filter types, it is good to remember the basic relationship that if the input signal to a stable system is $x(t) = A\cos(\omega_o t + \phi)$, then the steady state output of the system will be $y_{ss}(t) = A\,|\,H(j\omega_0)\,|\cos(\omega_o t + \phi + \angle H(j\omega_o))$. The filter type is determined by the magnitude of the transfer function at various frequencies, since this directly affects the output signal. Specifically, if the magnitude of the transfer function as a specific frequency is zero, the output signal is zero. Alternatively, if the magnitude of the transfer function at a specific frequency is one, then the input

signal passes without change. In talking about ideal filter types, this is generally what we are looking at: does the signal in a range of input frequencies *pass* or is the signal removed from the output.

The four major filter types are displayed in Figure 8.23, as ideal filters. Note that the magnitude of the transfer function is symmetric about the real axis, and we are plotting the magnitude response for both positive and negative frequencies. Although the phase of the filter is important in some applications, it is not important in classifying the filter type.

The first characterization is the *lowpass* filter, and this filter passes signals with low frequencies and removes high frequencies from the input signal. Note that the lowpass filter is centered at zero frequency, but the bandwidth is only measured from 0 to the edge of the passband. Next, a *highpass* filter removes all low frequencies and only allows high frequencies to pass. The bandwidth of a high pass filter is not really meaningful. A *bandpass* filter allows frequencies only in a range of frequencies to pass. Finally, a *bandreject* or *notch* filter removes only very specific frequencies. Notch filters are often used to remove 60 Hz noise from AC power systems.
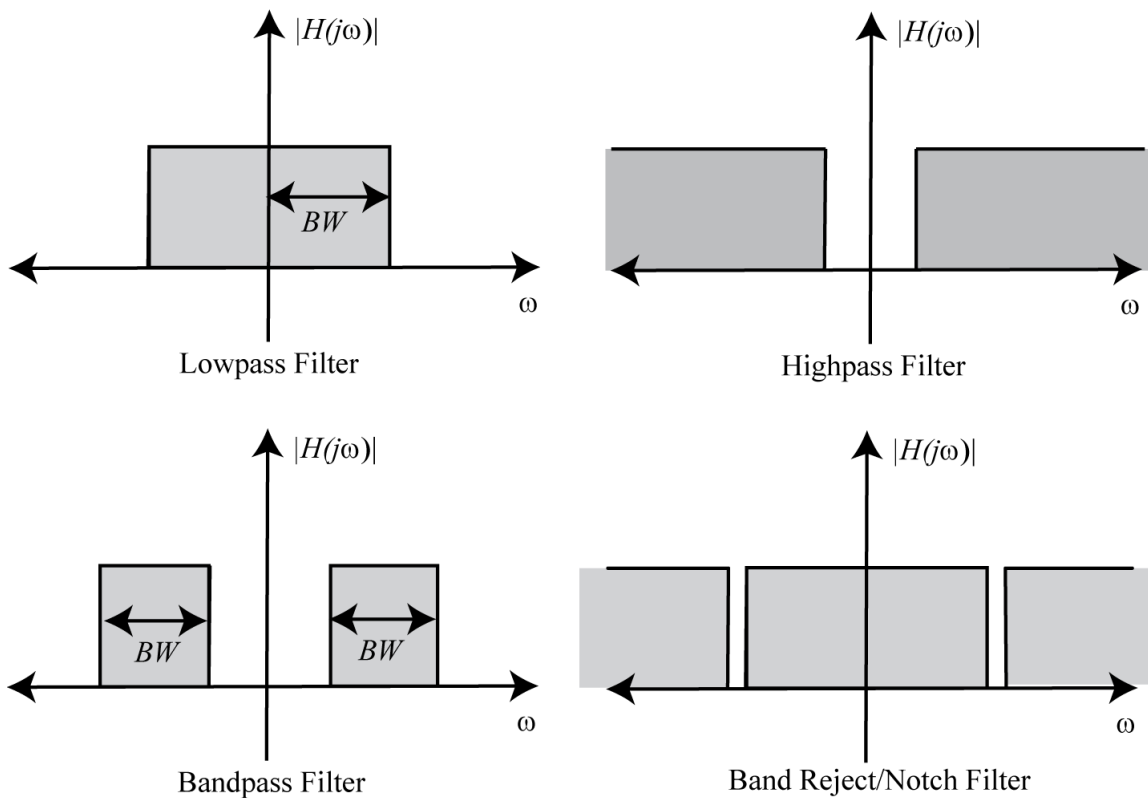
Figure 8.23. Ideal lowpass, highpass, bandpass, and bandreject filters.

8.6.3. Quality Factor of a Filter

The quality factor, or Q, of a bandpass filter is simply defined to be the ratio of the center frequency to the bandwidth of the filter.

$$Q = \frac{Center\ Frequency}{Band\ Width}$$

Note that the Q of a filter is dimensionless. The quality factor gives a measure of the width of the pass band, independent of the location of filter on the frequency axis. Figure 8.24 shows three filters with the same center frequency and different bandwidths, and hence different Q's. Note that the narrower the filter, the higher the Q (since the numerator is the same for all filters in this case). Figure 8.25 gives an example of three filters with the same Q, but at three different frequencies. Note that in this case, as the

center frequency is increased, the bandwidth must also increase to maintain the same Q.
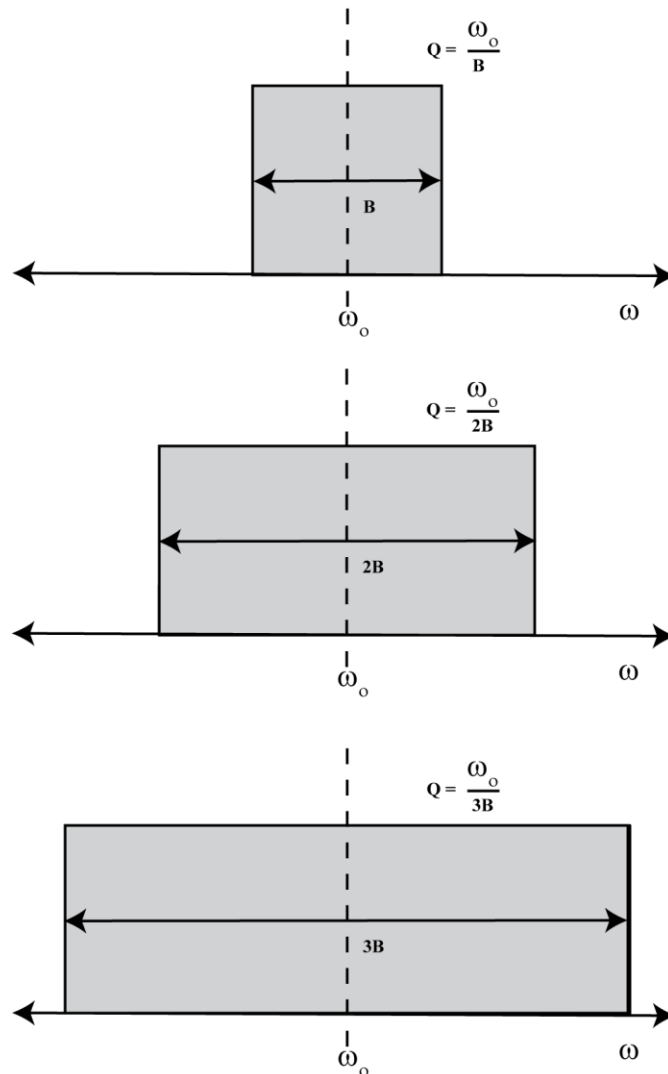
Figure 8.24. These three filters have the same center frequency but different quality (Q) factors. The filter quality factor decreases from the top filter to the bottom filter in this figure.
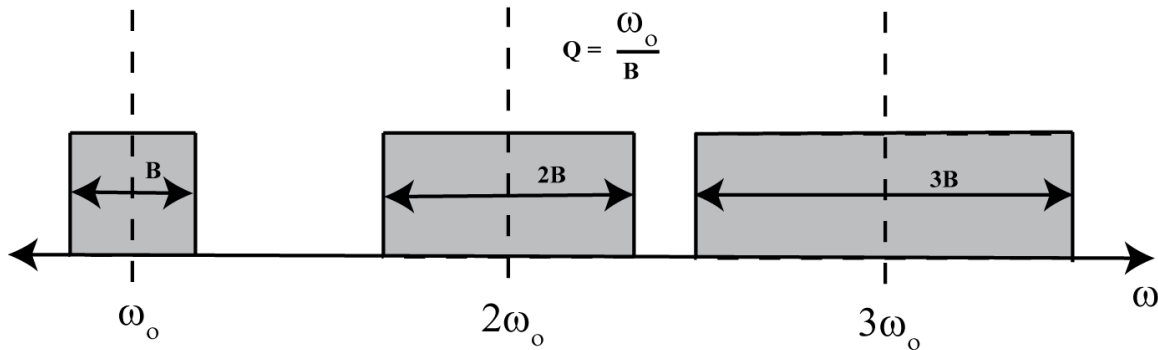
Figure 8.25. These three filters have the same quality factors.

8.7 Gain and Phase Margins

We now want to return to the concept of asymptotic stability, and determine if we can use the frequency response of a system to help us. Let's consider a closed loop system with transfer function given by

$$G_0(s) = \frac{G(s)}{1 + H(s)G(s)}$$

We know the system will be asymptotically stable if all of the poles of the closed loop system are in the (open) left half plane (the real parts of the poles are negative). When the system is asymptotically marginally stable, the poles are on the $j\omega$ axis. We can view this as the point at which the system is about to become unstable. If we have poles on the $j\omega$ axis, then we have

$$1 + H(j\omega)G(j\omega) = 0$$

We can rewrite this as

$$H(j\omega)G(j\omega) = -1$$

We can then break this into two different conditions, a *magnitude condition* and a *phase condition*:

$$magnitude\ condtion: \quad |H(j\omega)G(j\omega)| = 1$$
$$phase\ condition: \quad \angle H(j\omega)G(j\omega) = \pm 180^o$$

When both of these conditions are met, then the system is asymptotically marginally stable. However, we can also used these conditions to determine *how close to be unstable* is our system. The additional amplitude of $G(j\omega)H(j\omega)$ and the additional phase of $G(j\omega)H(j\omega)$ which result in purely imaginary poles of $G_0(s)$ are measures of the allowable tolerances or changes in $G(j\omega)H(j\omega)$ for system stability. This leads to the ideas of *gain margins* and *phase margins*.

We first define a *phase crossover frequency*, ω_p, to be any frequency at which we meet the phase condition. That is, at a phase crossover frequency we have

$$\measuredangle G(j\omega_p)H(j\omega_p) = \pm 180^o$$

We then define the _gain margin_ of a feedback system to be the additional amplitude necessary to make $|G(j\omega_p)H(j\omega_p)| = 1$. If there is more than one phase crossover frequency, we use the phase crossover frequency that produces the smallest gain margin. Specifically, we can write this as
\

$$\underbrace{|G(j\omega_p)H(j\omega_p)|}_{\substack{\textit{magnitude at phase} \\ \textit{crossover frequency}}} \times \text{Gain Margin} = 1$$

which we can rearrange as

$$\text{Gain Margin} = \frac{1}{|G(j\omega_p)H(j\omega_p)|}$$

We usually measure the gain margin in dB, so we have

$$\text{Gain Margin}_{dB} = 20\log_{10}(1) - 20\log_{10}\left(|G(j\omega_p)H(j\omega_p)|\right) = -20\log_{10}\left(|G(j\omega_p)H(j\omega_p)|\right)$$

or simply

$$\text{Gain Margin}_{dB} = -|G(j\omega_p)H(j\omega_p)|_{dB}$$

Note that if there is no phase crossover frequency, then the gain margin is infinite.

Next we define a *gain crossover frequency*, ω_g, to be any frequency at which we meet the magnitude condition. That is, at a gain crossover frequency we have

$$|G(j\omega_g)H(j\omega_g)| = 1$$

or,

$$|G(j\omega_g)H(j\omega_g)|_{dB} = 0$$

The _phase margin_ is then the additional negative phase necessary to make the phase of $G(j\omega_g)H(j\omega_g)$ equal to $\pm 180^o$, whichever is closer. We can write this condition as

$$\underbrace{\measuredangle G(j\omega_g)H(j\omega_g)}_{\substack{\textit{Phase at gain} \\ \textit{crossover frequency}}} - \text{Phase Margin} = \pm 180^o$$

Note that if there is no gain crossover frequency, then the phase margin is infinite.

In summary, the gain and phase margins are indications of how close to becoming asymptotically unstable our system is. Acceptable gain and phase margins will depend on the application. However, it can be shown *that if the system has no zeros in the right half plane, then the system is stable only if both*

the phase and gain margins are positive. The following examples illustrate determining the gain and phase margins using Bode plots of the frequency response.

Example 8.7.1. Consider the following feedback system, where we have $G(s) = \dfrac{20}{s+2}$ and

$H(s) = \dfrac{1}{s^2+4s+1}$. The bode plot of the open loop system $G(s)H(s)$ is displayed in Figure 8.26. In this figure, the gain crossover frequency is displayed as a triangle, and the phase crossover frequency is displayed as a circle. For this example, we have $\omega_g \approx 1.8$ rad/sec and $\omega_p \approx 3.0$ rad/sec. At the gain crossover frequency the phase of the system is approximately -150°, or $\angle G(j\omega_g)H(j\omega_g) \approx -150^o$. We then have

$$\angle G(j\omega_g)H(j\omega_g) - \text{Phase Margin} = -180^o \approx -150^o - \text{Phase Margin} = -180^o$$

which gives a phase margin of approximately 30°. Similarly, at the phase crossover frequency we have $|G(j\omega_p)H(j\omega_p)| \approx -9$ dB, so the gain margin is approximately 9 dB. Since both the phase and gain margins are positive, and there are no right hand plane zeros, this system is stable. In fact the poles of the closed loop system are at -5.09 and at $-0.46 \pm 2.03j$.

Example 8.7.2. Consider the following feedback system, where we have $G(s) = \dfrac{50}{s^2+s+100}$ and.
$H(s) = 1$ The bode plot of the open loop system $G(s)H(s)$ is displayed in Figure 8.27. In this figure, there are two gain crossover frequencies (displayed as triangles) and there is no phase crossover frequency. The first gain crossover frequency is $\omega_g \approx 7.1$ rad/sec. At this gain crossover frequency the phase of the system is approximately, -8° or $\angle G(j\omega_g)H(j\omega_g) \approx -8^o$. We then have

$$\angle G(j\omega_g)H(j\omega_g) - \text{Phase Margin} = -180^o \approx -8^o - \text{Phase Margin} = -180^o$$

which gives a phase margin of approximately 172°. The second gain crossover frequency is $\omega_g \approx 12.2$ rad/sec. At this gain crossover frequency the phase of the system is approximately -166°, or $\angle G(j\omega_g)H(j\omega_g) \approx -166^o$. We then have

$$\angle G(j\omega_g)H(j\omega_g) - \text{Phase Margin} = -180^o \approx -166^o - \text{Phase Margin} = -180^o$$

Which gives a phase margin of approximately 14°. The phase margin is the smaller of these two, so for our system the phase margin is 14°. The gain margin for this system is infinite. Since this system has no zeros in the right half plane and both the phase and gain margins are positive, the system is stable. The poles for this closed loop system are at $-0.50 \pm 12.24j$.
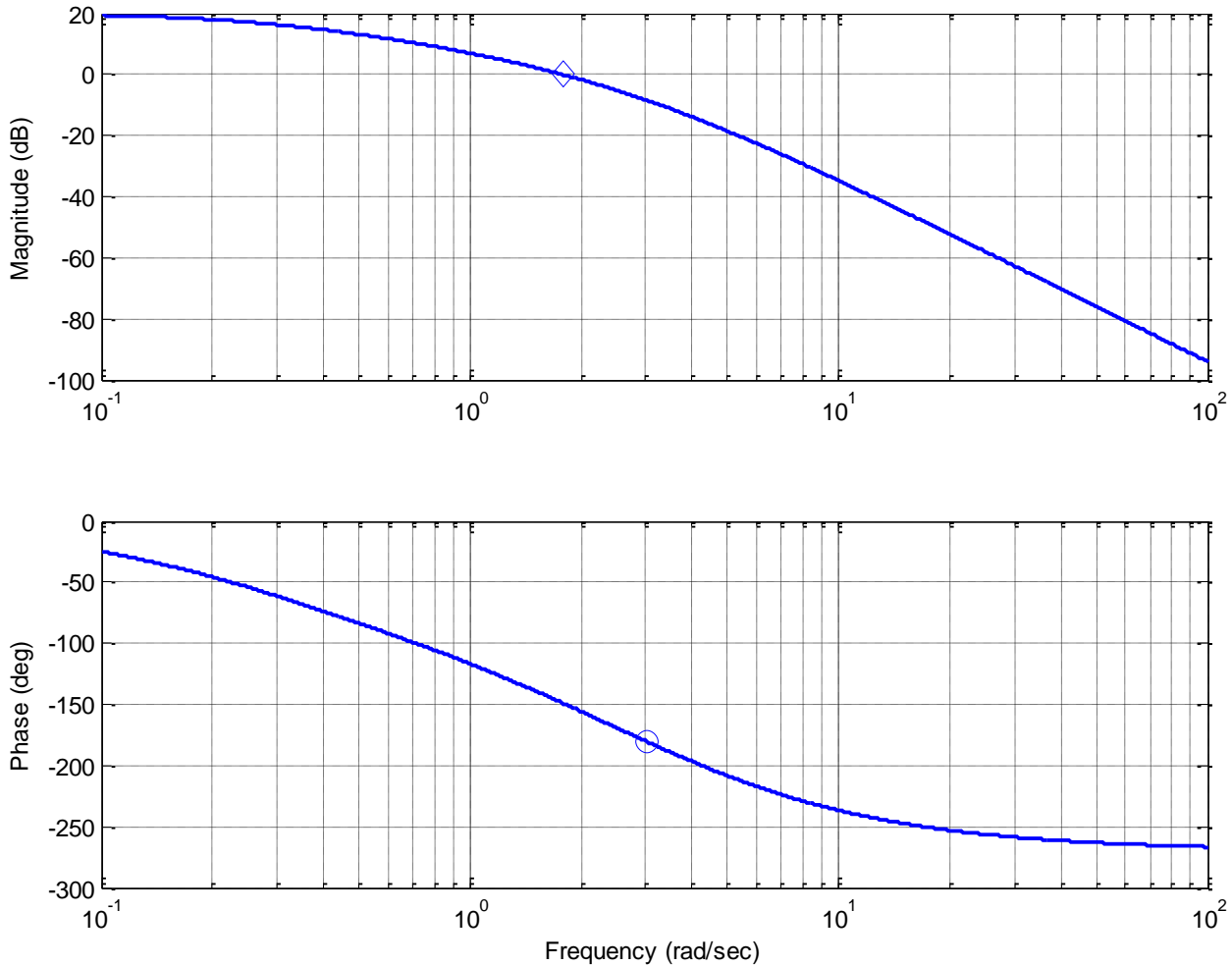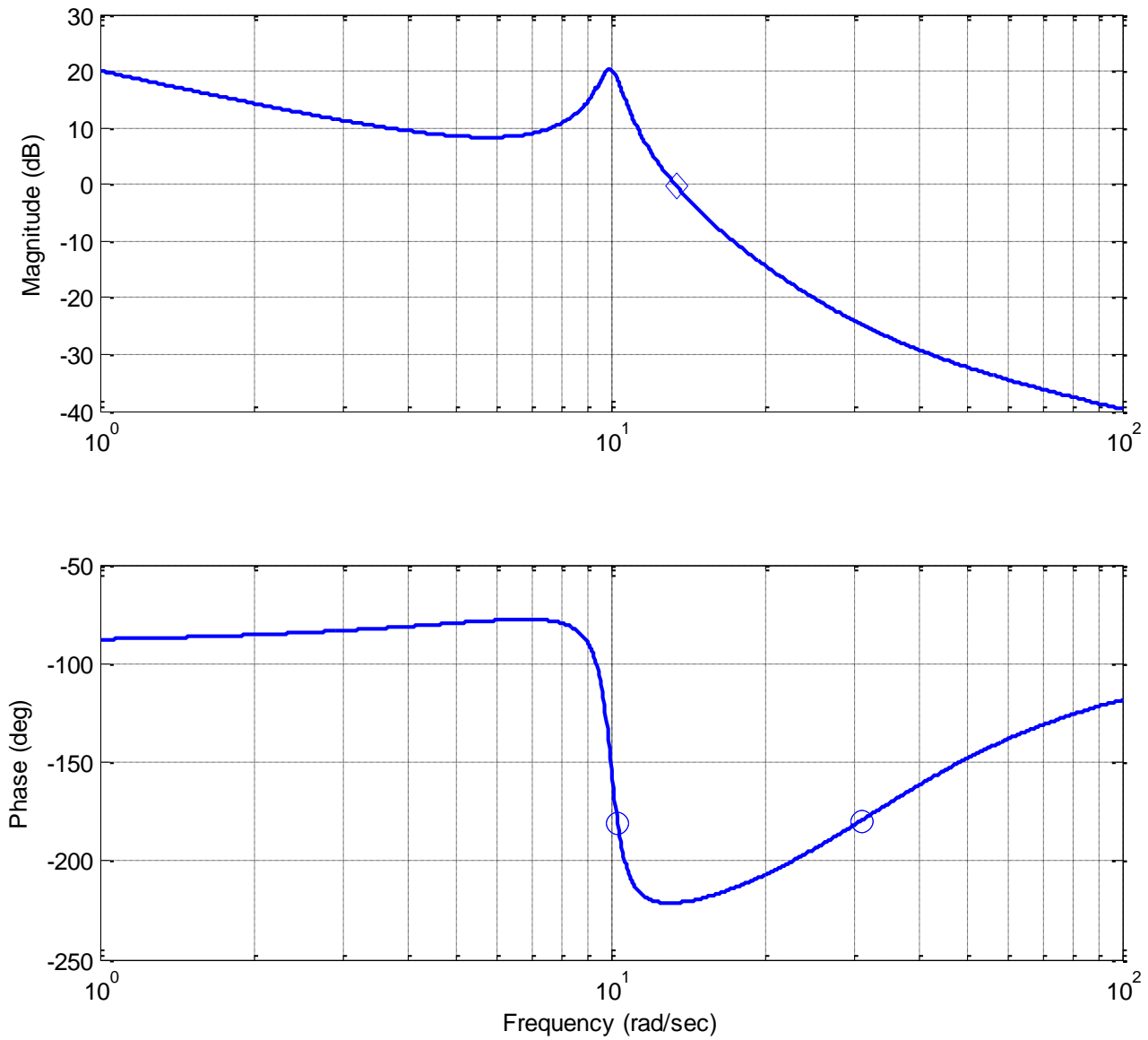
Figure 8.26. Bode plot for Example 8.7.1. The gain crossover frequency is shown as the diamond while the phase crossover frequency is shown as the circle. The phase margin for this system is approximately 30 degrees, and the gain margin is approximately 9 dB. Since the system has no poles in the right half plane and both the gain and phase margins are positive, the system is stable.

Example 8.8.3. Consider the following feedback system, where we have $G(s) = \dfrac{s^2 + 50s + 1000}{s^3 + s^2 + 100s}$ and. $H(s) = 1$ The bode plot of the open loop system $G(s)H(s)$ is displayed in Figure 8.28. In this figure, there are two phase crossover frequencies (displayed as circles) and there is one gain crossover frequency (displayed as a diamond). The gain crossover frequency is $\omega_g \approx 13$ rad/sec. At this gain crossover frequency the phase of the system is approximately -220°, or $\angle G(j\omega_g)H(j\omega_g) \approx -220^o$. We then have

$$\angle G(j\omega_g)H(j\omega_g) - \text{Phase Margin} = -180^o \approx -220^o - \text{Phase Margin} = -180^o$$

which gives a phase margin of approximately -40°. The two phase crossover frequencies are $\omega_p \approx 10$ rad/sec and $\omega_p \approx 30$ rad/sec. At the first phase crossover frequency we have $|G(j\omega_p)H(j\omega_p)| \approx -20$ dB, so the gain margin at this phase crossover frequency is approximately 20 dB. At the second phase crossover frequency we have $|G(j\omega_p)H(j\omega_p)| \approx -25$ dB, so the gain margin at this phase crossover frequency is approximately 25 dB. For this system then the phase margin is approximately -40° and the gain margin is approximately -25 dB. Since this system has no zeros in the right half plane the system is unstable and both the gain and phase margins are negative, the system is unstable. The poles of this system are in fact at -5.81 and $1.90 \pm 12.98j$.
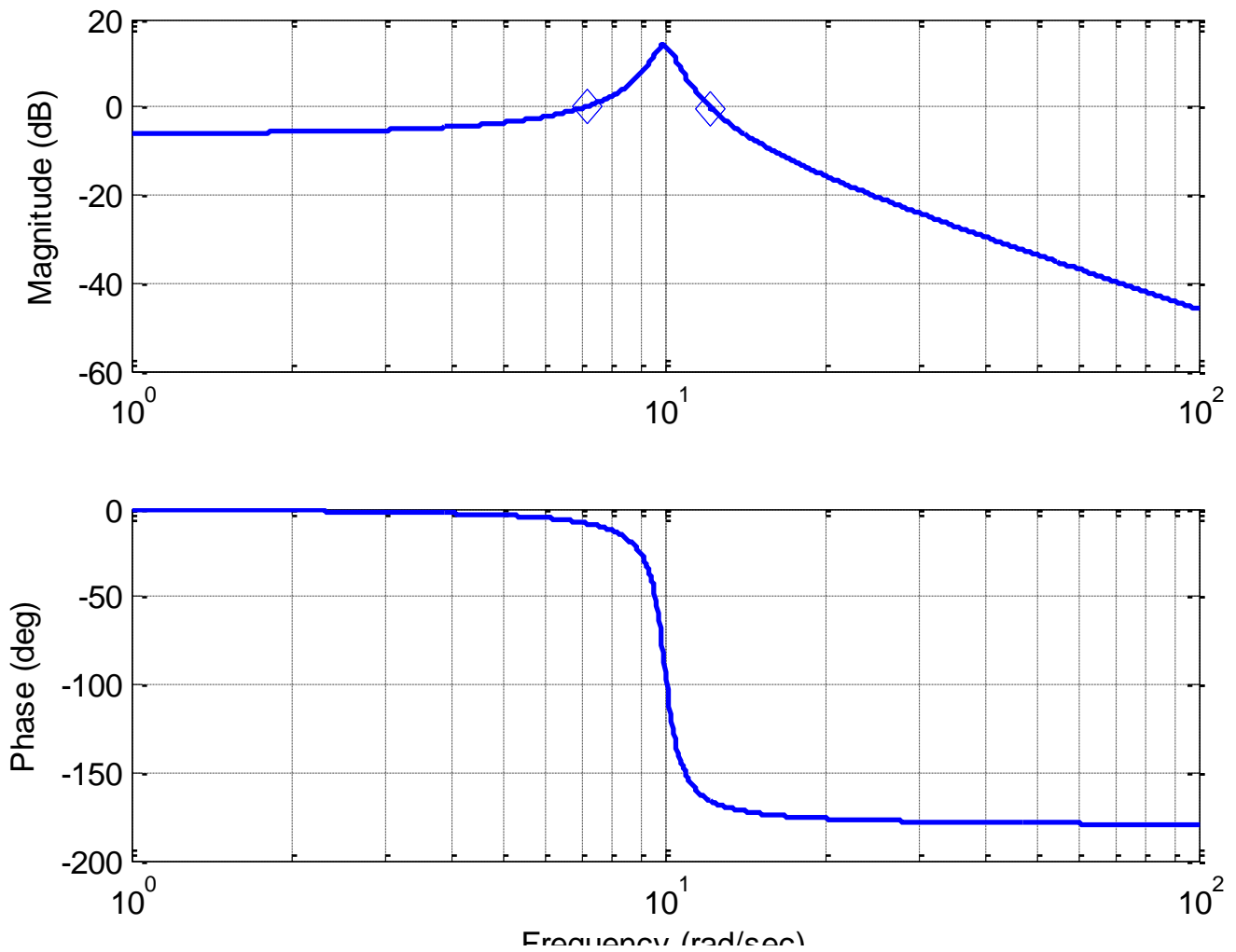
Figure 8.27. Bode plot for Example 8.7.2. The gain crossover frequencies are shown as diamonds. There is no phase crossover frequency. The phase margin for this system is approximately 14 degrees, and the gain margin is infinte.

Figure 8.28. Bode plot for Example 8.7.3. The gain crossover frequency is shown as a diamond, and the two phase crossoever frequency is shown as the open circles.

Chapter 8 Problems

1) For the following systems, determine

i) the settling time based on the pole locations

ii) the magnitude and phase of $H(j\omega_0)$ from the pole-zero diagram (graphically)

iii) the steady state output of the system

a) $H(s) = \dfrac{s+2}{(s+5+j)(s+5-j)}$, $x(t) = 4\cos(3t)u(t)$

b) $H(s) = \dfrac{s+5}{(s+2+6j)(s+2-6j)}$, $x(t) = 5\sin(4t+30^o)u(t)$

c) $H(s) = \dfrac{30}{(s+1)(s+4+4j)(s+4-4j)}$, $x(t) = 10\cos(2t-45^o)u(t)$

Answers:

$T_s = 0.8$, $y_{ss}(t) = 0.42\cos(3t-4.15^o)$

$T_s = 2$, $y_{ss}(t) = 1.11\sin(4t+34.97^o)$

$T_s = 4$, $y_{ss}(t) = 4.16\cos(2t-138.17^o)$

2) For the following transfer functions, using the provided Bode plots

i) determine the correct form for each building block, i.e, $K,(\tau s+1)^n,\left(\dfrac{1}{\omega_n^2}s^2 + \dfrac{2\zeta}{\omega_n}s+1\right)^n$

ii) sketch and label each building block (magnitude and phase)

iii) sketch the straight line approximation to the Bode plot (magnitude and phase) by adding the building block components

a) $H(s) = \dfrac{100}{s+10}$

b) $H(s) = \dfrac{(s+100)}{s}$

c) $H(s) = \dfrac{(s+10)}{s(s+100)}$

d) $H(s) = \dfrac{1000(s+10)}{(s+1000)^2}$

3) The following three figures display the magnitude of six transfer functions. All of the poles and zeros of these transfer functions are in the left half plane (these are minimum phase transfer functions). All of the magnitudes, poles, and zeros are either zero or simple powers of 10 ($10^{-1},1,10^1,10^2$ etc). Estimate the transfer functions.
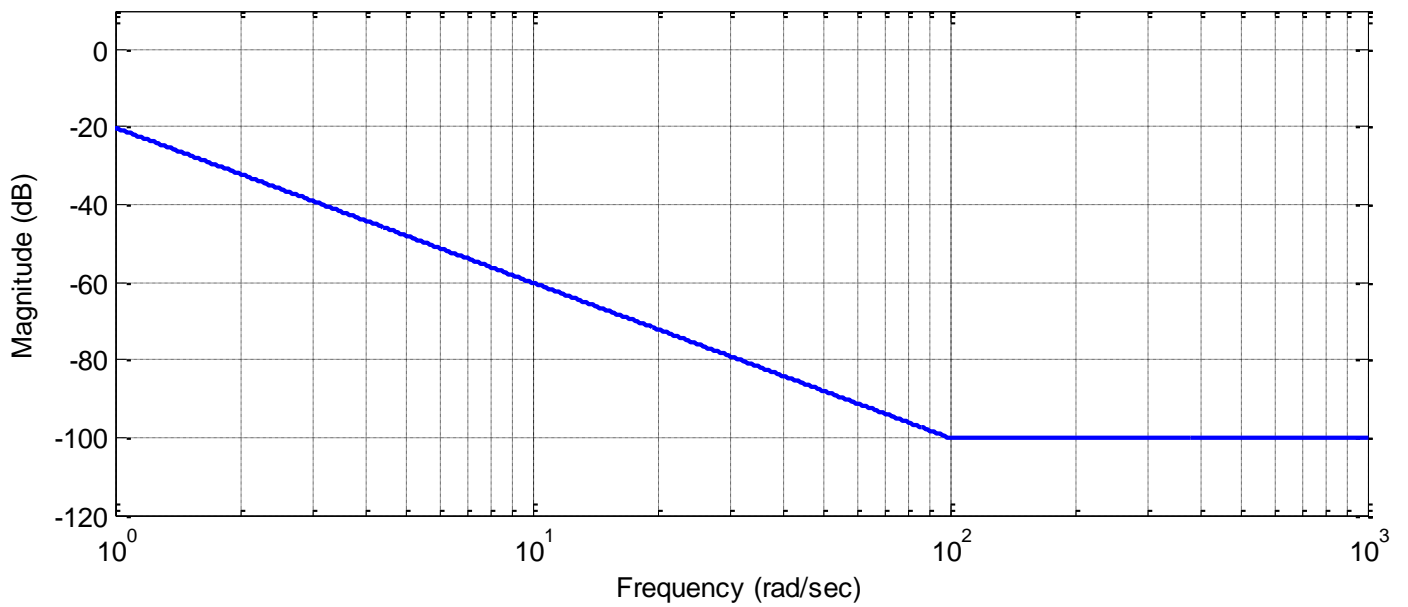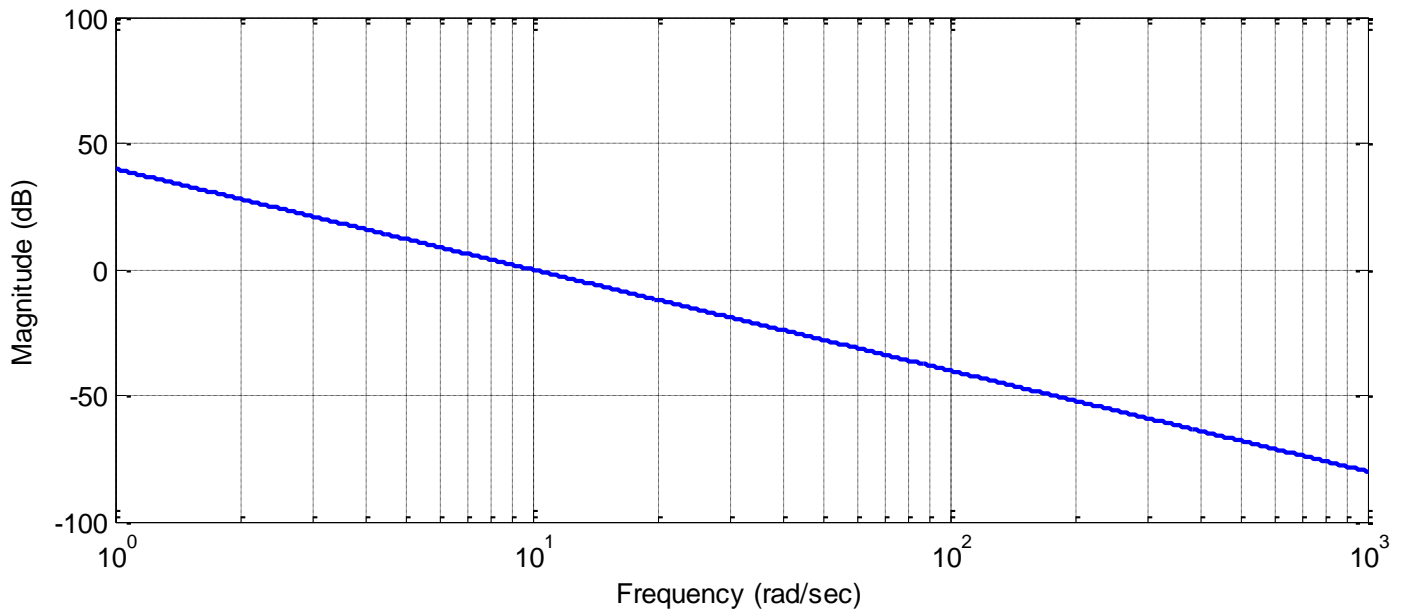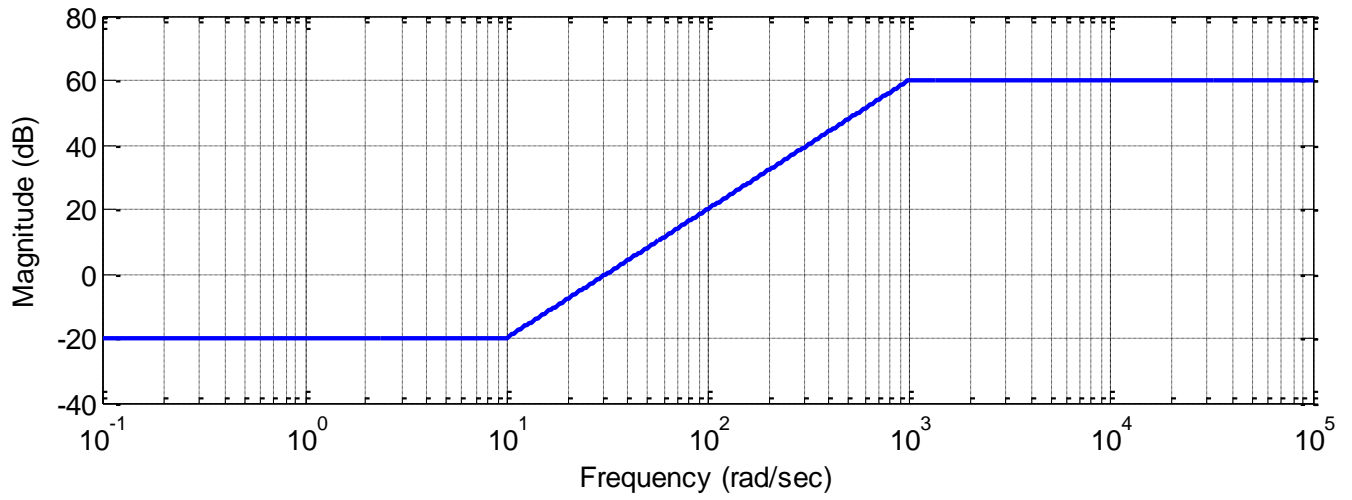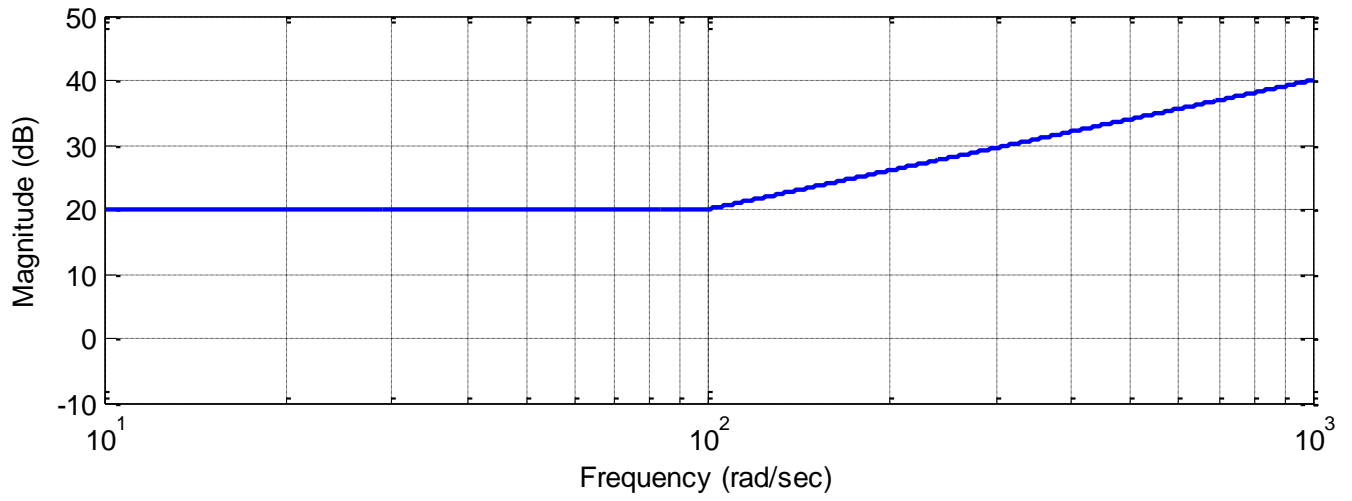
Figure 1: Problem 3, Systems a and b
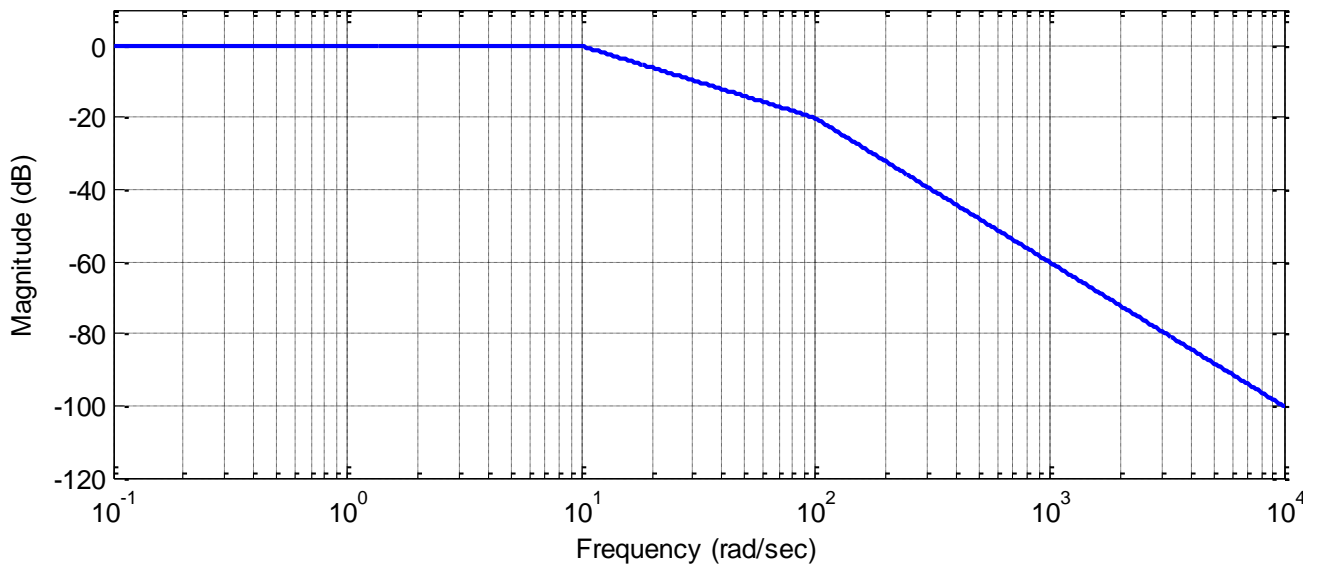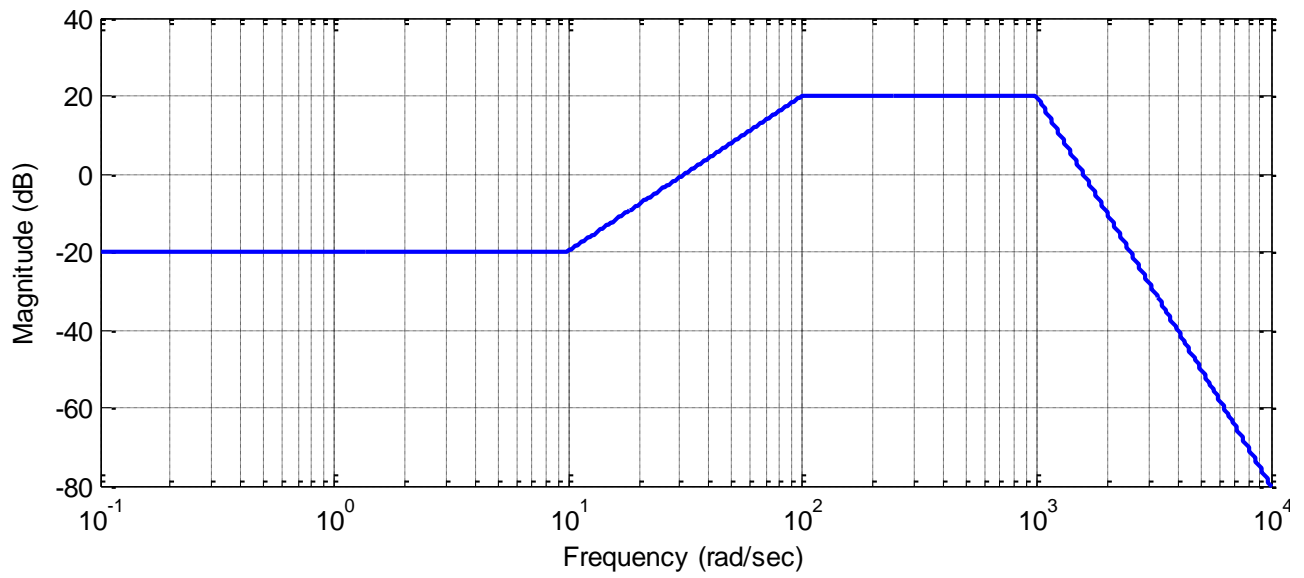
Figure 2: Problem 3, Systems c and d

Figure 3: Problem 3, Systems e and f

Answers:

$$H(s) = \frac{100}{s^2}, \quad H(s) = \frac{0.1\left(\frac{1}{100}s+1\right)^2}{s^2}, \quad H(s) = 10\left(\frac{1}{100}s+1\right)$$

$$H(s) = \frac{0.1\left(\frac{1}{10}s+1\right)^2}{\left(\frac{1}{1000}s+1\right)^2}, \quad H(s) = \frac{0.1\left(\frac{1}{10}s+1\right)^2}{\left(\frac{1}{100}s+1\right)^2\left(\frac{1}{1000}s+1\right)^5}, \quad H(s) = \frac{1}{\left(\frac{1}{10}s+1\right)\left(\frac{1}{100}s+1\right)}$$

4) The ***cutoff frequency*** of a filter is the frequency at which the amplitude of the transfer function is equal to $\frac{1}{\sqrt{2}}$ of its maximum value. This is the half power or 3 dB point. For each of the following systems, determine the cutoff frequency

$$H_a(s) = \frac{1}{s+p}, \quad H_b(s) = \frac{1}{as+p}, \quad H_c(s) = \frac{b}{as+p}$$

Answers: $\omega_c = p, \omega_c = \dfrac{p}{a}$

5) In this problem we will relate the settling time of a system with the bandwidth of a system if there are not complex poles. Consider a transfer function for a first order system, $G(s) = \dfrac{B}{s+p}$. Show that for this system, the product of the (2%) settling time and the bandwidth is equal to 4. Thus *to decrease the settling time you need to increase the bandwidth, and visa versa.*

**6)** Consider the following simple feedback control block diagram. The plant is $G_p(s) = \dfrac{2}{s+4}$.
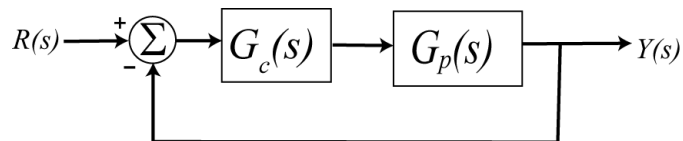

a) What is the bandwidth of the plant alone (assuming there is no feedback)

b) Assuming a proportional controller, $G_c(s) = k_p$, determine the closed loop transfer function, $G_0(s)$

c) Assuming a proportional controller, $G_c(s) = k_p$, determine the value of k_p so the bandwidth of the closed loop system is 24 rad/sec.

d) Assuming the proportional controller from problem c, determine the settling time and the steady state error for a unit step.

Partial Answers: 4, 10, 1/6

7) For the following four systems, determine the *bandwidth* of the system. For *lowpass* systems, the bandwidth is equal to the cutoff frequency. For *bandpass* systems, the bandwidth is the difference between the low and the high frequency cutoffs. For the systems with the magnitude portion of their Bode plots shown below, determine the bandwidth of the systems.

Answers: 65 Hz, 110 Hz, 470 Hz, 25 Hz
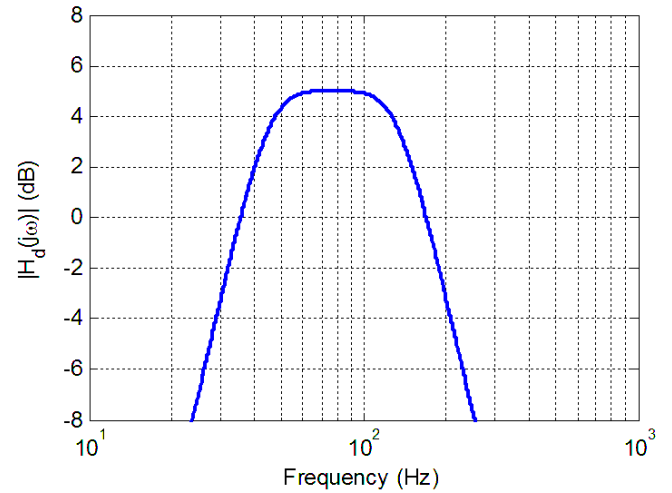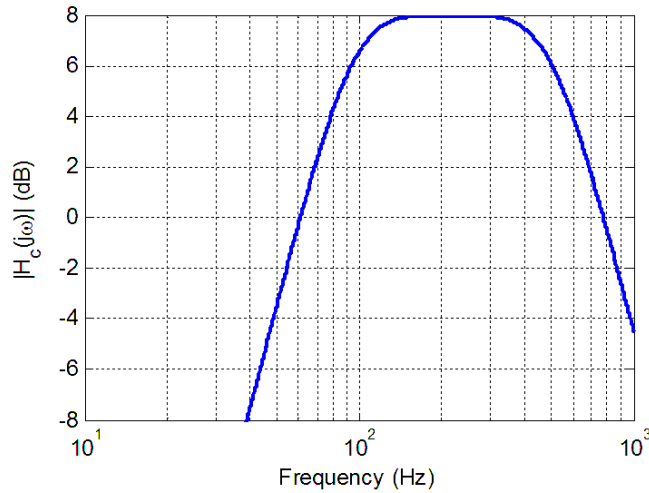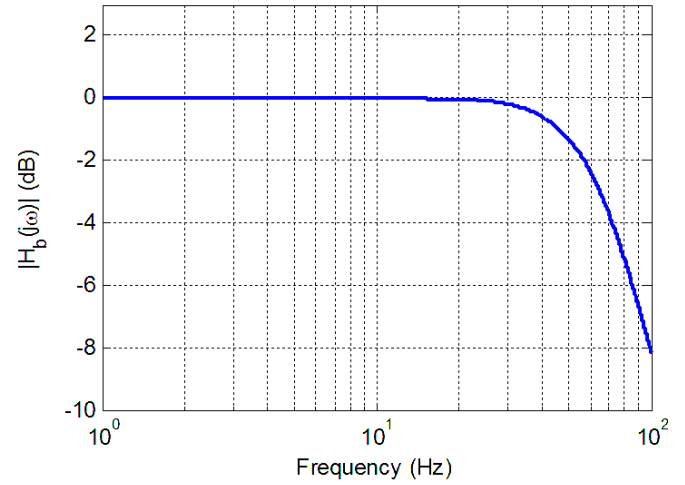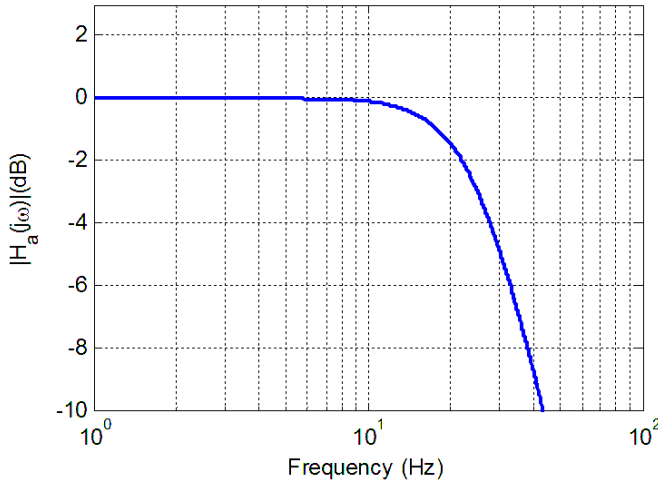
8) (Matlab Problem) Use Matlab to construct bode plots of the transfer functions you derived in problem 3. Print out each plot and turn it in.

Matlab Hints:

(1) If we want to plot a Bode plot of the transfer function

$$G(s) = \frac{0.1\left[\frac{1}{10}s+1\right]}{\left[\frac{1}{100}s+1\right]}$$

from $\omega = 10^{-2}$ to $\omega = 10^{5}$, then we would type the following in Matlab

G = tf(0.1*[1/10 1],[1/100 1]);

Bode(G,logspace(-2,5,1000); grid;

(2) If we have a transfer function with repeated poles, such as

$$G(s) = \frac{\left[\frac{1}{100}s+1\right]^{3}}{\left[\frac{1}{10}s+1\right]^{4}}$$

we want Matlab to do the work of multiplying these out. To multiply polynomials in Matlab, we use the command conv, which performs a discrete-time convolution. To construct the above transfer function then, we would type something like the following:

np = [1/100 1];

n = conv(np,np); % raised to the second power

n = conv(n,np); % raised to the third power

dp = [1/10 1];

d = conv(dp,dp); % raised to the second power

d = conv(d,dp); % raised to the third power

d = conv(d,dp); % raised to the fourth power

G = tf(n,d);

(3) If we want to make the transfer function $G(s) = \dfrac{1}{(s+1)(s+5)}$ we would type something like the

following:

G = tf(1,conv([1 1],[1 5]);

9) (Matlab Problem) The Matlab routine **homework10.m** plots the magnitude of the frequency response of the transfer function based on the location of the poles and zeros entered in lines 8 and 9 of the code. This magnitude plot is not in Db and the amplitude has been scaled to have a maximum value of 1 for all of the plots. All you are asked to do in this problem is change the location of the poles and the zeros, run the code, and turn in your plots. You can save your plots to a word file and then turn that in , but do not put more than three plots in a page, and the grader must be able to read them. However, try to think about the pole-zero plots you made in problem 1 as you do the problem.

a) Run the code with poles at $-10\pm10j, -5\pm10j, -1\pm10j$ and no zeros (leave the variable zeros as an empty array, do not put in a zero). Note that as the magnitude of the imaginary part of the pole gets significantly larger than the magnitude of the real part, we see well defined peaks in the frequency response. *Turn in all three plots.*
b) Leave the poles at $-1\pm10j$ and add zeros first at 0, then at 0 and $\pm5j$, then at 0 and $\pm15j$. Note that at the frequency where a zero of the transfer function occurs, the output of the transfer function is ….zero! *Turn in all three plots.*