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ECE-205

Exam 1

Winter 2010-2011

Calculators can only be used for simple calculations. Solving integrals, differential equations, systems of equations, etc. does not count as a simple calculation.

You must show your work to receive credit.

Problem 1 ________/20

Problem 2 ________/30

Problem 3 ________/18

Problem 4-11 ________/32

Total ____________

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1) (20 points) Assume we have a first order system with the governing differential equation

$$0.6\dot{y}(t) + y(t) = 2x(t)$$

The system has the initial value of 2, so $y(0) = 2$. The input to this system is

$$x(t) = \begin{cases} 0 & t < 0 \\ 1 & 0 \le t < 1 \\ -2 & 1 \le t < 3 \\ 3 & 3 < t \end{cases}$$

Determine the output of the system in each of the above time intervals. *Simplify your final answer as much as possible and box it. Be sure to include the correct initial value in the first interval!*

$$y(t) = [y(t_0) - y(\infty)]e^{-(t-t_0)/\tau} + y(\infty) \qquad K = 2,\ \tau = 0.6$$

$\underline{0 \le t \le 1}$ $\quad y(t_0) = 2 \quad y(\infty) = 2(1) = 2 \quad \tau = .6$

$$y(t) = [2-2]e^{-t/0.6} + 2 = \boxed{2 = y(t)}$$

$\underline{1 \le t \le 3}$ $\quad t_0 = 1 \quad y(1) = 2 \quad y(\infty) = 2(-2) = -4$

$$y(t) = [2-(-4)]e^{-(t-1)/0.6} - 4 = \boxed{6e^{-(t-1)/0.6} - 4 = y(t)}$$

$3 \le t$ $\quad t_0 = 3 \quad y(3) = 6e^{-2/0.6} - 4 = -3.786 \qquad y(\infty) = 2(3) = 6$

$$y(t) = [-3.786 - 6]e^{-(t-3)/0.6} + 6 = \boxed{-9.786e^{-(t-3)/0.6} + 6 = y(t)}$$

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2) (30 points) For the following three differential equations, assume the input is $x(t) = 4u(t)$ (the input is equal to four for time greater than zero), and the initial conditions are $y(0) = \dot{y}(0) = 0$

Determine the solution to each of the following differential equations and put your final answer in a box. Be sure to use the initial conditions to solve for all unknowns. You must show all your work to receive credit.

a) $\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = x(t)$ $\quad 2y_f = 4 \quad y_f = 2 \quad r^2 + 3r + 2 = 0 \quad (r+1)(r+2) = 0$

$y(t) = c_1 e^{-2t} + c_2 e^{-t} + 2 \qquad y(0) = 0 = c_1 + c_2 + 2 \qquad$ adding, $-c_1 + 2 = 0$

$\dot{y}(t) = -2c_1 e^{-2t} - c_2 e^{-t} \qquad \dot{y}(0) = 0 = -2c_1 - c_2 \qquad c_1 = 2$

$c_2 = -4$

$$\boxed{y(t) = 2e^{-2t} - 4e^{-t} + 2}$$

b) $\ddot{y}(t) + 4\dot{y}(t) + 4y(t) = 8x(t)$ $\quad 4y_f = 8 \cdot 4 = 32 \quad y_f = 8 \quad r^2 + 4r + 4 = 0 \quad (r+2)^2 = 0$

$y(t) = c_1 e^{-2t} + c_2 t e^{-2t} + 8 \qquad y(0) = 0 = c_1 + 0 + 8 \qquad c_1 = -8$

$\dot{y}(t) = -2c_1 e^{-2t} + c_2 e^{-2t} - 2c_2 t e^{-2t} \qquad \dot{y}(0) = 0 = -2c_1 + c_2 \qquad c_2 = -16$

$$\boxed{y(t) = 8 - 8e^{-2t} - 16te^{-2t}}$$

c) $\ddot{y}(t) + 4\dot{y}(t) + 16y(t) = 4x(t)$ $\quad 16y_f = 4 \cdot 4 = 16 \quad y_f = 1 \quad r^2 + 4r + 16 = 0$

$r = \frac{-4 \pm \sqrt{16 - 4(16)}}{2} = -2 \pm j2\sqrt{3}$

$y(t) = 1 + ce^{-2t}\sin(2\sqrt{3}t + \theta)$

$y(0) = 0 = 1 + c\sin(\theta) \qquad c = -1/\sin(\theta)$

$\dot{y}(t) = -2ce^{-2t}\sin(2\sqrt{3}t + \theta) + 2\sqrt{3}ce^{-2t}\cos(2\sqrt{3}t + \theta)$

$\dot{y}(0) = 0 = -2\sin(\theta) + 2\sqrt{3}\cos(\theta)$

$\frac{\sin(\theta)}{\cos(\theta)} = \tan(\theta) = \frac{2\sqrt{3}}{2} = \sqrt{3} \qquad \theta = 60^\circ \qquad c = -1/\sin(60^\circ) = -1.1547$

$$\boxed{y(t) = 1 - 1.1547e^{-2t}\sin(2\sqrt{3}t + 60^\circ)}$$

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3) (18 points) The form of the under damped ($0 < \zeta < 1$) solution to the second order differential equation

$$\ddot{y}(t) + 2\zeta\omega_n \dot{y}(t) + \omega_n^2 y(t) = K\,\omega_n^2 x(t)$$

for a step input $x(t) = Au(t)$ is

$$y(t) = KA + ce^{-\zeta\omega_n t}\sin(\omega_d t + \phi)$$

where c and ϕ are constants to be determined and the damped frequency $\omega_d = \omega_n\sqrt{1-\zeta^2}$

a) Using the initial condition $\dot{y}(0) = 0$ show that $\tan(\phi) = \dfrac{\sqrt{1-\zeta^2}}{\zeta}$

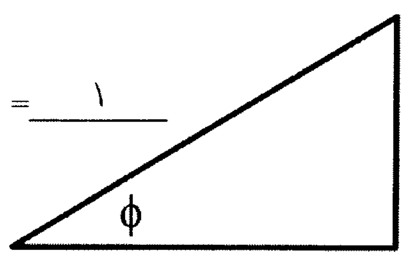
$$\dot{y}(t) = -\zeta\omega_n c e^{-\zeta\omega_n t}\sin(\omega_d t + \phi) + \omega_d c e^{-\zeta\omega_n t}\cos(\omega_d t + \phi)$$

$$\dot{y}(0) = 0 = -\zeta\omega_n \sin(\phi) + \omega_d\cos(\phi)$$

$$\frac{\sin(\phi)}{\cos(\phi)} = \tan(\phi) = \frac{\omega_d}{\zeta\omega_n} = \frac{\omega_n\sqrt{1-\zeta^2}}{\omega_n\zeta} \qquad \boxed{\tan(\phi) = \frac{\sqrt{1-\zeta^2}}{\zeta}}$$

b) We can express the relationship in part a using the following triangle. Fill in the blanks and then use this triangle determine an expression for $\sin(\phi)$.

r = 1



y = $\sqrt{1-\zeta^2}$

x = ζ

$$x^2 + y^2 = r^2 = \zeta^2 + 1 - \zeta^2 = 1$$

$$r = 1$$

$$\sin(\phi) = \frac{y}{r} = \frac{\sqrt{1-\zeta^2}}{1} \qquad \boxed{\sin(\phi) = \sqrt{1-\zeta^2}}$$

c) Use your answer to part b, and the initial condition $y(0) = 0$ to determine the remaining unknown constant, and write out the complete solution for $y(t)$.

$$y(0) = KA + c\sin(\phi) = 0 \qquad c = \frac{-KA}{\sin(\phi)} = \boxed{\frac{-KA}{\sqrt{1-\zeta^2}} = c}$$

$$\boxed{y(t) = KA\left[1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \phi)\right]}$$

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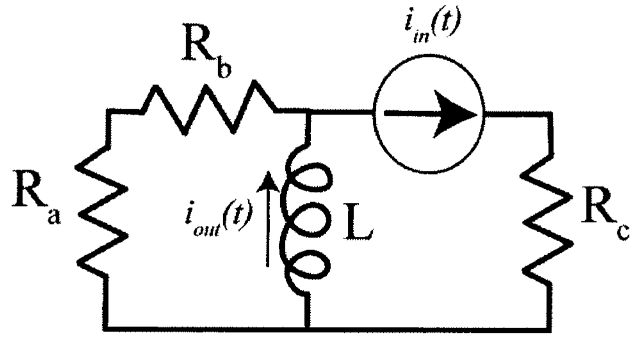
Problems 4-11, 4 points each, no partial credit (32 points)

Problems 4 and 5 refer to the following circuit



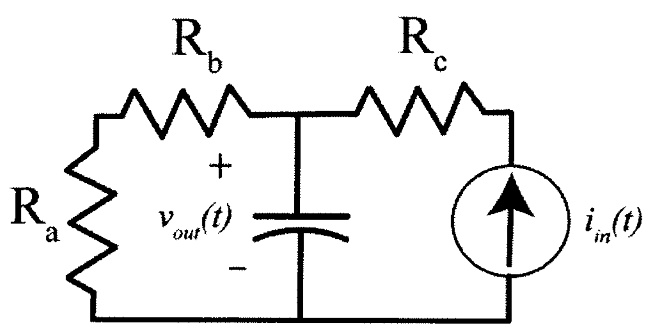
4) The Thevenin resistance seen from the ports of the inductor is

a) $R_{th}= R_c \,||\, (R_a + R_b)$ b) $R_{th}= R_c$ c) $R_{th}= R_a + R_b$ d) $R_{th}= R_a + R_b + R_c$ e) none of these

5) The static gain for the system is

a) $K=1$ b) $K=\dfrac{R_a+R_b}{R_a+R_b+R_c}$ c) $K=\dfrac{R_c}{R_a+R_b+R_c}$ d) $K=\dfrac{R_c}{R_a+R_b}$ e) none of these

Problems 6 and 7 refer to the following circuit



6) The Thevenin resistance seen from the ports of the capacitor is

a) $R_{th}=R_a+R_b$ b) $R_{th}=R_c$ c) $R_{th}=R_c \,||\, (R_a+R_b)$ d) $R_{th}=R_a+R_b+R_c$ e) none of these

7) The static gain for the system is

a) $K=1$ b) $K=R_c$ c) $K=R_a+R_b$ d) $K=R_c \,||\, (R_a+R_b)$ e) none of these

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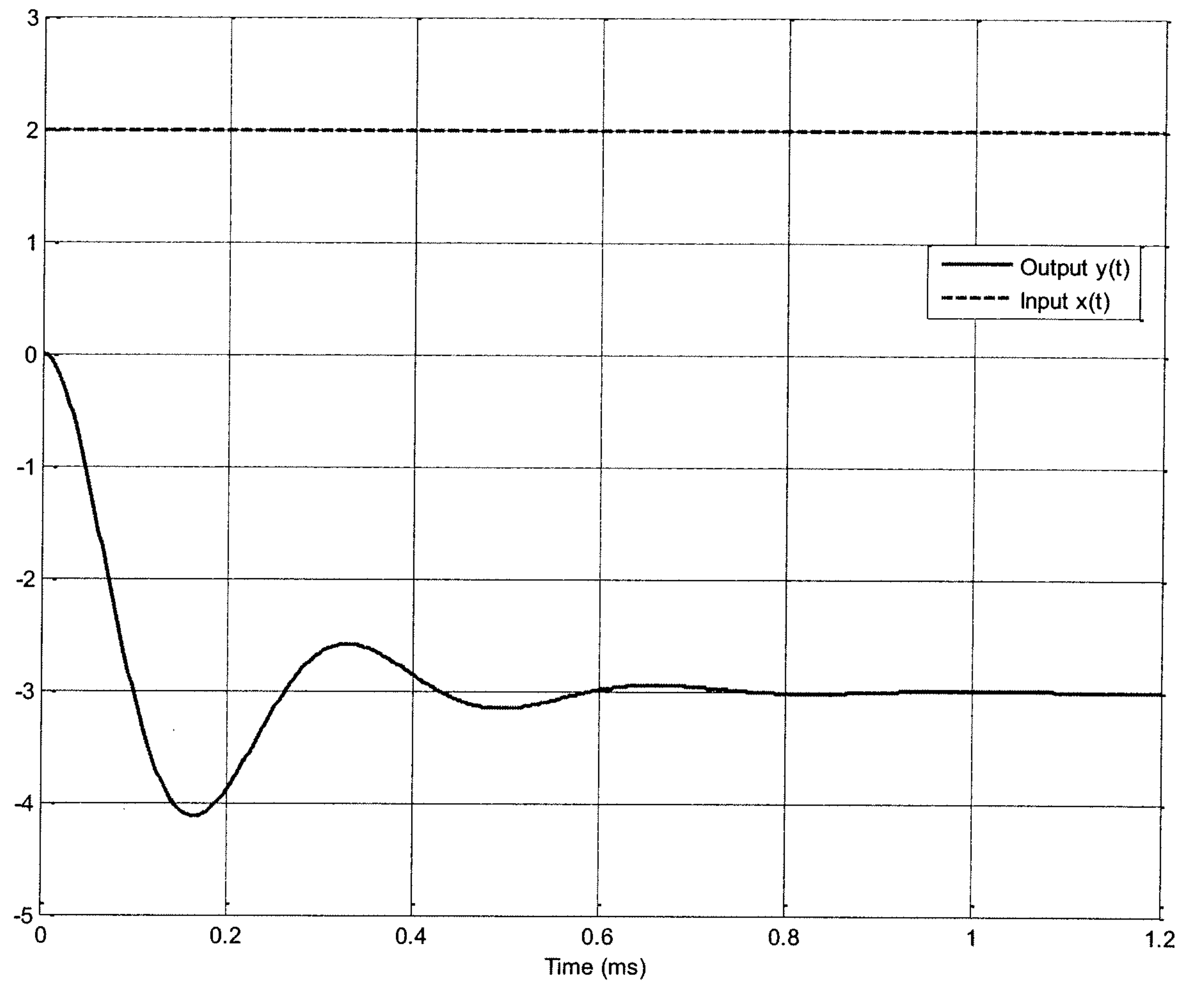
Problems 8 and 9 refer the following graph showing the response of a second order system to a step input.



8) The percent overshoot for this system is best estimated as

a) 400% b) -400 % c) 300% d) -300 % e) -33% f) 33%

$\frac{-4-(-3)}{-3} = \frac{-1}{-3} = \frac{1}{3}$

9) The static gain for this system is best estimated as

a) 1.5 b) 3 c) -1.5 d) -3

$K(2) = -3$

$K = -\frac{3}{2}$

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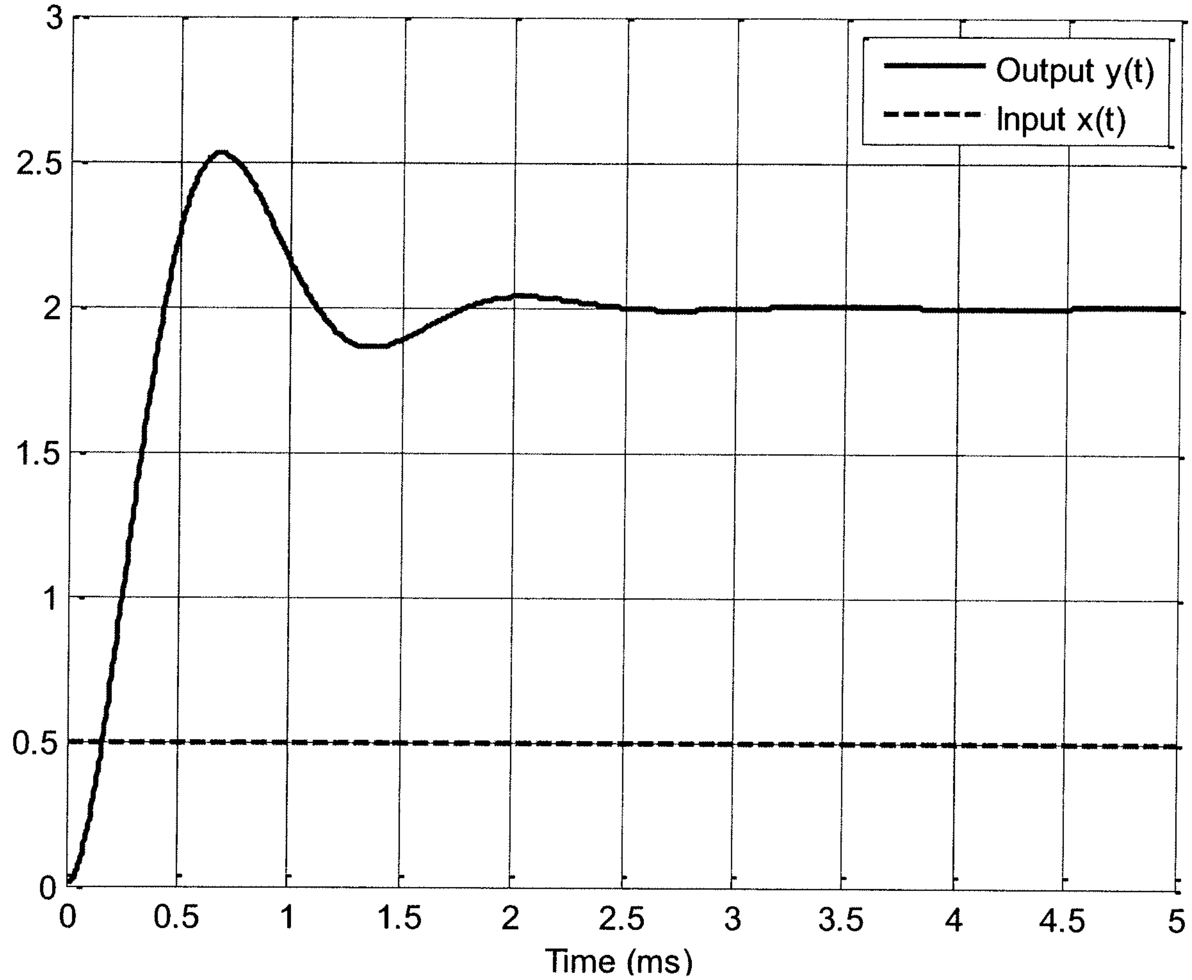
Problems 10-11 refer the following graph showing the response of a second order system to a step input.



10) The percent overshoot for this system is best estimated as

a) 400% b) 250 % c) 200% d) 150 % e) 100 % f) 25%

$\frac{2.5-2}{2} = \frac{0.5}{2} = \frac{1/2}{2} = 1/4$

11) The static gain for this system is best estimated as

a) 1 b) 2 c) 3 d) 4

$K(0.5) = 2$

$K = 4$