7.0 Laplace Transform Applications

In this chapter we will examine many applications of Laplace transforms. While it is possible to go back to the time-domain to determine properties of a system, it is often more convenient to be able to determine these properties in the *s*-domain directly.

7.1 Characteristic Polynomial, Characteristic Modes, and the Impulse Response

Consider a transfer function of the form

$$H(s) = \frac{N(s)}{D(s)}$$

where $N(s)$ and $D(s)$ are polynomials in s with no common factors. $D(s)$ is called the *characteristic polynomial of the system*. The poles of the system are determined from $D(s)$ and these give us most of the information we need to completely characterize the system. The time-domain functions that correspond to the poles of the transfer function are called the *characteristic modes of the system.* To determine the characteristic modes of a system it is often easiest to think of doing a partial fraction expansion and determining the resulting time functions. Finally, the *impulse response* is a linear combination of characteristic modes. A few examples will help.

Example 7.1.1. Consider the transfer function

$$H(s) = \frac{s+2}{s^s(s+1)(s+3)} = a_1\frac{1}{s} + \mathrm{a}_2\frac{1}{s^2} + \mathrm{a}_3\frac{1}{s+1} + a_4\frac{1}{s+3}$$

The characteristic polynomial is $D(s) = s^2(s+1)(s+3)$ and the characteristic modes are $u(t), tu(t), e^{-t}u(t),$ and $e^{-3t}u(t)$. The impulse response is a linear combination of these characteristic modes, $h(t) = a_1u(t) + \mathrm{a}_2tu(t) + \mathrm{a}_3e^{-t}u(t) + a_4e^{-3t}u(t)$.

Example 7.1.2. Consider the transfer function

$$H(s) = \frac{s-3}{s(s+1)^2(s+3)} = a_1\frac{1}{s} + a_2\frac{1}{s+1} + a_3\frac{1}{(s+1)^2} + a_4\frac{1}{s+3}$$

The characteristic polynomial is $D(s) = s(s+1)^2(s+3)$ and the characteristic modes are $u(t), e^{-t}u(t), te^{-t}u(t),$ and $e^{-3t}u(t)$. The impulse response is a linear combination of these characteristic modes, $h(t) = a_1u(t) + a_2e^{-t}u(t) + a_3te^{-t}u(t) + a_4e^{-3t}u(t)$.

Example 7.1.3. Consider the transfer function

$$H(s)=\frac{1}{s^2+s+1}=\frac{1}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}=a_1\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}+a_2\frac{\frac{\sqrt{3}}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

The characteristic polynomial is $D(s)=s^2+s+1$ and the characteristic modes are $e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t\right)u(t)$ and $e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{3}}{2}t\right)u(t)$. The impulse response is the linear combination $h(t)=a_1e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{2}t\right)u(t)+a_2e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{3}}{2}t\right)u(t)$.

There are a few things to keep in mind when finding the characteristic modes of a system:

- There are as many characteristic modes as there are poles in the transfer function
- For complex conjugate poles of the form $-\sigma \pm j\omega_d$ the characteristic modes will be of the form $e^{-\sigma t}\cos(\omega_d t)u(t)$ and $e^{-\sigma t}\sin(\omega_d t)u(t)$. Note that we could combine these into the form $e^{-\sigma t}\cos(\omega_d t+\theta)u(t)$

Example 7.1.4. If a transfer function has poles at $-1,-1,-2\pm 3j,$ and $-5\pm 2j$ the characteristic modes will be $e^{-t}u(t), te^{-t}u(t), e^{-2t}\cos(3t)u(t), e^{-2t}\sin(3t)u(t)$, $e^{-5t}\cos(2t)u(t)$, and $e^{-5t}\sin(2t)u(t)$.

7.2 Asymptotic Stability

We have previously introduced the concept of *Bounded Input Bounded Output*, or BIBO, stability. As we have seen, an LTI system is BIBO stable if

$$\int_{-\infty}^{\infty}|h(\lambda)|\,d\lambda<\infty$$

Another useful definition of stability, which is used often in control systems, is that of *asymptotic stability*. A system is defined to be *asymptotically stable* if all of its characteristic modes go to zero as $t\to\infty$, or equivalently, if $\lim_{t\to\infty}h(t)=0$. A system is defines to be *asymptotically marginally stable* is all of its characteristic modes are bounded as $t\to\infty$, or equivalently, if $\lim_{t\to\infty}|h(t)|\le M$ for some constant *M*. If a system is neither asymptotically stable or asymptotically marginally stable, the system is *asymptotically unstable*. In determining asymptotic stability, the following mathematical truths should be remembered:

$$\lim_{t\to\infty} t^n e^{-at} = 0 \qquad \textit{for all } n>0 \textit{ and } a>0$$
$$\lim_{t\to\infty} e^{-at} cos(\omega_d t+\theta) = 0 \qquad \textit{for all } a>0$$
$$u(t) \qquad \textit{is bounded}$$
$$\cos(\omega_d t+\theta) \qquad \textit{is bounded}$$
$$\sin(\omega_d t+\theta) \qquad \textit{is bounded}$$

Example 7.2.1. Assume a system has poles at -1, 0. and -2. Is the system asymptotically stable? The characteristic modes for this system are $e^{-t}u(t), u(t)$ and $e^{-2t}u(t)$. Both $e^{-t}u(t)$ and $e^{-2t}u(t)$ go to zero as $t\to\infty$. $u(t)$ does not go to zero as $t\to\infty$, but it is bounded. Hence the system is *asymptotically marginally stable.*

Example 7.2.2. Assume a system has poles at -1, 1, and $-2\pm 3j$. Is the system asymptotically stable? The characteristic modes for this system are $e^{-t}u(t)$, $e^{t}u(t)$, $e^{-2t}\cos(3t)u(t)$, and $e^{-2t}\sin(3t)u(t)$. All of the modes go to zero as $t\to\infty$ except for $e^{t}u(t)$, which goes to infinity. Hence the system is *asymptotically unstable.*

Example 7.2.3. Assume a system has poles at -1, -1, $-2\pm j$. Is the system asymptotically stable? The characteristic modes for this system are $e^{-t}u(t)$, $te^{-t}u(t)$, $e^{-2t}\cos(t)u(t)$, and $e^{-2t}\sin(t)u(t)$. All of the characteristic modes of the system go to zero as $t\to\infty$, so the system *is asymptotically stable*.

From these examples, if should be clear that *a system will be asymptotically stable if all of the poles of the system are in the left half plane (all of the poles have negative real parts).* This is a very easy test to remember.

7.3 Settling Time and Dominant Poles

Once we think about representing the impulse response as a linear combination of characteristic modes, we can define asymptotic stability in terms of the way these modes behave as $t\to\infty$. Another benefit of this approach is that we can think of the settling time of a system, i.e., the time the system takes to reach 2% of its final value, in terms of the settling time of each of its characteristic modes. When we talk about the settling time of a system, we assume

- the system is asymptotically stable
- the poles of the system are distinct (no repeated poles)
- the input to the system is a step

Let's assume our system has transfer function $H(s)$ with corresponding impulse response

$$h(t) = a_1\phi_1(t) + a_2\phi_2(t) + \cdots + a_n\phi_n(t)$$

Here the a_i are the coefficients we determine using the partial fraction expansion, and the $\phi_i(t)$ are the characteristic modes, i.e.,

$$\frac{a_i}{s+p_i} \leftrightarrow a_1\phi_i(t)$$

Now let's assume the input to our system is a step of amplitude *A* and we want to use partial fractions to determine the output,

$$Y(s) = H(s)\frac{A}{s} = B\frac{1}{s} + b_1\frac{1}{s+p_1} + b_2\frac{1}{s+p_2} + \cdots + b_n\frac{1}{s+p_n}$$

In the time-domain this will have the form

$$y(t) = Bu(t) + b_1\phi_1(t) + b_2\phi_2(t) + \cdots + b_n\phi_n(t)$$

The primary difference between this and the impulse response of the system is the term $Bu(t)$, which represents the final value of the system due to the step, and a different linear combination of the characteristic modes (the a_i are now b_i, but the $\phi_i(t)$ remain the same.)

Recall that the 2% settling time for an exponential function $\phi_i(t) = e^{-t/\tau_i}u(t)$ is equivalent to four time constants, $T_s = 4\tau$. In order to determine the settling of a system with multiple characteristic modes, we determine the settling time corresponding to each characteristic mode. *The longest settling time determines the settling time of the system.* Note that this is not an exact formula, since the actual settling time of the system is also a function of the coefficients. However, for most instances this gives a reasonably good first estimate. *The pole that produces the longest settling time is called the dominant pole. If a system has two complex conjugate poles that produce the longest settling time, those poles are the dominant poles.*

Example 7.3.1. Consider a system with transfer function

$$H(s) = \frac{20}{(s+1)(s+5)(s+10)}$$

and corresponding impulse response

$$h(t) = \frac{5}{9}e^{-t}u(t) - e^{-5t}u(t) + \frac{4}{9}e^{-10t}u(t)$$

The unit step response of the system is

$$y(t) = \frac{2}{5}u(t) - \frac{5}{9}e^{-t}u(t) + \frac{1}{5}e^{-5t}u(t) - \frac{2}{45}e^{-10t}u(t)$$

Note that except for the first term, which comes from the input, each of the other time functions is the same as for the impulse response. The time constant for each of the characteristic modes is 1, 0.2, and 0.1 with corresponding settling times of 4, 0.8, and 0.4 seconds. Hence we estimate the settling time of the system to be the largest of these, or $T_s \approx 4$ seconds. Figure 7.1 plots the system output and each of the characteristic modes. As the plot indicates, the settling time is approximately 4 seconds, though this is not exact.

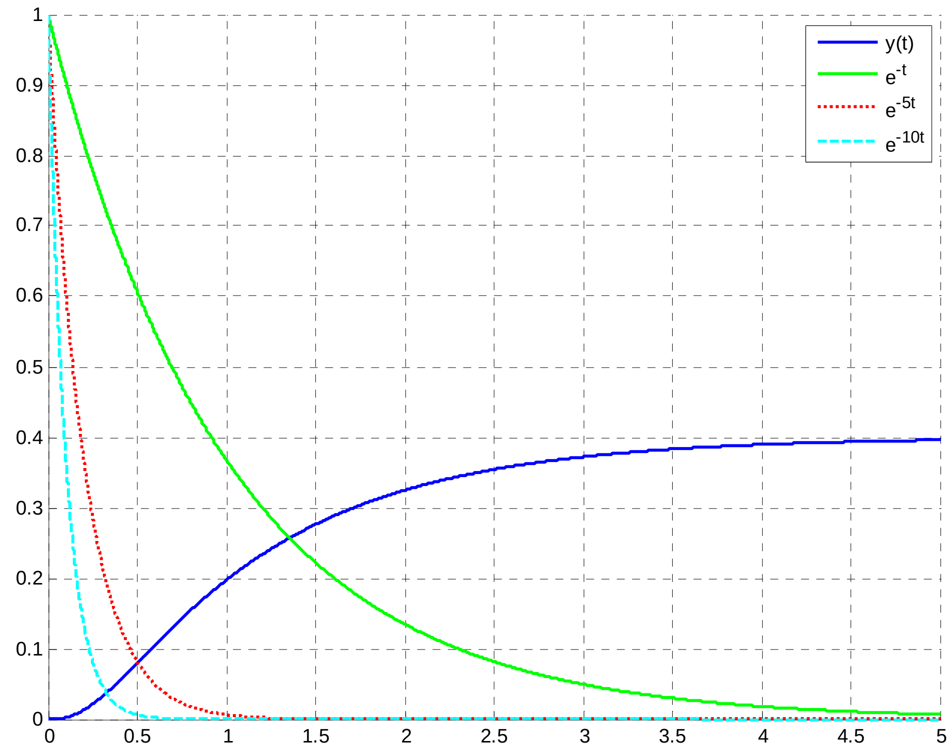


Figure 7.1. Response of system from Example 7.3.1.

Example 7.3.2. Consider a system with transfer function

$$H(s) = \frac{50}{(s+10)\left[(s+2)^2+9\right]}$$

and corresponding impulse response

$$h(t) = 0.68493e^{-10t}u(t) - 0.68493e^{-2t}\cos(3t)u(t) + 1.82648e^{-2t}\sin(3t)u(t)$$

The unit step response of the system is given by

$$y(t) = 0.38462u(t) - 0.06849e^{-10t}u(t) - 0.31613e^{-2t}\cos(3t)u(t) - 0.43905e^{-2t}\sin(3t)u(t)$$

The time constant for each of the characteristic modes is 0.1, 0.5, and 0.5 with corresponding settling times of 0.4, 2.0, and 2.0 seconds. Hence we estimate the settling time of the system to be the largest of these, or $T_s \approx 2$ seconds. Figure 7.2 plots the system output and each of the characteristic modes. As the plot indicates, the settling time is approximately 2 seconds, though this is not exact.

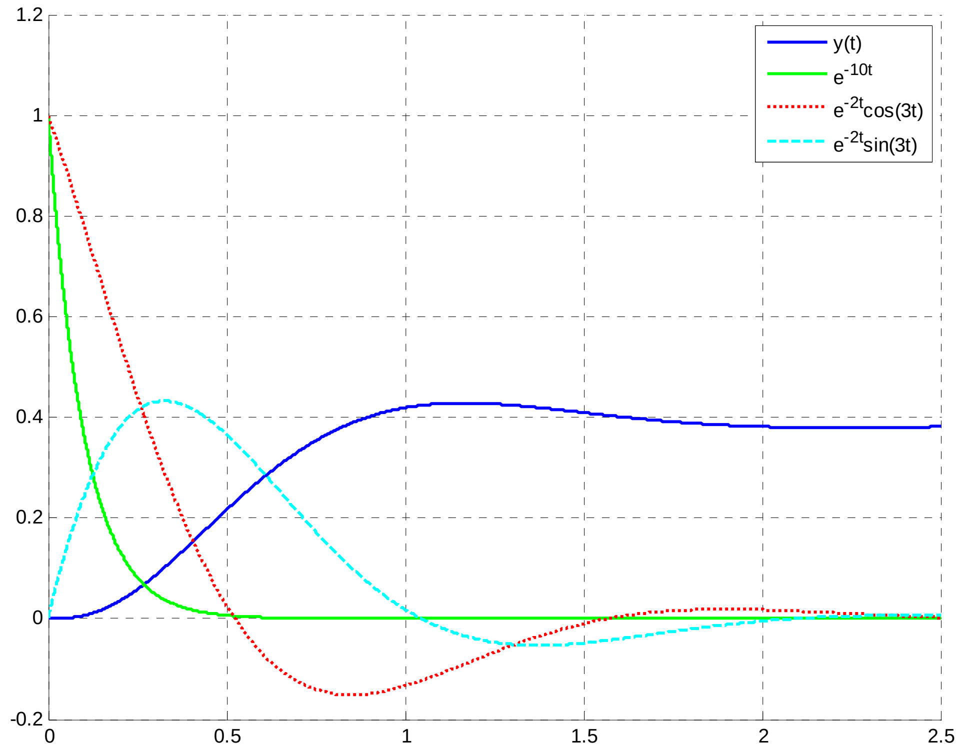


Figure 7.2. Response of system from Example 7.3.2.

Example 7.3.3. Consider a system with a transfer function with poles at -2, -4, and -6. Estimate the settling time of the system. We do not have to actually determine the characteristic functions of the impulse response to do this problem. The corresponding time constants will be ½. ¼, and 1/6. The estimated settling times corresponding to each of these time constants is 2, 1, and 0.667 seconds. The largest settling time is 2 seconds, so that is our estimate of the settling time of the system. In this case, the dominant pole is the pole at -2, since it leads to the largest settling time.

Example 7.3.4. Consider a system with a transfer function with poles at $-4, -6$, and $-2 \pm 5j$. Estimate the settling time of the system. The time constants that correspond to these pole locations are ¼, 1/6, and ½ seconds. The settling time associate with each of these time constants is 1, 0.667, and 2 seconds. Hence the estimated settling time is again 2 seconds, and the dominant poles are at $-2 \pm 5j$.

Example 7.3.5. Determine the estimated settling time for a transfer function with poles at $-4 \pm 5j$, $-8 \pm j$, and -10. The corresponding time constants are 0.25, 0.125, and 0.1 seconds. The associated settling times are 1, 0.5, and 0.4 seconds. Hence the estimated settling time is 1 second and the dominant poles are at $-4 \pm 5j$.

It should be obvious by now that the dominant poles are those poles with the real part closest to the $j\omega$ axis.

7.4 Initial and Final Value Theorems

It is often necessary to determine the initial ($t = 0$) or final ($t \to \infty$) value of a function represented in the *s*-domain. While it is possible to perform partial fractions and determine the time domain representation, this is often tedious and we would like to be able to perform this computation directly in the *s*-domain. Instead we generally use the initial and final value Theorems, which are stated below:

Initial Value Theorem: If $x(t) \leftrightarrow X(s)$ and $X(s)$ is asymptotically stable, then

$$\lim_{t \to 0^+} x(t) = \lim_{s \to \infty} sX(s)$$

Final Value Theorem: If $x(t) \leftrightarrow X(s)$ and $X(s)$ is asymptotically stable, then

$$\lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s)$$

Note that these final value theorems are also valid if there is a single pole at the origin.

Example 7.4.1. Consider the transform pair

$$X(s) = \frac{1}{(s+1)(s+3)} \leftrightarrow x(t) = \frac{1}{2}e^{-t}u(t) - \frac{1}{2}e^{-3t}u(t)$$

Clearly all of the poles are in the left half plane, so we can use the initial value theorem,

$$\lim_{t \to 0^+} x(t) = \frac{1}{2} - \frac{1}{2} = 0$$

$$\lim_{s \to \infty} sX(s) = 0$$

Example 7.4.2. Consider the transform pair

$$X(s) = \frac{s+1}{(s+3)(s+4)} \leftrightarrow x(t) = -2e^{-3t}u(t) + 3e^{-4t}u(t)$$

Clearly all of the poles are in the left half plane, so we can use the initial value theorem,

$$\lim_{t \to 0^+} x(t) = 2 - 3 = 1$$

$$\lim_{s \to \infty} sX(s) = 1$$

Example 7.4.3. Consider the transform pair

$$X(s) = \frac{3s^2 + 16s + 71}{(s+1)\left[(s+3)^2 + 25\right]} \leftrightarrow x(t) = 2e^{-t}u(t) + e^{-3t}\cos(5t)u(t)$$

Clearly all of the poles are in the left half plane, so we can use the initial value theorem,

$$\lim_{t \to 0^+} x(t) = 2 + 1 = 3$$

$$\lim_{s \to \infty} sX(s) = 3$$

Example 7.4.4. Consider the transform pair

$$X(s) = \frac{1}{s} \leftrightarrow x(t) = u(t)$$

Clearly all of the poles are in the left half plane, with the exception of a single pole at the origin, so we can use the final value theorem,

$$\lim_{t\to\infty} x(t) = 1$$

$$\lim_{s\to 0} sX(s) = 1$$

Example 7.4.5. Consider the transform pair

$$X(s) = \frac{1}{s(s+3)} \leftrightarrow x(t) = \frac{1}{3}u(t) - \frac{1}{3}e^{-3t}u(t)$$

Clearly all of the poles are in the left half plane, with the exception of a single pole at the origin, so we can use the final value theorem,

$$\lim_{t\to\infty} x(t) = \frac{1}{3}$$

$$\lim_{s\to 0} sX(s) = \frac{1}{3}$$

Example 7.4.6. Consider the transform pair

$$X(s) = \frac{-2s^2 + 26}{s\left[(s+3)^2 + 4\right]} \leftrightarrow x(t) = 2u(t) - 4e^{-3t}\cos(2t)u(t)$$

Clearly all of the poles are in the left half plane, with the exception of a single pole at the origin, so we can use the final value theorem,

$$\lim_{t\to\infty} x(t) = 2$$

$$\lim_{s\to 0} sX(s) = 2$$

As you will see as we go through this chapter, the initial value theorem is often used to determine the initial amount of “effort” needed for a system, while the final value theorem is used for determining the final value of a system and the steady state error.

7.5 Static Gain

The *static gain* of a system is basically the steady state gain of a system when the input is a step function. Obviously the concept of the static gain of a system only makes sense for systems that are asymptotically stable. Since the static gain of the system is measured during steady state, all of the transients have died out.

The final value theorem is generally used to find the static gain of a system. Assume we have an asymptotically stable system with transfer function $H(s)$ and the input $x(t)$ to our system is a step of amplitude A, $x(t) = Au(t)$. The we can determine the output of the system as

$$Y(s) = H(s)X(s) = H(s)\frac{A}{s}$$

Using the final value theorem, we can then determine the steady state value of the output as

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} sH(s)\frac{A}{s} = \lim_{s\to 0} H(s)A = H(0)A$$

The gain of the system is the output amplitude divided by the input amplitude, so we can compute the static gain

K as

$$K = \frac{\text{Output Amplitude}}{\text{Input Amplitude}} = \frac{H(0)A}{A} = H(0)$$

So we have the result (for asymptotically stable systems) that $K = H(0)$ and $\lim_{t\to\infty} y(t) = KA$ for a step input of amplitude A.

Example 7.5.1. Determine the static gain and steady state value of the output for the system

$$H(s) = \frac{s+2}{s^2 + 0.4s + 3}$$

for a step input of amplitude 5. Here $K = \frac{2}{3}$ and $\lim_{t\to\infty} y(t) = KA = \frac{2}{3}\times 5 = \frac{10}{3}$. The response of this system is displayed in Figure 7.3.

Example 7.5.2. Determine the static gain and steady state value of the output for the system

$$H(s) = \frac{2}{s^2 + s + 1}$$

for a step input of amplitude 5. Here $K = 2$ and. $\lim_{t\to\infty} y(t) = KA = 2\times 5 = 10$ The response of this system is displayed in Figure 7.4.

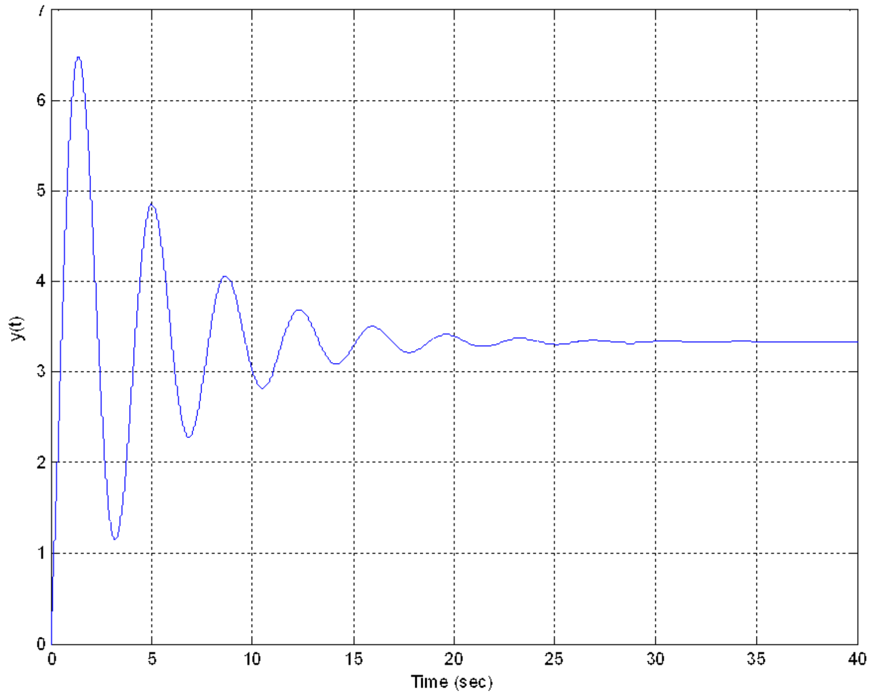


Figure 7.3. Response of system for Example 7.5.1 to a step of amplitude 5. The static gain of this system is 2/3 and the final value of the output is then 10/3 = 3.3, as the figure shows.

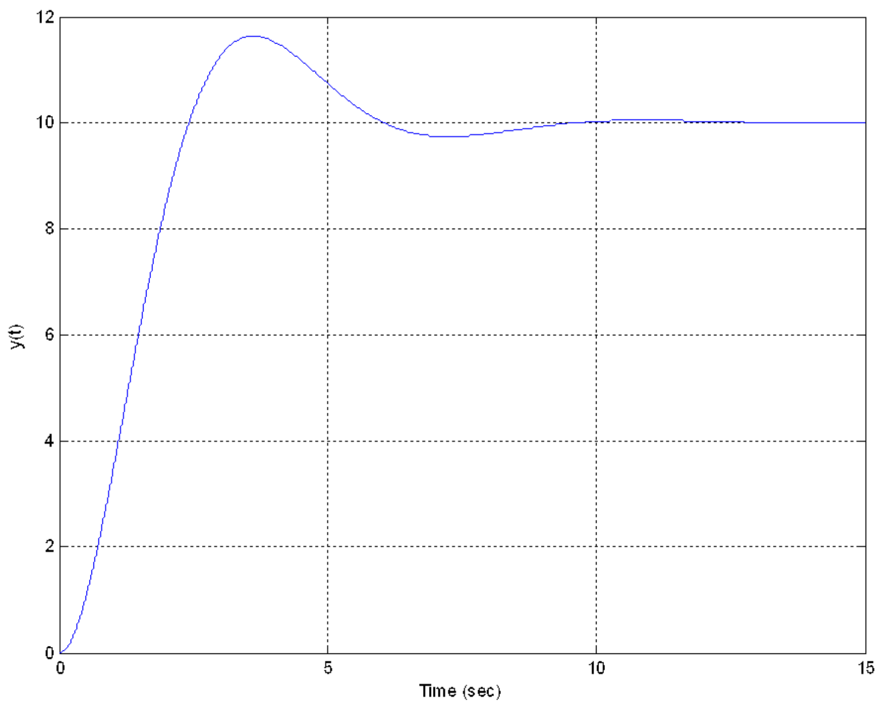


Figure 7.4. Response of system for Example 7.5.2 to a step of amplitude 5. The static gain of this system is 2 and the final value of the output is then 10, as the figure shows.

7.6 Ideal Second Order Systems (Again)

Recall that the differential equation that describes an ideal second order system is of the form

$$\frac{d^2y(t)}{dt^2} + 2\zeta\omega_n \frac{dy(t)}{dt} + \omega_n^2 y(t) = K\omega_n^2 x(t)$$

where ζ is the damping ratio, ω_n is the natural frequency, and K is the static gain of the system. To determine the transfer function we assume all of the initial conditions are zero and take the Laplace transform of each term,

$$\mathcal{L}\left\{\frac{d^2y(t)}{dt^2}\right\} = s^2Y(s)$$

$$\mathcal{L}\left\{2\zeta\omega_n \frac{dy(t)}{dt}\right\} = 2\zeta\omega_n sY(s)$$

$$\mathcal{L}\left\{\omega_n^2 y(t)\right\} = \omega_n^2 Y(s)$$

$$\mathcal{L}\left\{K\omega_n^2 x(t)\right\} = K\omega_n^2 X(s)$$

Combining these we get,

$$\left[s^2 + 2\zeta\omega_n s + \omega_n^2\right]Y(s) = \left[K\omega_n^2\right]X(s)$$

and transfer function

$$H(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{K}{\frac{1}{\omega_n^2}s^2 + \frac{2\zeta}{\omega_n}s + 1}$$

The characteristic equation for this system is $s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$, and for the under damped case ($0 < \zeta < 1$)the poles of the system are given by

$$s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -\zeta\omega_n \pm j\omega_d = -\sigma \pm j\omega_d$$

Figure 7.5 displays the relationship between these parameters in the complex plane. We have previously determined that the response of the system to a step input of amplitude A is given by

$$y(t) = KA + ce^{-\zeta\omega_n t}\sin(\omega_d t + \phi)$$

where the constants c and ϕ are determined by the initial conditions. What we would like to examine now are how we can specify system pole locations to achieve a desired settling time, percent overshoot, and time to peak.

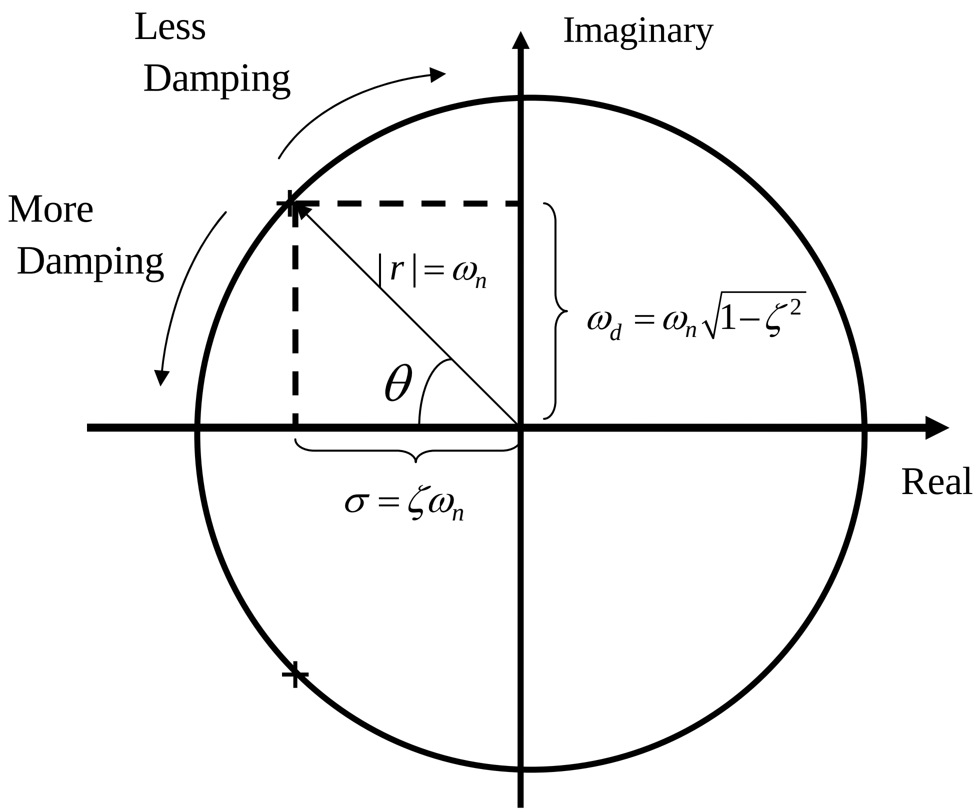


Figure 7.5. Relationship between damping ratio (ζ), natural frequency (ω_n), and damped frequency (ω_d) in the complex plane for an under damped system ($0 < \zeta < 1$). The poles of the system are located at $s = -\sigma \pm j\omega_d$.

Settling Time

We have approximated the settling time for a system with distinct poles to be the time it takes for the slowest characteristic mode to reach four time constants. The time constant for each characteristic mode is the reciprocal of the magnitude of the real part of the pole. For example, for a system with complex poles at $s = -\sigma \pm j\omega$. The corresponding characteristic modes are of the form $e^{-\sigma t}\cos(\omega t), e^{-\sigma t}\sin(\omega t)$, and the time constant is $\tau = 1/\sigma$. Thus if we want our system to have a settling time of T_s^{max}, then we have

$$T_s = 4\tau = \frac{4}{\sigma} < T_s^{max}$$

or

$$\frac{4}{T_s^{max}} < \sigma$$

This means the magnitude of the real part of the pole must be greater than $\frac{4}{T_s^{max}}$, or equivalently, all poles of the system must be to the left of $-\frac{4}{T_s^{max}}$. Although this relationship was derived for an ideal second order system, it is generally a good initial approximation for any system without repeated poles.

Example 7.6.1. Determine the allowed pole locations in the *s*-plane that correspond to a settling time of less than or equal to 2.5 seconds. Here $T_s^{max} = 2.5$ and

$$\sigma = \frac{4}{T_s^{max}} = \frac{4}{2.5} = 1.6$$

Hence all poles must be to the left of -1.6. The allowable pole locations are shown as the shaded region in Figure 7.6.

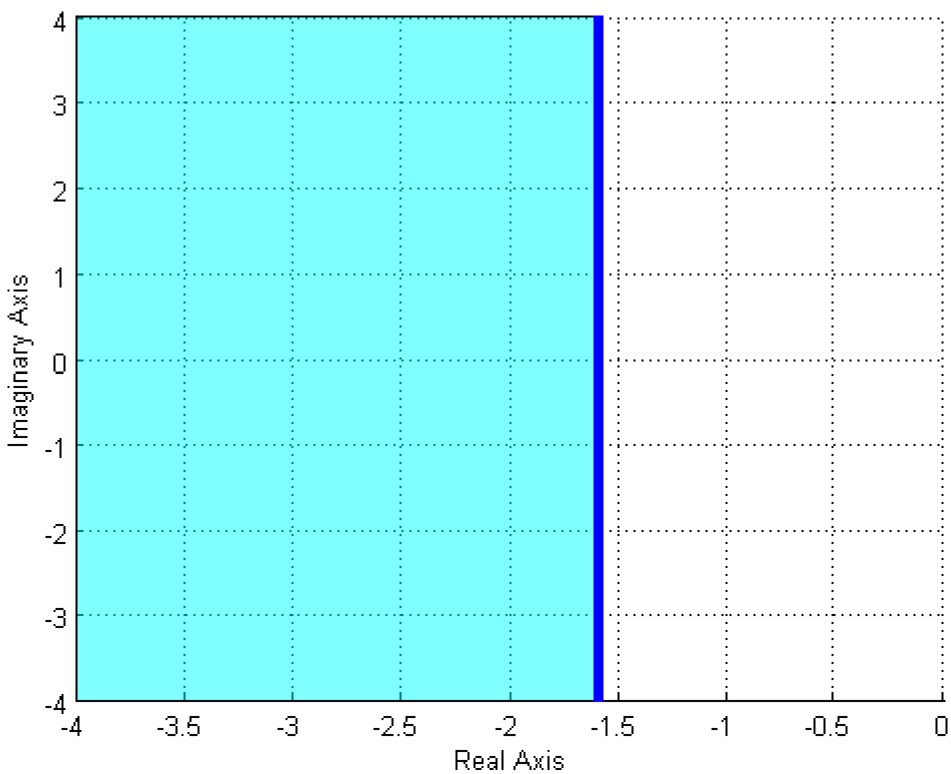


Figure 7.6. The pole locations corresponding to a settling time less that 2.5 seconds are shown in the shaded region, to the left of -1.6 on the real axis. Note that the settling time constraint only affects the real part of the pole, not the imaginary part of the poles.

Example 7.6.2. Determine the allowed pole locations in the *s*-plane that correspond to a settling time of less than or equal to 0.5 seconds. Here $T_s^{max} = 0.5$ and

$$\sigma = \frac{4}{T_s^{max}} = \frac{4}{0.5} = 8$$

Hence all poles must be to the left of -8. The allowable pole locations are shown as the shaded region in Figure 7.7.

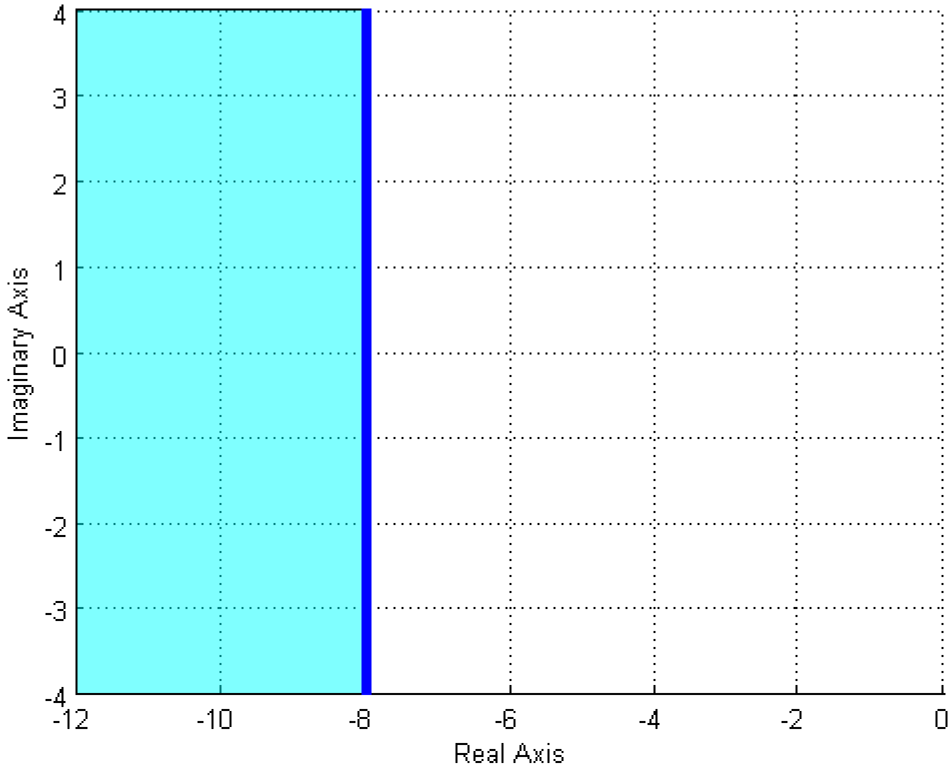


Figure 7.7. The pole locations corresponding to a settling time less that 0.5 seconds are shown in the shaded region, to the left of -8 on the real axis. Note that the settling time constraint only affects the real part of the pole, not the imaginary part of the poles.

Percent Overshoot

From our previous analysis, the percent overshoot (*PO*) is computed as

$$PO = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100\%$$

In order to determine the pole locations in the s-plane that produce an acceptable percent overshoot, we need to do some simple algebra. Let's define

$$O^{max} = \frac{PO^{max}}{100}$$

Here O^{max} is the maximum overshoot, not expressed as a percentage. To set an upper bound on the allowed percent overshoot we have

$$PO \le PO^{max}$$

or

$$e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \le \frac{PO^{max}}{100} = O^{max}$$

Next we need to solve for ζ ,

$$e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \le O^{max}$$

$$\frac{-\zeta\pi}{\sqrt{1-\zeta^2}} \le \ln(O^{max})$$

$$\frac{\zeta}{\sqrt{1-\zeta^2}} \ge \frac{-\ln(O^{max})}{\pi}$$

Squaring both sides and expanding we have

$$\frac{\zeta^2}{1-\zeta^2} \ge \left[\frac{-\ln(O^{max})}{\pi}\right]^2$$

$$\zeta^2 \ge \left[\frac{-\ln(O^{max})}{\pi}\right]^2 - \zeta^2$$

$$\zeta^2 \left\{1+\left[\frac{-\ln(O^{max})}{\pi}\right]^2\right\} \ge \left[\frac{-\ln(O^{max})}{\pi}\right]^2$$

$$\zeta \ge \frac{\frac{-\ln(O^{max})}{\pi}}{\sqrt{1+\left[\frac{-\ln(O^{max})}{\pi}\right]^2}}$$

Finally, we use the relationship depicted in Figure 7.5, $\cos(\theta) = \zeta$, or $\cos^{-1}(\zeta) = \theta$, where the angle θ is measured from the negative real axis. Note that since we usually specify a maximum allowed percent overshoot we have determined the maximum allowed value of the damping ratio, ζ . Since the damping ratio increases (and the damping decreases) as the poles get closer to the real axis, specifying the maximum allowed percent overshoot determines the maximum angle allowed. Thus the allowable pole locations to achieve a maximum percent overshoot will have the shape of a wedge. Finally, note that this relationship is only valid for ideal second order systems.

Example 7.6.3. Determine the allowable pole locations so an ideal second order system will have a maximum percent overshoot of 25%. We have

$$O^{max} = \frac{25}{100} = 0.25$$

and then

$$\zeta \geq \frac{\frac{-\ln(0.25)}{\pi}}{\sqrt{1+\left[\frac{-\ln(0.25)}{\pi}\right]^2}} = 0.404$$

$$\cos^{-1}(\zeta) = \cos^{-1}(0.404) = 66.2^o$$

This means the maximum angle with the real axis is 66.2 degrees. The allowable pole locations are shown as the shaded region in Figure 7.8.

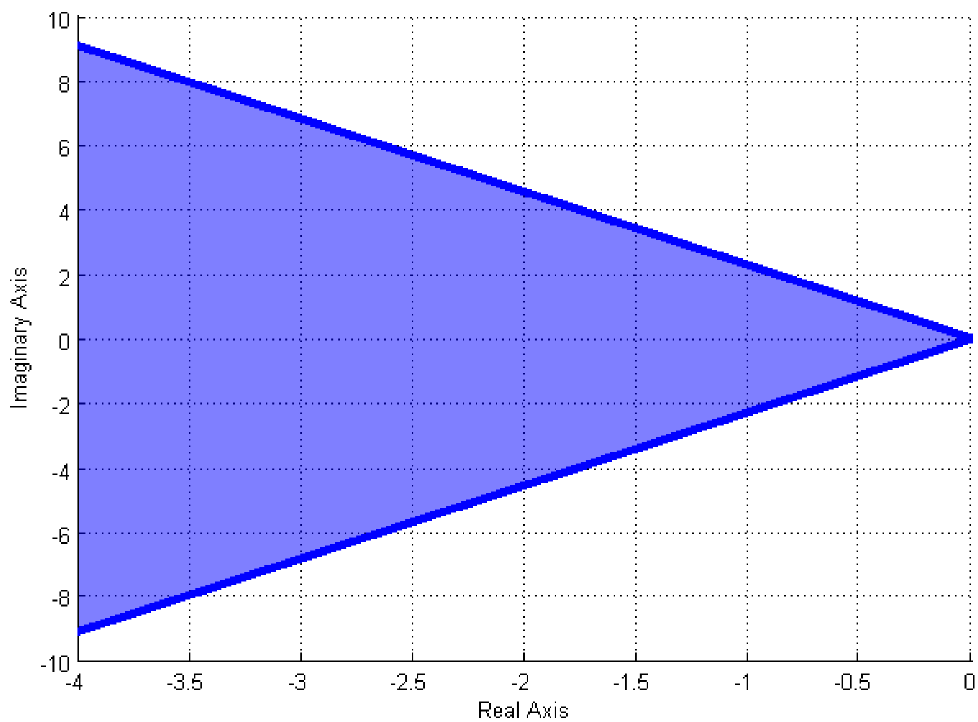


Figure 7.8. The pole locations corresponding to a maximum percent overshoot of 25% are shown in the shaded region, within the wedge.. Note that the percent overshoot affects both the real and imaginary parts of the poles.

Example 7.6.4. Determine the allowable pole locations so an ideal second order system will have a maximum percent overshoot of 10%. We have

$$O^{max} = \frac{10}{100} = 0.10$$

and then

$$\zeta \geq \frac{\frac{-\ln(0.10)}{\pi}}{\sqrt{1+\left[\frac{-\ln(0.10)}{\pi}\right]^2}} = 0.591$$

$$\cos^{-1}(\zeta) = \cos^{-1}(0.591) = 53.8^o$$

This means the maximum angle with the real axis is 53.8 degrees. The allowable pole locations are shown as the shaded region in Figure 7.9.

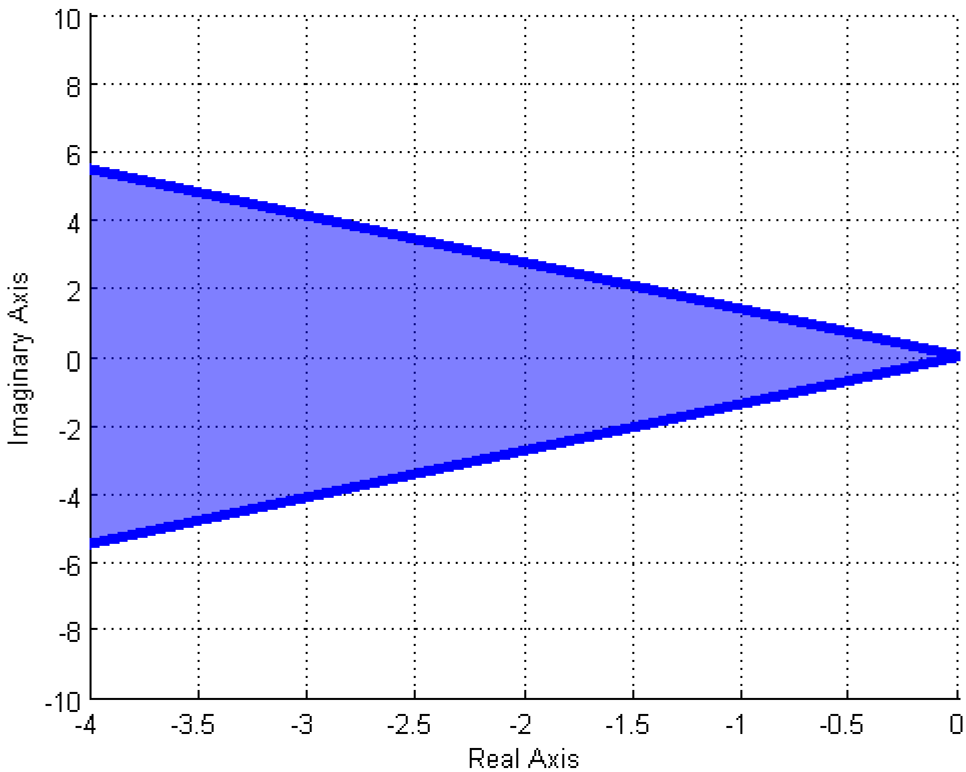


Figure 7.9. The pole locations corresponding to a maximum percent overshoot of 10% are shown in the shaded region, within the wedge. Note that the percent overshoot affects both the real and imaginary parts of the poles. Note also that the angle of this wedge is narrower than that in Figure 7.8 since the allowed percent overshoot in that case is smaller (10% compared to 25% in Figure 7.8).

Example 7.6.5. Determine the allowable pole locations for an ideal second order system so the response to a step will have a percent overshoot less than 5% and a settling time of less than 2 seconds. To solve this problem we need to determine the acceptable regions for each constraint, and then determine if there is any overlapping region so that both constraints will be satisfied. To meet the percent overshoot requirement we have

$$O^{max} = \frac{5}{100} = 0.05$$

$$\zeta \geq \frac{\frac{-\ln(0.05)}{\pi}}{\sqrt{1+\left[\frac{-\ln(0.05)}{\pi}\right]^2}} = 0.690$$

$$\cos^{-1}(\zeta) = \cos^{-1}(0.690) = 46.4^o$$

To meet the settling time requirement we have $T_s^{max} = 2.0$ and

$$\sigma = \frac{4}{T_s^{max}} = \frac{4}{2} = 2.0$$

Hence the region of the s-plane that meets both constraints is to the left of -2 and within a wedge with an angle of 46.4 degrees. Each of these individual regions, and the overlapping region are displayed in Figure 7.10.

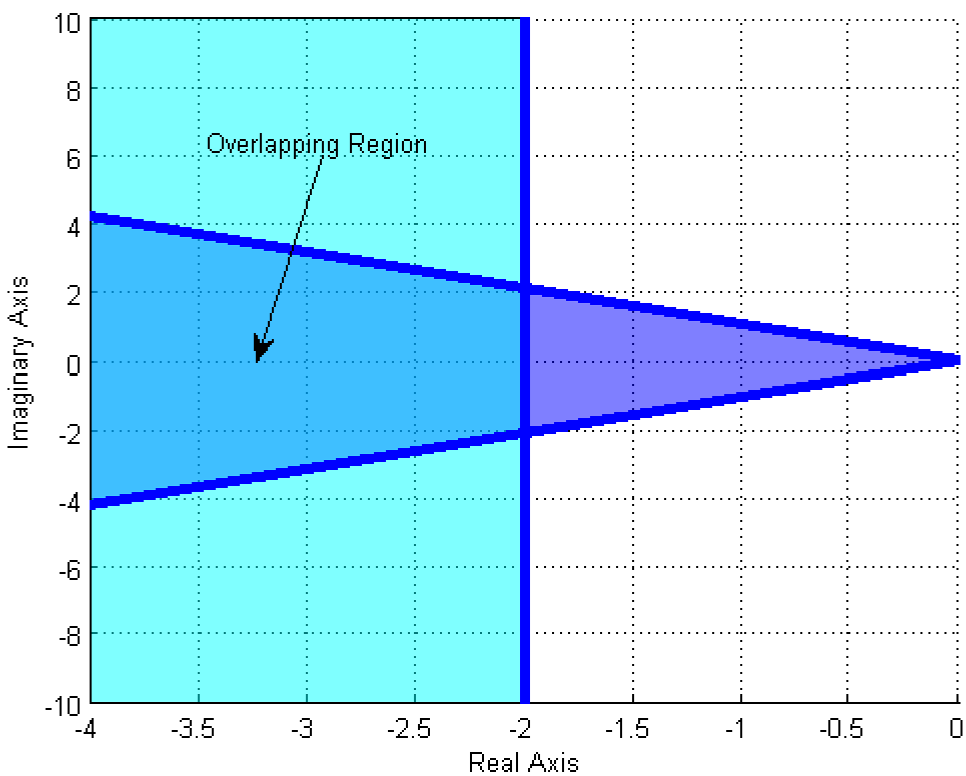


Figure 7.10. The pole locations corresponding to a maximum percent overshoot of 5% and a settling time of 2 seconds are shown in the shaded regions. The overlapping region shows the location in the *s*-plane where both conditions are met.

Time to Peak

From our previous analysis, the time to peak for an under damped ideal second order system is given by

$$T_p = \frac{\pi}{\omega_d}$$

From Figure 7.5 we can easily see that ω_d, the damped frequency, is just the imaginary part of the poles. Hence this constraint will only constrain the imaginary parts of the poles. If we define the maximum allowable time to peak as T_p^{max}, then we have

$$T_p = \frac{\pi}{\omega_d} \leq T_P^{max}$$

or

$$\frac{\pi}{T_p^{max}} \leq \omega_d$$

which indicates the imaginary part of the pole must be larger than π / T_p^{max}.

Example 7.6.6. Determine the pole locations for an ideal second order system that correspond to a time to peak of less than 0.5 seconds for an ideal second order system. We have $T_p^{max} = 0.5$ and then

$$\frac{\pi}{T_p^{max}} = \frac{\pi}{0.5} = 6.283 < \omega_d$$

This means the imaginary part of the poles must be larger than 6.283. The acceptable pole locations are shown as the shaded region in Figure 7.11.

Example 7.6.7. Determine the pole locations for an ideal second order system that correspond to a maximum time to peak of less than or equal to 1.5 seconds and a settling time of less than or equal to 1 seconds. We have $T_p^{max} = 1.5$ and then

$$\frac{\pi}{T_p^{max}} = \frac{\pi}{1.5} = 2.09 < \omega_d$$

To meet the settling time requirement we have $T_s^{max} = 1.0$ and

$$\sigma = \frac{4}{T_s^{max}} = \frac{4}{1} = 4.0$$

Hence to meet both requirements we need the real parts of the poles to the left of -4, and the imaginary parts greater than 2.09 (or less than -2.09). The acceptable pole locations are shown in Figure 7.12.

Example 7.6.7. Determine the pole locations for an ideal second order system that corresponds to a time to peak of less than or equal to 3 seconds, a settling time of less than 2 seconds, and a percent overshoot of less than 20%. We have $T_p^{max} = 3.0$ and then

$$\frac{\pi}{T_p^{max}} = \frac{\pi}{3.0} = 1.05 < \omega_d$$

To meet the settling time requirement we have $T_s^{max} = 2.0$ and

$$\sigma = \frac{4}{T_s^{max}} = \frac{4}{2} = 2.0$$

Finally to meet the percent overshoot requirement we have $O^{max} = \frac{20}{100} = 0.2$ and

$$\zeta \geq \frac{\frac{-\ln(0.2)}{\pi}}{\sqrt{1+\left[\frac{-\ln(0.2)}{\pi}\right]^2}} = 0.456$$

$$\cos^{-1}(\zeta) = \cos^{-1}(0.456) = 62.9^o$$

Figure 7.13 displays the acceptable regions for each of the three requirements, and the overlapping region where all three requirements are met.

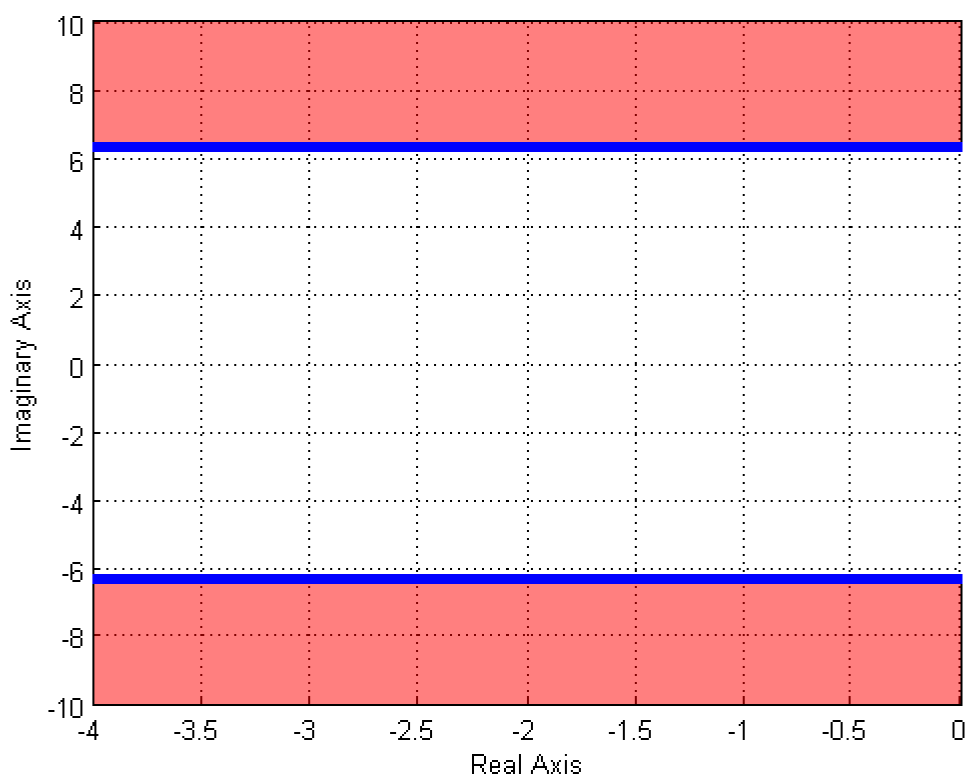


Figure 7.11. The pole locations corresponding to a maximum time to peak of 0.5 seconds. This corresponds to the imaginary part of the poles being larger than 6.28. Note that this constraint affects only the imaginary parts of the pole.

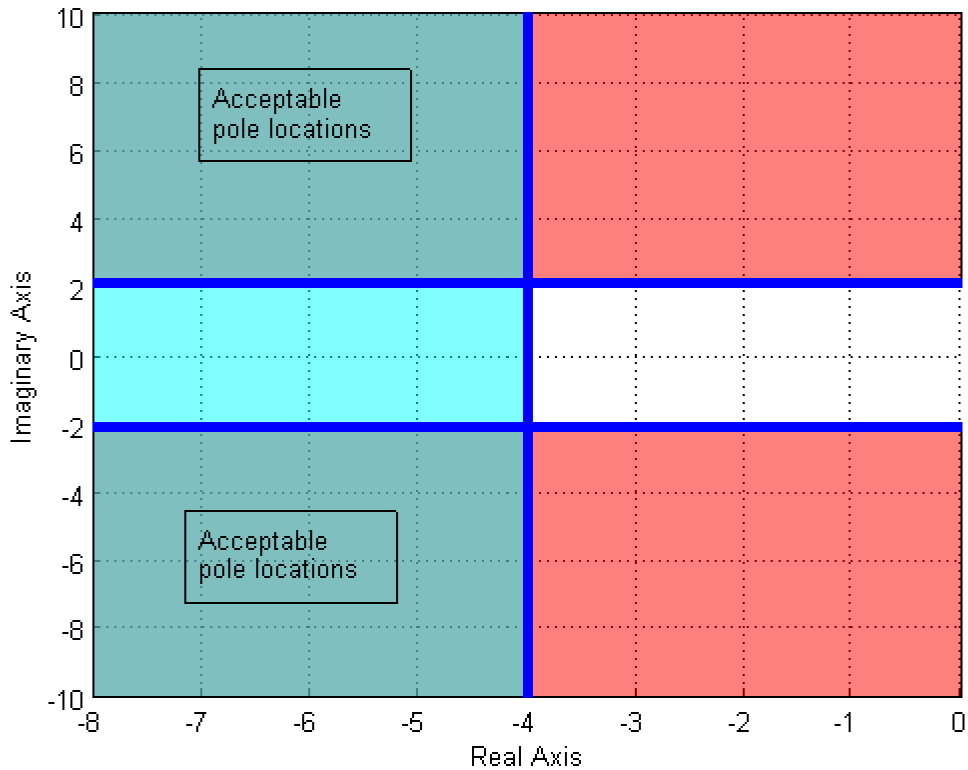


Figure 7.12. The pole locations corresponding to a maximum time to peak of 1.5 seconds and a settling time less than 1 seconds. The settling time constraint means the real part of the poles must be let than -4, and the peak time constraint means the absolute value of the imaginary part of the pole must be greater than 2.09. The pole locations that meet both of these constraints is the overlapping regions, labeled as “Acceptable pole locations”.

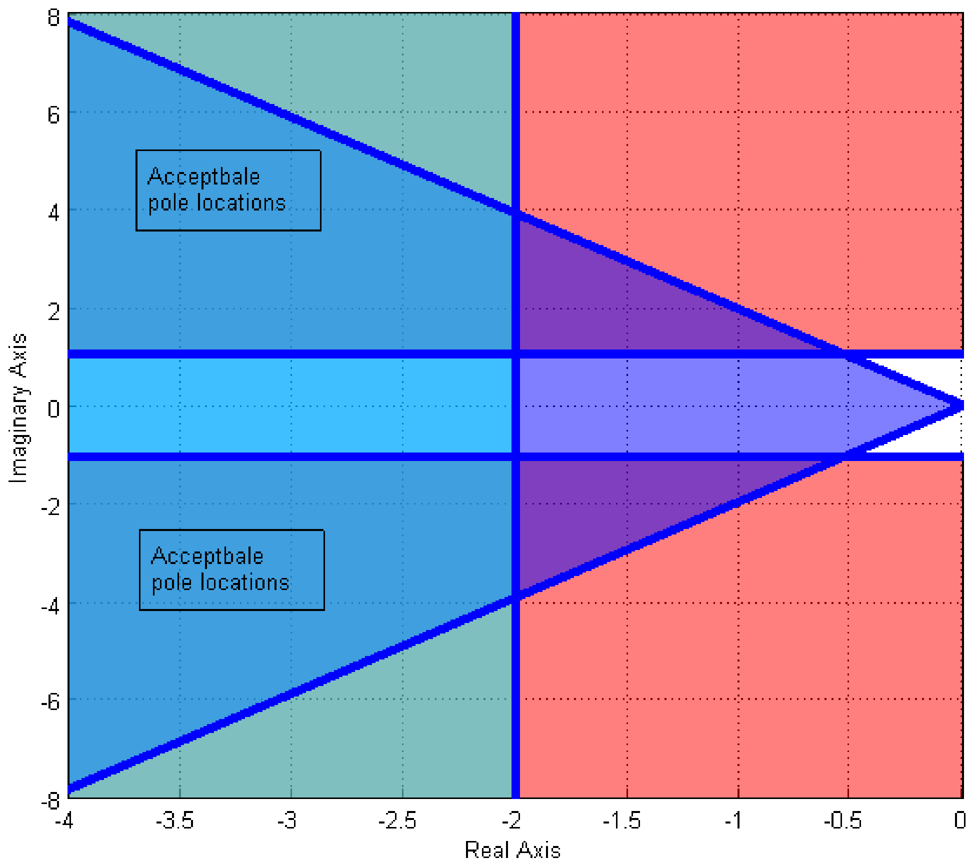


Figure 7.13. The acceptable pole locations for an ideal second order system that corresponds to a time to peak of less than or equal to 3 seconds, a settling time of less than 2 seconds, and a percent overshoot of less than 20%. The pole locations that meet all of these constraints is the overlapping regions, labeled as "Acceptable pole locations".

7.7 Block Diagrams

We often need to analyze and design interconnected systems. When we introduced convolution for LTI systems, we demonstrated some simple interconnected systems. However, using convolution techniques for these systems is often difficult. Instead we utilize transfer functions relating the output of one system to the input of another system. We then use the fact that the time domain relationship

$$y(t) = h_1(t) \star h_2(t) \star h_3(t) \star x(t)$$

is equivalent to the *s*-domain algebraic relationship

$$Y(s) = H_1(s)H_2(s)H_3(s)X(s)$$

Example 7.7.1. Consider the op-amp circuit shown in Figure 7.14. The input to the circuit is $v_{in}(t)$, the output is $v_{out}(t)$, and $v_m(t)$ is the voltage depicted in the figure as the output of the first op amp.

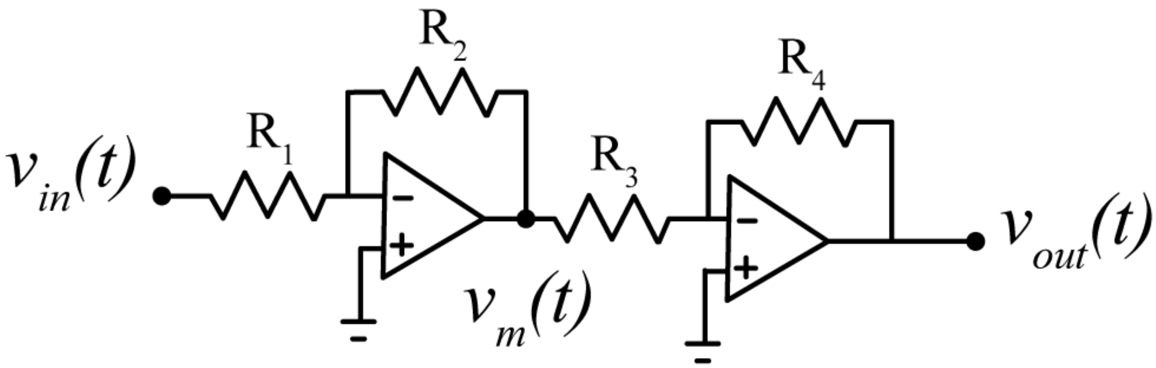


Figure 7.14. Proportional gain circuit for Example 7.7.1.

At the negative terminal of the first op amp we have

$$\frac{V_{in}(s)}{R_1} + \frac{V_m(s)}{R_2} = 0$$

or

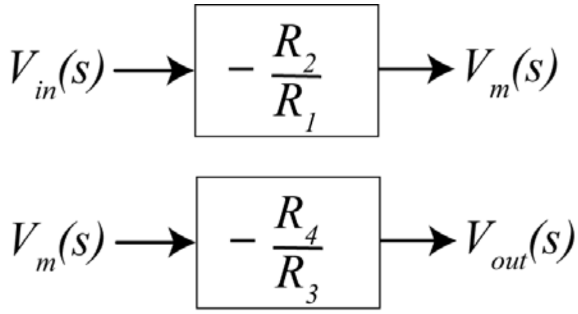
$$V_m(s) = -\frac{R_2}{R_1} V_{in}(s)$$

So the transfer function between input $V_{in}(s)$ and output $V_m(s)$ is $H_1(s) = -\frac{R_2}{R_1}$

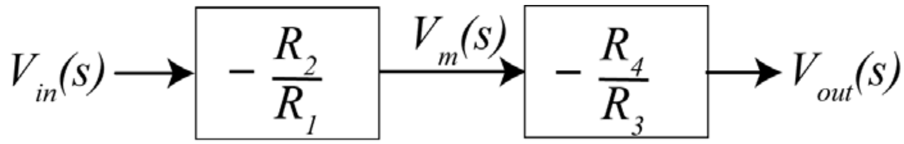
Similarly we have

$$V_{out}(s) = -\frac{R_4}{R_3} V_m(s)$$

So the transfer function between input $V_m(s)$ and output $V_{out}(s)$ is $H_2(s) = -\frac{R_4}{R_3}$. We can depict these relationships graphically as



We can then combine these blocks as follows:



This block diagram indicates graphically that

$$V_m(s) = -\frac{R_2}{R_1}V_{in}(s)\,,\;\; V_{out}(s) = -\frac{R_4}{R_3}V_m(s)\,,\;\; \text{and} \;\; V_{out}(s) = \frac{R_2R_4}{R_1R_3}V_{in}(s)$$

Finally, we can write this as

$$V_{out}(s) = \mathrm{k}_pV_{in}(s)$$

where k_p is a proportionality constant.

Example 7.7.2. Consider the op-amp system shown in Figure 7.15.

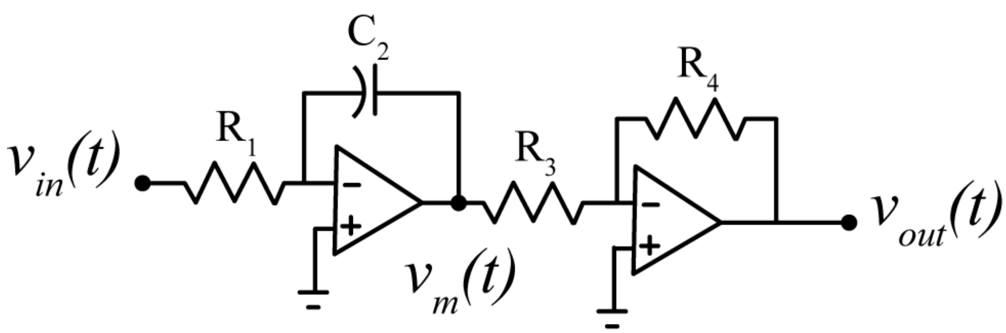


Figure 7.15. Integral and gain op amp circuit for Example 7.7.2.

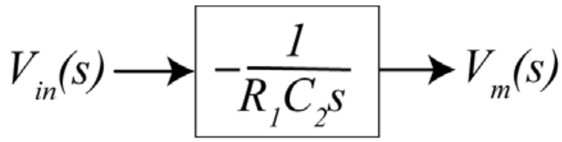
At the negative node of the first op amp we have

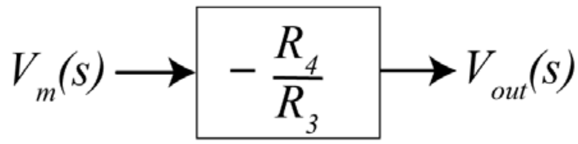
$$\frac{V_{in}(s)}{R_1} + \frac{V_m(s)}{\frac{1}{C_2 s}} = 0\,, \text{ or } V_m(s) = -\frac{1}{R_1C_2s}V_{in}(s)$$

At the negative node of the second op amp we again have

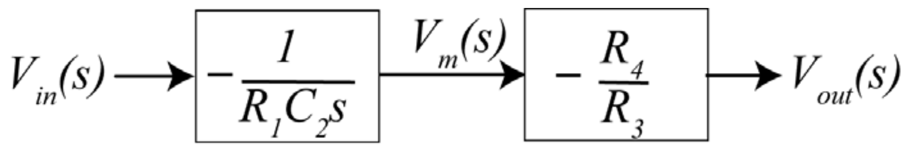
$$V_{out}(s) = -\frac{R_4}{R_3}V_m(s)$$

We can depict these relationships graphically as





Again, we can combine these as



This block diagram indicates graphically that

$$V_{out}(s) = \left(-\frac{1}{R_1C_2s}\right)\left(-\frac{R_4}{R_3}\right)V_{in}(s) = \left(\frac{R_4}{R_1R_3C_2s}\right)V_{in}(s) = \frac{k_i}{s}V_{in}(s)$$

Here we have written the proportionality constant as

$$k_i = \frac{R_4}{R_1R_3C_2}$$

Example 7.7.3. Consider the *differential amplifier* circuit shown in Figure 7.16. We assume $v_{out}(t) = v_a(t) - v_b(t)$. At the negative input terminal we have

$$\frac{V_f(s) - V^-(s)}{R} + \frac{V_a(s) - V^-(s)}{R_g} = 0$$

and at the positive terminal we have

$$\frac{V_r(s) - V^+(s)}{R} + \frac{V_b(s) - V^+(s)}{R_g} = 0$$

Since under the ideal op amp assumption $V^+(s) = V^-(s)$, we can rearrange these as

$$\frac{V_f(s)}{R} + \frac{V_a(s)}{R_g} = V^-(s)\left[\frac{1}{R} + \frac{1}{R_g}\right] = V^+(s)\left[\frac{1}{R} + \frac{1}{R_g}\right] = \frac{V_r(s)}{R} + \frac{V_b(s)}{R_g}$$

Simplifying this we get

$$V_a(s) - V_b(s) = V_{out}(s) = \frac{R_g}{R}\left[V_r(s) - V_f(s)\right]$$

or

$$V_{out}(s) = k\left[V_r(s) - V_f(s)\right]$$

This relationship is depicted graphically in Figure 7.17 and is commonly used in feedback systems. Note that if $R_g = R$ then $k = 1$.

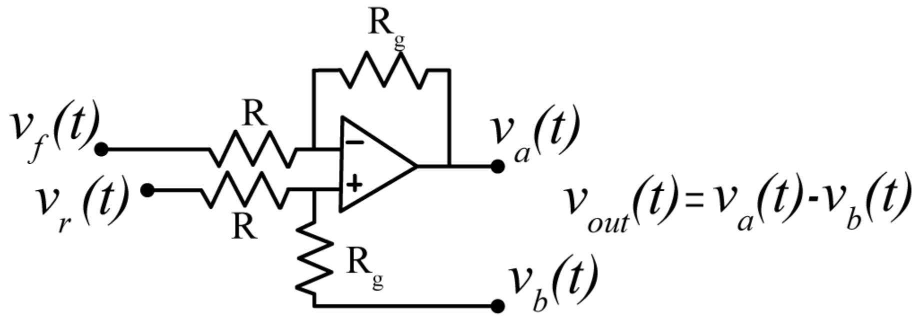


Figure 7.16. Differential amplifier circuit used in Example 7.7.3.

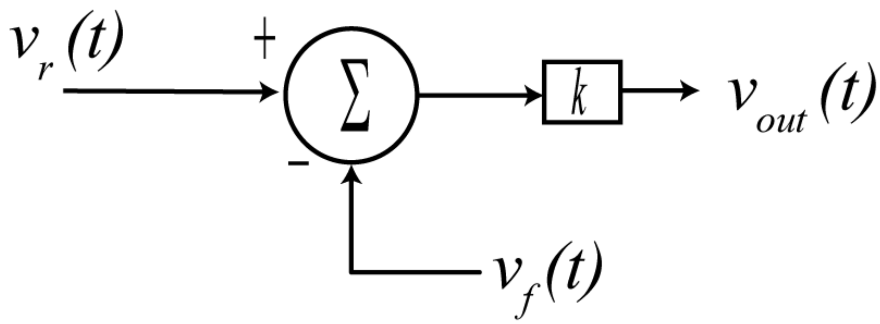


Figure 7.17. Block diagram for the differential amplifier circuit. This configuration is the basis for feedback.

Example 7.7.4. Consider the model of an armature controlled DC motor shown in Figure 7.18. Te armature (the part that does the work) is located on the rotor, while the field (the part that creates the magnetic field) is located on the stator. The field source is constant and hence the strength of the field does not vary, this means the constants K_e and K_t do not vary. The system input is the applied voltage $v_a(t)$. The developed motor torque, $T_m(t)$, is proportional to the current flowing in the loop, $i_a(t)$, so $T_m(t) = K_t i_a(t)$. The motor also develops a "back emf", $e_b(t)$, which is proportional to the speed of the motor, $e_b(t) = K_e \omega(t)$. The motor is used to drive a load with moment of inertia J, damping B, and a load torque $T_l(t)$.

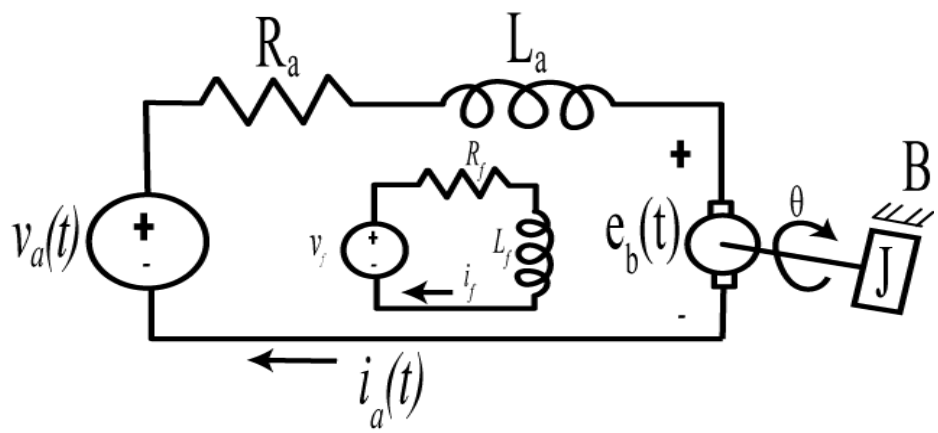


Figure 7.18. Model of an armature controlled DC motor.

To determine the current we have

$$V_a(s) - I_a(s)[R_a + L_a s] = E_b(s)$$

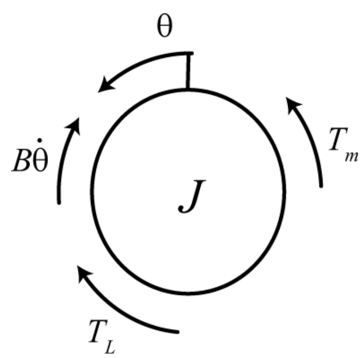
or

$$I_a(s) = \frac{V_a(s) - E_b(s)}{R_a + L_a s}$$

The torque developed by the motor is then

$$T_m(s) = K_t I_a(s)$$

This torque is then used to spin the load and overcome frictional forces and any external applied loads. The free body diagram for the load is shown below.



Conservation of angular momentum gives

$$J\ddot{\theta} = T_m - B\dot{\theta} - T_L$$

In the Laplace domain we get

$$s^2 J\Theta(s) + Bs\Theta(s) = T_m(s) - T_L(s)$$

or

$$s\Theta(s) = \Omega(s) = \frac{T_m(s) - T_L(s)}{sJ + B}$$

Finally, we have

$$E_b(s) = K_e\Omega(s)$$

The block diagram for this system is shown in Figure 7.19. Note that this system includes feedback.

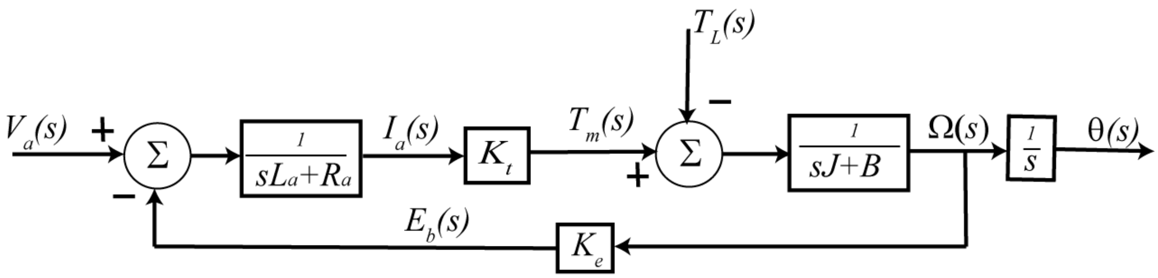


Figure 7.19. Block diagram representation for the armature controlled DC motor from Example 7.7.4.

Although block diagrams and transfer functions are widely used in various engineering disciplines for showing the interconnection of subsystems, one needs to be careful to avoid *loading effects* when representing a system with a block diagram. Consider the simple resistive circuit shown in Figure 7.20 with input $v_{in}(t)$ and output $v_{out}(t)$.

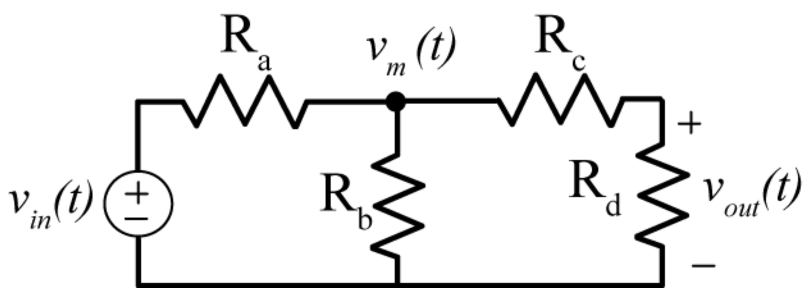


Figure 7.20. Circuit used to demonstrate loading.

At the node with voltage $v_m(t)$ we can write

$$\frac{V_{in}(s)-V_m(s)}{R_a} = \frac{V_m(s)}{R_b} + \frac{V_m(s)}{R_c+R_d}$$

This can be simplified to be

$$V_m(s) = \left[\frac{R_b(R_c+R_d)}{(R_a+R_b)(R_c+R_d)+R_aR_b}\right]V_{in}(s)$$

and finally we get

$$V_{out}(s) = \left[\frac{R_bR_d}{(R_a+R_b)(R_c+R_d)+R_aR_b}\right]V_{in}(s)$$

Now suppose we decided to write the circuit as two interconnected subsystems, as shown in Figure 7.21.

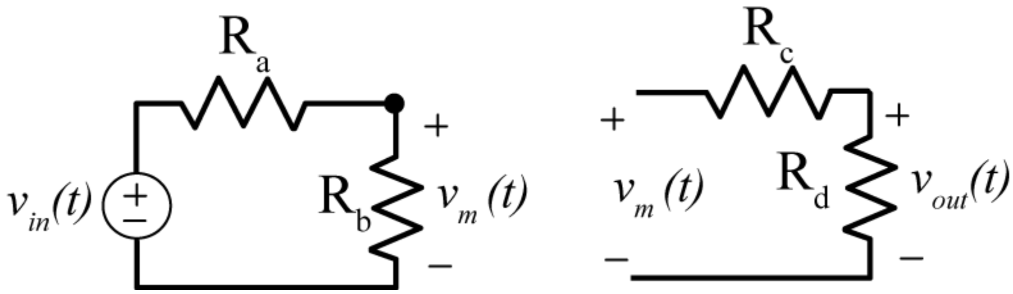


Figure 7.21. Circuit from Figure 7.20 written (incorrectly) as two subsystems.

The transfer function for the first system is clearly

$$\frac{V_m(s)}{V_{in}(s)} = \frac{R_b}{R_a + R_b}$$

and the transfer function for the second subsystem is

$$\frac{V_{out}(s)}{V_m(s)} = \frac{R_d}{R_c + R_d}$$

From this we can determine the system transfer function to be

$$\frac{V_{out}(s)}{V_{in}(s)} = \frac{R_b R_d}{(R_a + R_b)(R_c + R_d)}$$

and the output is

$$V_{out}(s) = \left[\frac{R_b R_d}{(R_a + R_b)(R_c + R_d)} \right] V_{in}(s)$$

This is clearly the wrong answer, so what when wrong? In general, if a transfer function changes after a system is connected to it, the connections is said to have a *loading effect.* For our circuit the relationship between the system input $v_{in}(t)$ and the output $v_m(t)$ changes when the second half of the circuit is added to the system. For electronic system we can often insert an isolating amplifier to remove loading between subsystems. However, loading can occur for non-electrical systems also, so you need to be aware of it.

7.8 Feedback Systems

Assume we have a transfer function, $G_p(s)$, that represents a system, and we want to make this system behave in a certain manner. Typically we call any system we are trying to control a *plant.* For example, assume we have the mass-spring-damper system shown in Figure 7.22. In this system, the input is the voltage applied to a motor (which is modeled as a simple gain) and the motor output is a force applied to the cart. The system output is the displacement of the cart from equilibrium.

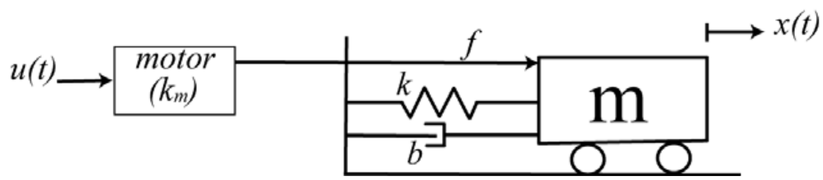
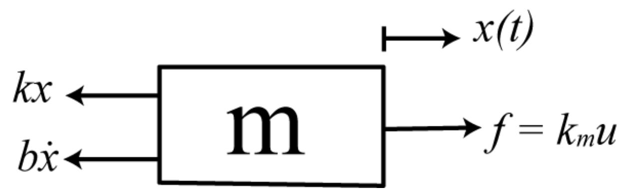


Figure 7.22. Spring-mass-damper system. The motor is modeled as a simple gain.

The input to our plant is the control signal $u(t)$ and the output is the displacement of the cart $x(t)$. Note that in this context $u(t)$ is not necessarily a unit step. It can be any allowable input. However, it is conventional in control systems to label the input to a plant as $u(t)$. A free body diagram of our system is as follows:



Applying conservation of linear momentum we get the equation of motion,

$$m\ddot{x}(t) = k_m u(t) - kx(t) - b\dot{x}(t)$$

which gives us the transfer function for the plant as

$$G_p(s) = \frac{k_m}{s^2 m + sb + k}$$

There are two general methods for trying to control the behavior of a plant, *open loop control* and *closed loop control.* These two methods are displayed in Figure 7.23. For the open loop control system, we have the system transfer function

$$G_o(s) = \frac{Y(s)}{R(s)} = \mathrm{G}_c(s) G_p(s)$$

For the closed loop system we need to do a bit more work. In analyzing a closed loop system we usually look at an intermediate signal that relates input and output and then try to eliminate any intermediate signals. So for this system we have

$$E(s) = R(s) - Y(s)H(s)$$

We can then write the output in terms of the error signal as

$$Y(s) = E(s)G_c(s)G_p(s)$$

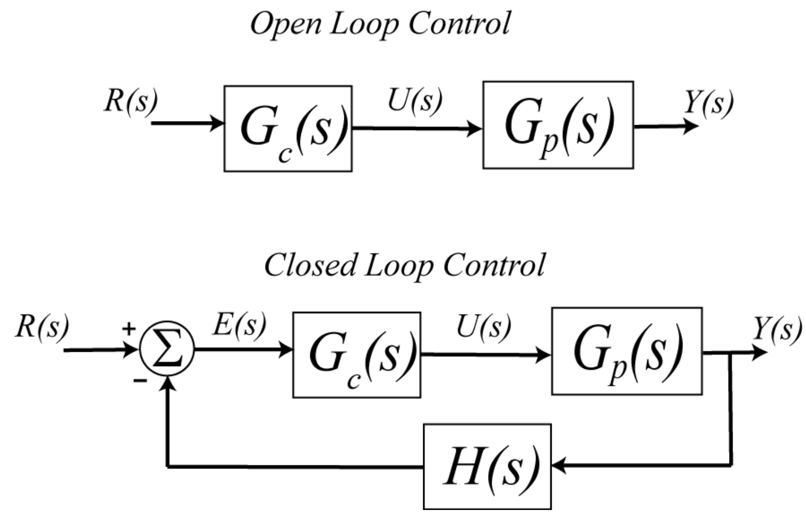


Figure 7.23. Open loop and closed loop control of a plant. Here $r(t)$ is the reference input, $u(t)$ is the control effort or control signal, $y(t)$ is the system output, and $e(t)$ is the error signal. Often the transfer function in the feedback loop, $H(s)$, is some type of transducer which converts the output to the same form at the input.

Finally we need to remove the error signal from these two equations,

$$Y(s) = [R(s) - Y(s)H(s)]G_c(s)G_p(s)$$

Rearranging these we get the closed loop transfer function.

$$G_o(s) = \frac{Y(s)}{R(s)} = \frac{G_c(s)G_p(s)}{1 + H(s)G_c(s)G_p(s)}$$

At this point, it should be obvious to you that using this transfer function to determine properties of the system is much easier than the equivalent time-domain convolution based representation,

$$y(t) \star [\delta(t) + h(t) \star g_c(t) \star g_p(t)] = r(t) \star g_c(t) \star g_p(t)$$

Most control systems are closed loop control systems. As you will see, a closed loop system has the ability to correct for errors in modeling the plant, or if the plant changes over time as components age.

7.9 Steady State Errors

Often we design a control system to track, or follow, the reference input. How well the system tracks the reference input is usually then divided into two parts: the *transient (time-varying) response*, and the *steady state response*. The most common reference input is a step input, and we can use our previously defined measures of settling time, percent overshoot, and rise time to measure how well our system tracks the step input during the transient time. Once the system has reached steady state we often want to use the steady state error as a measure of how well our system tracks the input.

The steady state error, e_{ss}, is usually defined as the difference between the reference input , $r(t)$, and output of the system, $y(t)$, in steady state, or

$$e_{ss} = \lim_{t\to\infty}\left[r(t) - y(t)\right]$$

While we can use partial fractions and inverse Laplace transforms to compute this, it is often easier to do this computation in the *s*-domain using the final value Theorem. If we assume the system is asymptotically stable, then we have

$$e_{ss} = \lim_{t\to\infty}\left[r(t) - y(t)\right] = \lim_{s\to 0} s\left[R(s) - Y(s)\right]$$

If we assume our system has transfer function $G_0(s)$, then we have

$$e_{ss} = \lim_{s\to 0} s\left[R(s) - Y(s)\right] = \lim_{s\to 0} s\left[R(s) - G_0(s)R(s)\right] = \lim_{s\to 0} sR(s)\left[1 - G_0(s)\right]$$

Finally, if we assume our input is a step of amplitude A, $r(t) = Au(t)$, then

$$R(s) = \frac{A}{s}$$

and

$$e_{ss} = \lim_{s\to 0} A\left[1 - G_0(s)\right]$$

Clearly for a steady state error of zero, we want $G_0(0) = 1$. Note also that this means we want the *static gain* of the system to be one.

Example 7.9.1. Assume we have the system transfer function

$$G_0(s) = \frac{1}{(s+1)(s+2)}$$

and the input to the system is a step of amplitude 3. Determine the steady state error in the time and *s*-domain. In the time-domain we can use partial fractions,

$$Y(s) = G_0(s)R(s) = \frac{1}{(s+1)(s+2)} \times \frac{3}{s} = \frac{3}{2}\frac{1}{s} - 3\frac{1}{s+1} + \frac{3}{2}\frac{1}{s+2}$$

or

$$y(t) = \frac{3}{2}u(t) - 3e^{-t}u(t) + \frac{3}{2}e^{-2t}u(t)$$

Then

$$e_{ss} = \lim_{t\to\infty}[3u(t) - y(t)] = \lim_{t\to\infty}[3u(t) - \frac{3}{2}u(t) + 3e^{-t}u(t) - \frac{3}{2}e^{-2t}u(t)] = \frac{3}{2}$$

In the *s*-domain we have

$$e_{ss} = \lim_{s\to 0} A\left[1 - G_0(s)\right] = 3[1 - G_0(0)] = \frac{3}{2}$$

Example 7.9.2. For the system depicted in Figure 7.14, determine the value of the *prefilter gain* , G_{pf} , so the steady state error for a step is zero. For this system we have the closed loop transfer function

$$G_0(s) = \frac{\frac{1}{s+2}\frac{3}{s+1}G_{pf}}{1+5\frac{1}{s+2}\frac{3}{s+1}} = \frac{3G_{pf}}{(s+2)(s+1)+15}$$

For zero steady state error we need

$$G_0(0) = 1 = \frac{3G_{pf}}{17}$$

So we need $G_{pf} = 17/3$. Note that in this case we do not really need to simplify the transfer function, we can directly evaluate the transfer function at $s = 0$,

$$G_0(s) = \frac{\frac{1}{2}\frac{3}{1}G_{pf}}{1+5\frac{1}{2}\frac{3}{1}} = \frac{\frac{3}{2}G_{pf}}{1+\frac{15}{2}} = \frac{3G_{pf}}{17}$$

Clearly in this example we want the prefilter gain to be 17/3.

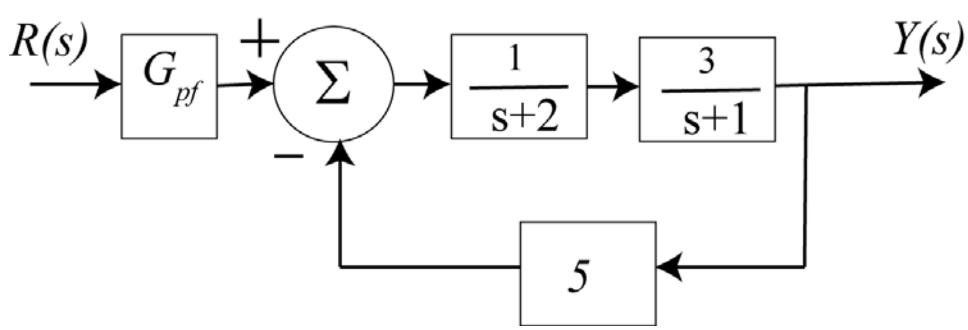


Figure 7.14. Block diagram for Example 7.9.2. For a unit step input, the prefilter should be 17/3 to produce zero steady state error.

Example 7.9.3. For the system depicted in Figure 7.15, determine the value of the prefilter so the steady state error for a step is zero. For this system we have the closed loop transfer function

$$G_0(s) = \frac{\frac{1}{s}\frac{2}{s^2+2s+1}G_{pf}}{1+\frac{1}{s}\frac{2}{s^2+2s+1}}$$

Note that we cannot immediately set $s=0$ in this form. We could multiply the transfer function out, but it is easier to just multiply the top and bottom by s,

$$G_0(s) = \frac{\frac{2}{s^2+2s+1}G_{pf}}{s+\frac{2}{s^2+2s+1}}$$

Now we can set $s=0$ to get $G_0(0) = G_{pf}$, so for a zero steady state error we need the prefilter to be 1.

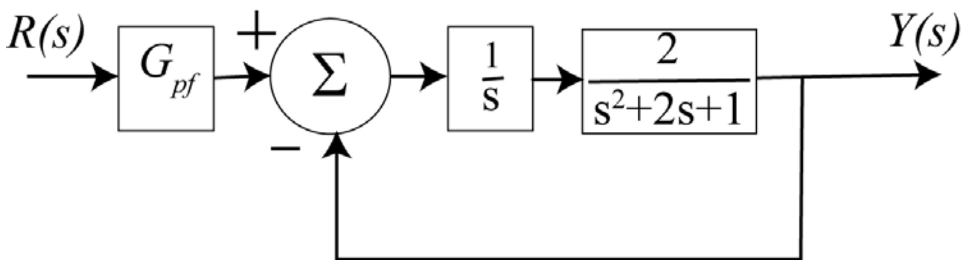


Figure 7.15. Block diagram for Example 7.9.3. For a unit step input, the prefilter should be one to produce zero steady state error.

Example 7.9.4. For the system depicted in Figure 7.16, determine the value of the parameter k so the steady state error for a unit step is less than or equal to 0.1, $e_{ss} \le 0.1$. For this system we have the closed loop transfer function

$$G_0(s) = \frac{k\frac{3s+2}{4s^2+5s+1}}{1+k\frac{3s+2}{4s^2+5s+1}}$$

and

$$G_0(0) = \frac{2k}{1+2k}$$

Then we want

$$e_{ss} = 1 - G_0(0) = 1 - \frac{2k}{1+2k} = \frac{1}{1+2k} \le \frac{1}{10}$$

or

$$1+2k \ge 10$$

This means we need $k \geq 4.5$.

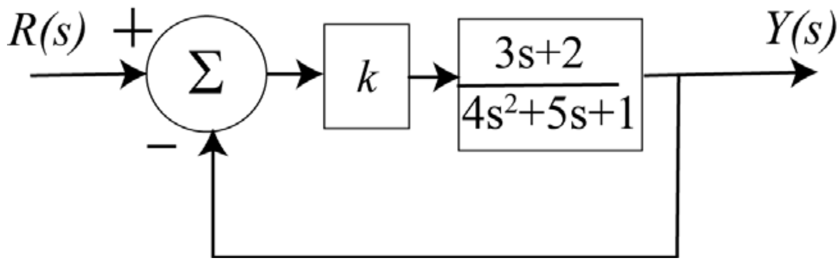


Figure 7.16. Block diagram for Example 7.9.4. For a unit step input, if we want the steady state error less than or equal to 0.1, we need $k > 4.5$.

7.10 Initial Control Effort

Although it is often straightforward to design a control system to produce a given steady state error or an acceptable transient response, sometimes these controllers require a control effort that is not possible to produce. In many, though not all, instances, the initial control effort is the largest control effort when the input is a step. In order to quickly determine the initial control effort we use the initial value Theorem. Recall that the initial value Theorem stated that if $x(t) \leftrightarrow X(s)$ and $X(s)$ is asymptotically stable, then

$$\lim_{t \to 0^+} x(t) = \lim_{s \to \infty} sX(s)$$

For our standard closed loop system in Figure 7.17, the control effort is denoted by $U(s)$. We can solve for this as follows:

$$E(s) = R(s) - H(s)Y(s)$$
$$Y(s) = E(s)G_c(s)G_p(s)$$

and

$$U(s) = E(s)G_c(s)$$

Hence

$$Y(s) = U(s)G_p(s)$$
$$E(s) = \frac{U(s)}{G_c(s)}$$

Combining these we have

$$\frac{U(s)}{G_c(s)} = R(s) - H(s)G_p(s)U(s)$$
$$U(s) = R(s)G_c(s) - H(s)G_c(s)G_p(s)U(s)$$

This yields the following expression for the control effort

$$U(s) = \frac{G_c(s)R(s)}{1 + H(s)G_c(s)G_p(s)}$$

Finally, to determine the initial control effort, we have

$$u(0^+) = \lim_{s\to\infty} sU(s) = \lim_{s\to\infty} \frac{sG_c(s)R(s)}{1 + H(s)G_c(s)G_p(s)}$$

If we assume the input is a step of amplitude A then we have

$$u(0^+) = \lim_{s\to\infty} \frac{G_c(s)A}{1 + H(s)G_c(s)G_p(s)}$$

As an example, let's consider a plant with the transfer function

$$G_p(s) = \frac{5}{s^2 + 2s + 2}$$

We will assume the closed loop control configuration shown in Figure 7.17, and look at the results using three different controllers. We will assume our reference input is a unit step for all three examples.

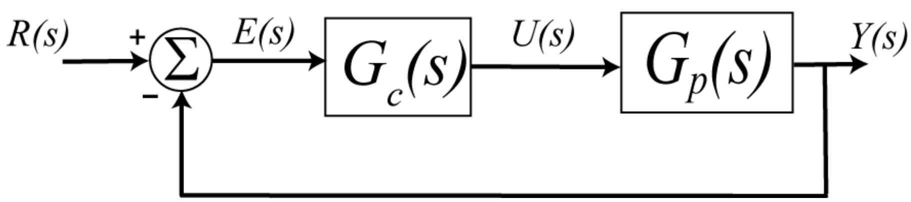


Figure 7.17. Block diagram used for controlling the plant $G_p(s) = \frac{5}{s^2 + 2s + 2}$.

In the first case, we will assume we have a *proportional (P) controller,* where the control effort is proportional to the error signal. Here we will have

$$G_c(s) = k_p,\ k_p = 20$$

where we have assigned $k_p = 20$. The steady state error for this system can easily be determined to be $e_{ss} = 0.02$. The closed loop poles are at $-1 \pm j10.05$ which gives an approximate settling time of 4 seconds.

In the second case, we will assume we have a *proportional plus integral (PI) controller*. Here the control effort is made up of two components, one is proportional to the error signal, and one is proportional to the integral of the error signal. Here we will have

$$G_c(s) = k_p + \frac{k_i}{s},\ k_p = 0.04, k_i = 0.2$$

where we have assigned $k_p = 0.04$ and $k_i = 0.2$. The steady state error for this system is zero, $e_{ss} = 0$. The closed loop poles are at (approximately) $-0.60 \pm j0.94$ and -0.81, which gives an approximate settling time of 6.7 seconds.

In the third case, we will assume we have a *proportional plus derivative (PD) controller*. Here the control effort is again made up of two components, one is again proportional to the error signal and the other is proportional to the derivative of the error signal. This controller will have the form

$$G_c(s) = k_p + k_d s,\ k_p = 8, k_d = 0.4$$

where we have assumed $k_p = 8$ and $k_d = 0.4$. The steady state error for this system is approximately 0.05. The closed loop poles are at (approximately) $-2.0 \pm j6.16$ which gives an approximate settling time of 2 seconds.

The response of the plant to each of these controllers is shown in Figure 7.18.

Next let's look at the initial control effort for each of these controllers. For the proportional controller we have

$$u(0^+) = \lim_{s\to\infty} \frac{k_p}{1 + (1)(k_p)\left(\frac{5}{s^2 + 2s + 1}\right)} = k_p = 20$$

For the proportional plus integral controller we have

$$u(0^+) = \lim_{s\to\infty} \frac{k_p + \frac{k_i}{s}}{1 + (1)\left(k_p + \frac{k_i}{s}\right)\left(\frac{5}{s^2 + 2s + 1}\right)} = k_p = 0.04$$

Finally, for the proportional plus derivative controller we have

$$u(0^+) = \lim_{s\to\infty} \frac{k_p + k_d s}{1 + (1)(k_p + k_d s)\left(\frac{5}{s^2 + 2s + 1}\right)} = \infty$$

For these three controllers, the PI controller requires the least initial control effort. While the control effort for the P controller is finite, it may be more difficult to implement this controller using op amps with fixed voltage sources. Finally, the initial control effort is infinite for the PD controller. However, often this means the source will just saturate and not reach infinity. However, it is something you will need to be aware of.

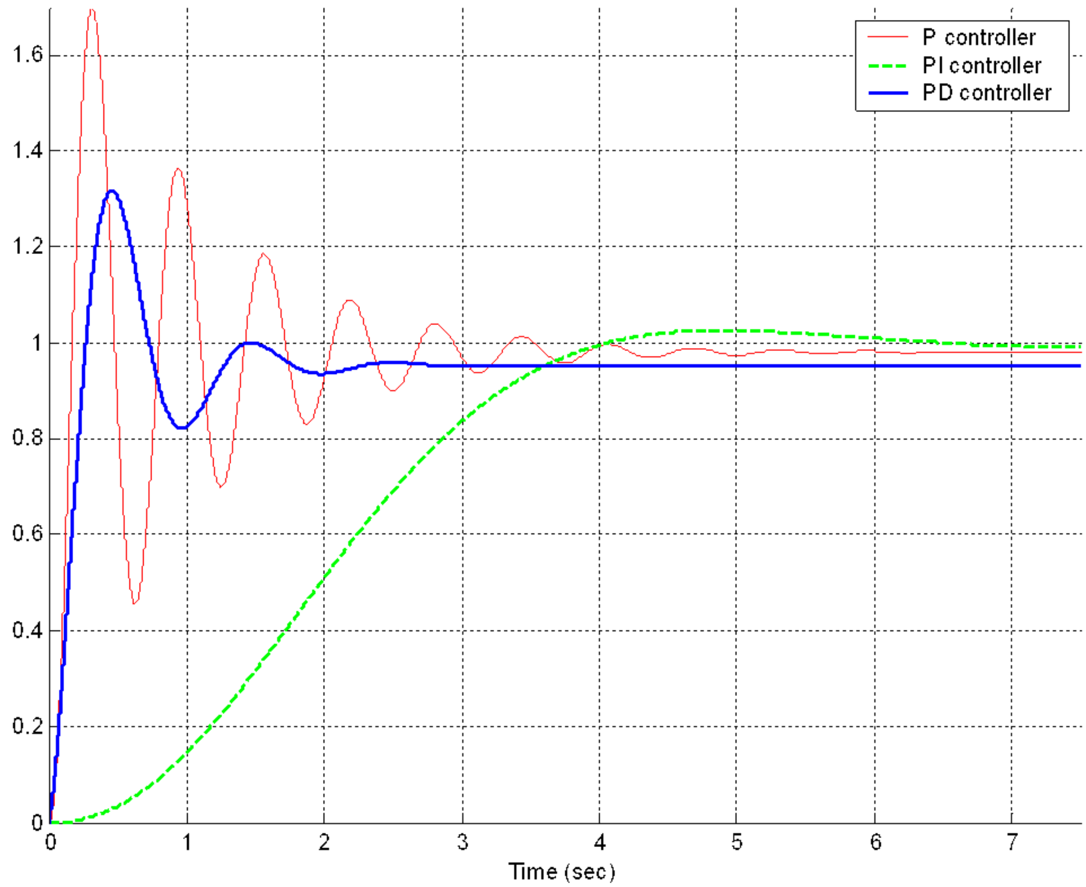


Figure 7.18. The response of the plant $G_p(s) = \frac{5}{s^2 + 2s + 2}$ to the proportional (P) controller $G_c(s) = 20$, the proportional plus integral (PI) controller $G_c(s) = 0.04 + \frac{0.2}{s}$, and the proportional plus derivative controller $G_c(s) = 8 + 0.4s$. The input was a unit step.