6.0 Laplace Transforms

In many applications we have interconnections of LTI systems. We can determine the output of the system by using convolution in the time-domain, but this often proves to be difficult when we have more than just a few interconnecting systems. Sometimes we don't want to just compute the output, but rather we want to be able to determine properties of the system in a simple way. We will utilize Laplace transforms for this, though in some applications the use of Fourier transforms may be more appropriate.

6.1 Laplace Transform Definitions

The *two-sided* Laplace transform of a signal $x(t)$ is defined to be

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

while the *one-sided* Laplace transform is defined as

$$X(s) = \int_{0^-}^{\infty} x(t)e^{-st}dt$$

The only difference between the two definitions is the lower limit, and one and two sided refers to integrating one on side of zero or both sides of zero. In this course we will only use the one-sided transform, and when we refer to the Laplace transform we mean the one sided transform. However, you should be aware the two-sided transform has some use when we are dealing with *noncausal* systems. Looking at the form of the integral, since the exponent must be dimensionless, we can conclude that the variable s has the units of 1/time.

There are a few conventions we need to know about. First of all, the lower limit on the one sided Laplace transform is generally written as 0^-, or starting at a time just before zero. This is particularly useful when determining the Laplace transform of an impulse centered at zero. Secondly, a very common convention is to use lower case letters for time-domain functions, and capital letters for the corresponding transform domain. We usually write $x(t) \leftrightarrow X(s)$ or $\mathcal{L}\{x(t)\} = X(s)$ to show that $x(t)$ and $X(s)$ are transform pairs. We dill denote the Laplace transform operator as $\mathcal{L}$. Finally, the complex variable s is sometimes written in terms of its real and imaginary parts as $s = \sigma + j\omega$. This is particularly useful in determining if the integral is finite (or can be made to be finite) or is infinite.

6.2 Basic Laplace Transforms

Let's start off by determining the Laplace transform of some basic signals. As you will see sometimes we need to put conditions on σ to be sure the integral converges. This condition defines the *region of convergence.*

Example 6.2.1. For $x(t) = \delta(t)$ we have

$$X(s) = \int_{0-}^{\infty} \delta(\lambda) e^{-s\lambda} d\lambda = e^{s0} \int_{0^-}^{\infty} \delta(\lambda) d\lambda = 1$$

since the delta function is contained in the region of integration. Hence we have the transform pair

$$x(t) = \delta(t) \leftrightarrow X(s) = 1$$

Example 6.2.2. For $x(t) = \delta(t - t_0)$, where $t_0 \geq 0$, we have

$$X(s) = \int_{0^-}^{\infty} \delta(\lambda - t_0) e^{-s\lambda} d\lambda = e^{-st_0} \int_{0-}^{\infty} \delta(\lambda - t_0) d\lambda = e^{-st_0}$$

since again the delta function is located in the region of integration. Hence we have the transform pair

$$x(t) = \delta(t - t_0) \leftrightarrow X(s) = e^{-st_0}$$

Note that if $t_0 < 0$ the integral will be zero.

Example 6.2.3. For $x(t) = u(t)$ we have

$$X(s) = \int_{0^-}^{\infty} u(\lambda) e^{-s\lambda} d\lambda = \int_{0}^{\infty} e^{-s\lambda} d\lambda = \left. \frac{e^{-s\lambda}}{-s} \right|_{\lambda=0}^{\lambda=\infty}$$

At this point we cannot really evaluate this integral unless we put some conditions on σ. Let's make the substitution $s = \sigma + j\omega$ and we have

$$X(s) = \left. \frac{e^{-(\sigma + j\omega)\lambda}}{-s} \right|_{\lambda=0}^{\lambda=\infty} = \left. \frac{e^{-\sigma\lambda + j\omega\lambda}}{-s} \right|_{\lambda=0}^{\lambda=\infty} = \left. \frac{e^{-\sigma\lambda} e^{j\omega\lambda}}{-s} \right|_{\lambda=0}^{\lambda=\infty}$$

Using Euler's identity

$$e^{j\omega\lambda} = \cos(\omega\lambda) + j\sin(\omega\lambda)$$

We can determine the magnitude of $e^{j\omega\lambda}$ as

$$|e^{j\omega\lambda}| = \sqrt{\cos^2(\omega\lambda) + \sin^2(\omega\lambda)} = 1$$

Hence the term $e^{j\omega\lambda}$ does not contribute to the convergence of the integral. That means that ω does not contribute to the convergence of the integral. That leaves us with σ. If $\sigma > 0$, then the exponent in the exponential is negative, when we evaluate it at the limit of infinity we get zero. Hence we have the transform pair

$$x(t) = u(t) \leftrightarrow X(s) = \frac{1}{s}, \quad \sigma > 0$$

Note that the condition $\sigma > 0$, or the real part of s must be positive, defines the region of convergence. Hence the integral will converge if $\Re\{s\} > 0$ and we can rewrite the transform pair as

$$x(t)=u(t)\leftrightarrow X(s)=\frac{1}{s},\quad \Re\{s\}>0$$

Example 6.2.4. For $x(t)=u(t-t_0)$ we have

$$X(s)=\int_{0^-}^{\infty}u(\lambda-t_0)e^{-s\lambda}d\lambda=\int_{t_0}^{\infty}e^{-s\lambda}d\lambda=\left.\frac{e^{-s\lambda}}{-s}\right|_{\lambda=t_0}^{\lambda=\infty}=\frac{e^{-st_0}}{s},\quad \Re\{s\}>0$$

The region of convergence is the same as for the non-delayed unit step. So we have the transform pair

$$x(t)=u(t-t_0)\leftrightarrow X(s)=\frac{e^{-st_0}}{s},\quad \Re\{s\}>0$$

Example 6.2.5. For $x(t)=e^{-at}u(t)$ we have

$$X(s)=\int_{0^-}^{\infty}e^{-a\lambda}u(\lambda)e^{-s\lambda}d\lambda=\int_{0}^{\infty}e^{-(s+a)\lambda}\mathrm{d}\lambda=\left.\frac{e^{-(s+a)\lambda}}{s+a}\right|_{\lambda=0}^{\lambda=\infty}$$

Now in order for the integral to converge, we need $\Re\{s+a\}>0$ .We can rewrite this as $\Re\{s\}+\Re\{a\}>0$ or $\Re\{s\}>-\Re\{a\}$. Hence the region of convergence is defined as

$\Re\{s\}>-\Re\{a\}$ and we have the Laplace transform pair

$$x(t)=e^{-at}u(t)\leftrightarrow X(s)=\frac{1}{s+a},\ \Re\{s\}>-\Re\{a\}$$

Example 6.2.6. For $x(t)=\cos(\omega_0 t)u(t)$ we will need to use Euler's identity in the form

$$\cos(\omega_0 t)=\frac{e^{j\omega_0 t}+e^{-j\omega_0 t}}{2}$$

Computing the Laplace transform we have

$$X(s)=\int_{0^-}^{\infty}\cos(\omega_0\lambda)u(\lambda)e^{-s\lambda}d\lambda=\int_{0}^{\infty}\frac{1}{2}\left[e^{j\omega_0\lambda}+e^{-j\omega_0\lambda}\right]e^{-s\lambda}d\lambda$$

$$=\frac{1}{2}\int_{0}^{\infty}e^{(j\omega_0-s)\lambda}d\lambda+\frac{1}{2}\int_{0}^{\infty}e^{-(j\omega_0+s)\lambda}d\lambda=\frac{1}{2}\left.\frac{e^{(j\omega_0-s)\lambda}}{j\omega_0-s}\right|_{\lambda=0}^{\lambda=\infty}-\frac{1}{2}\left.\frac{e^{-(j\omega_0+s)\lambda}}{(j\omega_0+s)}\right|_{\lambda=0}^{\lambda=\infty}$$

Both integrals will converges if $\Re\{s\}>0$, and then we have

$$X(s)=\frac{1}{2}\frac{1}{s-j\omega_0}+\frac{1}{2}\frac{1}{s+j\omega_0}=\frac{1}{2}\frac{s+j\omega_0+s-j\omega_0}{s^2+\omega_0^2}=\frac{s}{s^2+\omega_0^2}$$

The Laplace transform pair is then

$$x(t) = \cos(\omega_0 t)\, u(t) \leftrightarrow X(s) = \frac{s}{s^2 + \omega_0^2},\ \Re\{s\} > 0$$

Example 6.2.7. Let's assume we want to find the Laplace transform of the derivative of $x(t)$. Then we have

$$\mathcal{L}\left\{\frac{dx(t)}{dt}\right\} = \int_{0^-}^{\infty} \frac{dx(\lambda)}{d\lambda} e^{-s\lambda} d\lambda$$

In order to evaluate this we will need to use integration by parts. For two functions $u(\lambda)$ and $v(\lambda)$, we can write $d(uv) = vdu + udv$. Rearranging this we get the usual form for integration by parts, $\int udv = uv - \int vdu$. For our integral we have

$$\frac{dv(\lambda)}{d\lambda} = \frac{dx(\lambda)}{d\lambda}$$

so $dv = dx$ or $v(\lambda) = x(\lambda)$. We also have $u(\lambda) = e^{-s\lambda}$, so

$$\frac{du(\lambda)}{d\lambda} = -se^{-s\lambda}$$

or

$$du = -se^{-s\lambda} d\lambda$$

Combining these we have

$$\mathcal{L}\left\{\frac{dx(t)}{dt}\right\} = \int_{0^-}^{\infty} \frac{dx(\lambda)}{d\lambda} e^{-s\lambda} d\lambda = x(\lambda)e^{-s\lambda}\Big|_{\lambda=0^-}^{\lambda=\infty} + s\int_{0^-}^{\infty} x(\lambda)e^{-s\lambda} d\lambda = -x(0^-) + sX(s)$$

So we have the transform pair

$$\mathcal{L}\left\{\frac{dx(t)}{dt}\right\} = sX(s) - x(0^-)$$

Table 6.1 summarizes some common Laplace transforms.

6.3 Laplace Transforms of RLC Circuits

In this section we will determine the Laplace transforms of resistors, capacitors, and inductors, and then use these relationships in some examples to determine the output of some RLC circuits in the Laplace domain. In the next sections we will review the use of partial fractions to go back from the Laplace domain to the time domain.

Consider a resistor element shown in the left panel of Figure 6.1. If $v(t)$ is the voltage across the resistor, $i(t)$ is the current through the resistor, and R is the resistance, then we have by Ohm's law $v(t) = i(t)R$. Taking Laplace transforms of both sides of this

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equation we have $\mathcal{L}\{v(t)\} = \mathcal{L}\{i(t)\mathrm{R}\}$ or $V(s) = I(s)R$. Thus, in the Laplace domain, the equivalent impedance is still just R and the resistive circuit element in the left panel of Figure 6.3.1 is replaced with the circuit element in the right panel of Figure 6.1.

$$\mathcal{L}\{\delta(t)\} = 1$$

$$\mathcal{L}\{u(t)\} = \frac{1}{s}$$

$$\mathcal{L}\{tu(t)\} = \frac{1}{s^2}$$

$$\mathcal{L}\left\{\frac{t^{m-1}}{(m-1)!}u(t)\right\} = \frac{1}{s^m}$$

$$\mathcal{L}\{e^{-at}u(t)\} = \frac{1}{s+a}$$

$$\mathcal{L}\{te^{-at}u(t)\} = \frac{1}{(s+a)^2}$$

$$\mathcal{L}\left\{\frac{t^{(m-1)}}{(m-1)!}e^{-at}u(t)\right\} = \frac{1}{(s+a)^m}$$

$$\mathcal{L}\{\cos(\omega_0 t)u(t)\} = \frac{s}{s^2+\omega_0^2}$$

$$\mathcal{L}\{\sin(\omega_0 t)u(t)\} = \frac{\omega_0}{s^2+\omega_0^2}$$

$$\mathcal{L}\{e^{-\alpha t}\cos(\omega_0 t)u(t)\} = \frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}$$

$$\mathcal{L}\{e^{-\alpha t}\sin(\omega_0 t)u(t)\} = \frac{\omega_0}{(s+\alpha)^2+\omega_0^2}$$

$$\mathcal{L}\left\{\frac{dx(t)}{dt}\right\} = sX(s) - x(0^-)$$

$$\mathcal{L}\left\{\frac{d^2x(t)}{dt^2}\right\} = s^2X(s) - sx(0^-) - \dot{x}(0^-)$$

$$\mathcal{L}\{x(t-a)\} = e^{-as}X(s)$$

$$\mathcal{L}\{e^{-at}x(t)\} = X(s+a)$$

$$\mathcal{L}\left\{x\left(\frac{t}{a}\right), a>0\right\} = aX(as)$$

Table 6.1. Some common Laplace transform pairs.

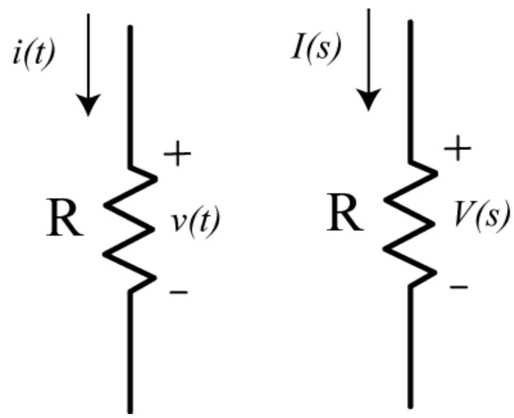

Figure 6.1. Resistive element in the time domain (left) and Laplace domain (right).

Consider the capacitive circuit element shown in the left panel of Figure 6.2. If $v(t)$ is the voltage across the capacitor, $i(t)$ is the current through the capacitor, and C is the capacitance, then we have by Ohm's law $i(t) = C\frac{dv(t)}{dt}$. Taking Laplace transforms of both sides of this equation we have

$$\mathcal{L}\{i(t)\} = I(s) = \mathcal{L}\left\{C\frac{dv(t)}{dt}\right\} = C\mathcal{L}\left\{\frac{dv(t)}{dt}\right\} = C\left\{sV(s) - v(0^-)\right\}$$

or

$$I(s) = CsV(s) - Cv(0^-)$$

Rewriting this we have

$$V(s) = I(s)\frac{1}{Cs} + \frac{v(0^-)}{s}$$

If for now we ignore the initial conditions we have

$$V(s) = I(s)\frac{1}{Cs}$$

which means the capacitor has an equivalent impedance of $\frac{1}{Cs}$. If we make the substitution $s = j\omega$, we have the equivalent impedance $\frac{1}{j\omega C}$ which is identical to what you used in sinusoidal (phasor) analysis. Thus, in the Laplace domain, the capacitor circuit element in the left panel of Figure 6.2 is replaced with the two circuit elements in the right panel of Figure 6.2.

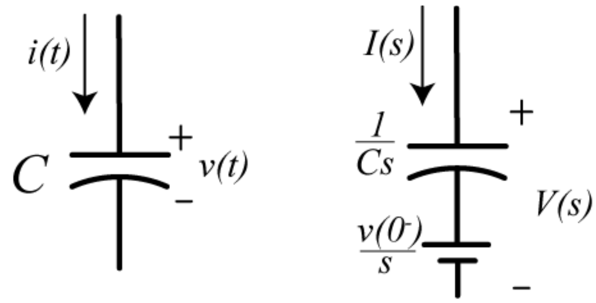

Figure 6.2. Capacitive element in the time domain (left) and Laplace domain (right).

Consider the inductive circuit element shown in the left panel of Figure 6.3. If $v(t)$ is the voltage across the inductor, $i(t)$ is the current through the inductor, and L is the inductance, then we have by Ohm's law. $v(t) = L\frac{di(t)}{dt}$ Taking Laplace transforms of both sides of this equation we have

$$\mathcal{L}\left\{v(t)\right\} = V(s) = \mathcal{L}\left\{L\frac{di(t)}{dt}\right\} = L\mathcal{L}\left\{\frac{di(t)}{dt}\right\} = L\left\{sI(s) - i(0^-)\right\}$$

or

$$V(s) = LsI(s) - Li(0^-)$$

We can rearrange this equation into its more common form as

$$I(s) = V(s)\frac{1}{Ls} + \frac{i(0^-)}{s}$$

Again, if we ignore the initial condition term we have $I(s) = V(s)\frac{1}{Ls}$ which means the inductor has the equivalent impedance sL. Making the substitution $s = j\omega$ we have the equivalent impedance $j\omega L$ which is identical the what you used in sinusoidal analysis. Thus, in the Laplace domain, the inductor circuit element in the left panel of Figure 6.3 is replaced with the circuit elements in the right panel of Figure 6.3.

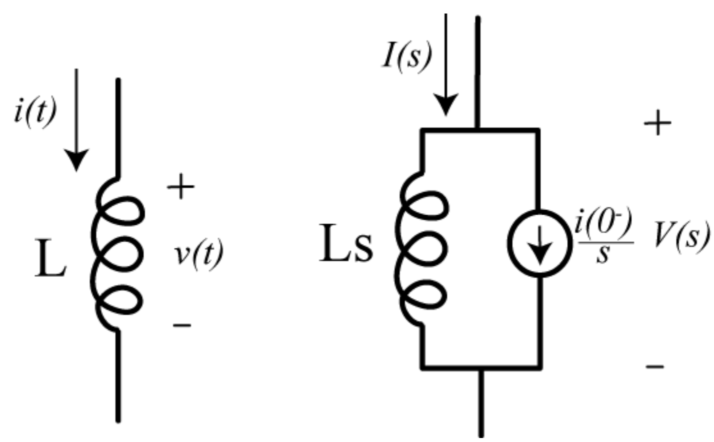

Figure 6.3. Inductive element in the time domain (left) and Laplace domain (right).

Note that the substitution $s = j\omega$ and ignoring the initial conditions gives the same equivalent impedances for these circuit elements as you used before in your sinusoidal steady state analysis. We will have more to say about this substitution when we talk about frequency response. However, this is only one possible value of s.

In the following examples we use Laplace transforms of the circuit elements to write the output of the circuit in terms of the input and initial conditions

Example 6.3.1. Consider the RC circuit shown in Figure 6.4. The output voltage is the voltage across the resistor, and the input is voltage $v_{in}(t)$. We assume $i(t)$ is the current flowing in the circuit. We want to write the output of the circuit in terms of the input and initial voltage on the capacitor, $v(0^-)$, in the Laplace domain. The circuit is redrawn in the Laplace domain in Figure 6.5. Going around the loop we have

$$V_{in}(s) - I(s)R - I(s)\frac{1}{Cs} = \frac{v(0^-)}{s}$$

Solving for the current we have

$$I(s) = \frac{V_{in}(s) - \frac{v(0^-)}{s}}{R + \frac{1}{Cs}}$$

The system output is the voltage across the resistor, so we have

$$V_{out}(s) = I(s)R = \frac{\left[V_{in}(s) - \frac{v(0^-)}{s}\right]R}{R + \frac{1}{Cs}}$$

Finally we can write the output as the sum of two different parts. The ***Zero State Response*** (ZSR) is the response of the system to the input alone, assuming no initial conditions. The ***Zero Input Response*** (ZIR) is the response of the system to the initial conditions alone, assuming there is no input. Hence our final solution is

$$V_{out}(s) = \frac{V_{in}(s)R}{R + \frac{1}{Cs}} - \frac{\frac{v(0^-)}{s}R}{R + \frac{1}{Cs}} = \underbrace{\left[\frac{V_{in}(s)RCs}{RCs+1}\right]}_{ZSR} + \underbrace{\left[-\frac{v(0^-)RC}{RCs+1}\right]}_{ZIR}$$

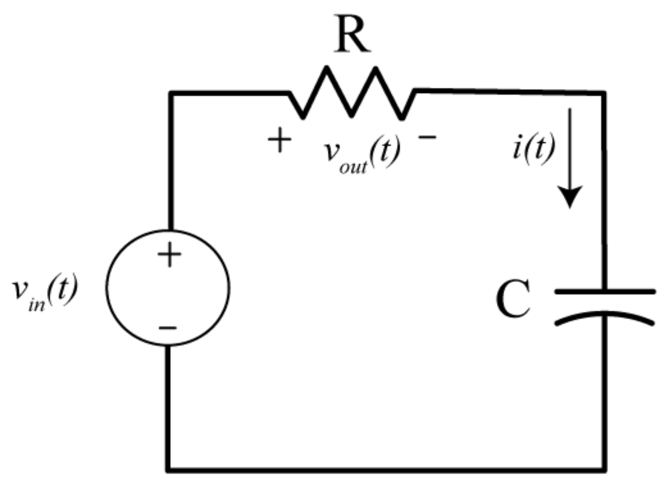

Figure 6.4. Circuit for Example 6.3.1.

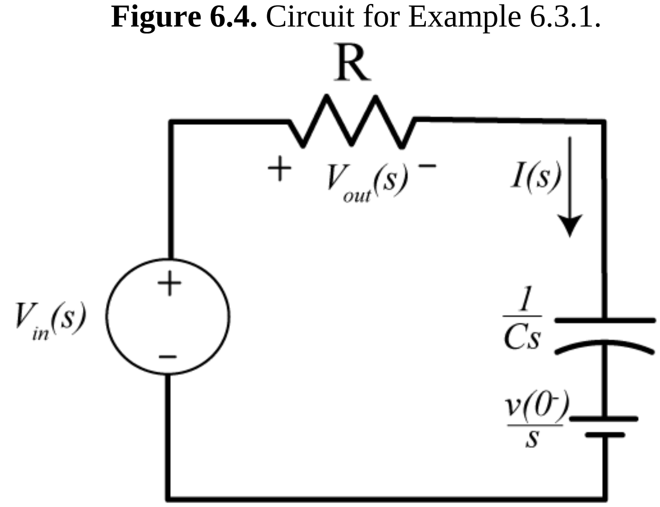

Figure 6.5. Circuit from Example 6.3.1 in the Laplace domain.

Example 6.3.2. Consider the RL circuit shown in Figure 6.6. We again assume the output of the system is the voltage across the resistor, and want to determine the output of the system in terms of the input voltage and the initial current in the inductor $i(0^-)$ in the Laplace domain. In Figure 6.7 the circuit has been redrawn in the Laplace domain. To analyze this circuit, let's define the voltage across the inductor as $V_L(s)$, so we have $V_L(s) = V_{in}(s) - V_{out}(s)$. Equating currents we then have

$$\frac{V_{out}(s)}{R} = \frac{V_L(s)}{Ls} + \frac{i(0^-)}{s} = \frac{V_{in}(s) - V_{out}(s)}{Ls} + \frac{i(0^-)}{s}$$

Rearranging we get

$$V_{out}(s)\left[\frac{1}{R} + \frac{1}{Ls}\right] = V_{out}(s)\left[\frac{R+Ls}{RLs}\right] = \frac{V_{in}(s)}{Ls} + \frac{i(0^-)}{s}$$

and finally

$$V_{out}(s) = \underbrace{\left[\frac{R}{R+Ls}V_{in}(s)\right]}_{ZSR} + \underbrace{\left[\frac{RL}{R+Ls}i(0^-)\right]}_{ZIR}$$

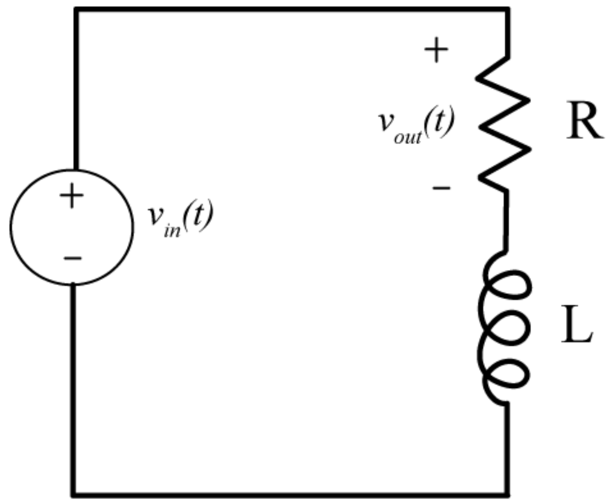

Figure 6.6. Circuit for Example 6.3.2.

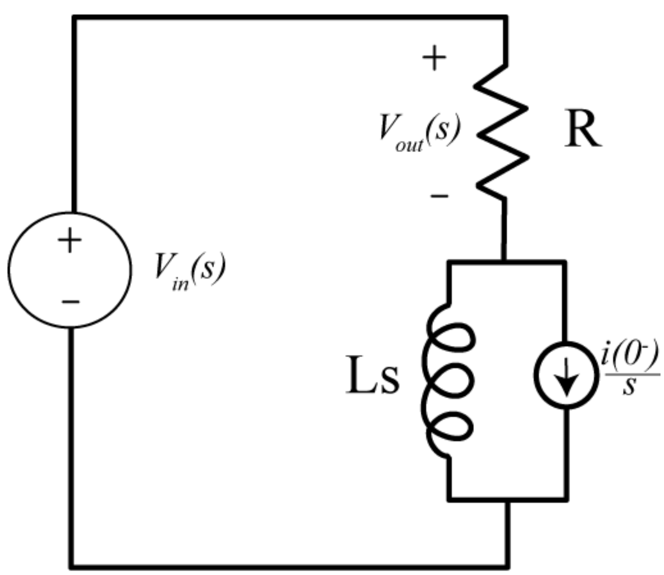

Figure 6.7. Circuit from Example 6.3.2 in the Laplace domain.

Example 6.3.3. Consider the RLC circuit shown in Figure 6.8. We again assume the output of the system is the voltage across the resistor, and want to determine the output of the system in terms of the input current and both the initial voltage across the capacitor $v(0^-)$ and the initial current in the inductor $i(0^-)$ in the Laplace domain. In Figure 6.9 the circuit has been redrawn in the Laplace domain. To analyze this circuit we need to equate all of the currents, as follows:

$$I_{in}(s) = \frac{V_{out}(s)}{R} + \frac{\left[V_{out}(s) - \frac{v(0^-)}{s}\right]}{\frac{1}{Cs}} + \frac{V_{out}(s)}{Ls} + \frac{i(0^-)}{s}$$

Combining terms we have

$$I_{in}(s) = V_{out}(s)\left[\frac{1}{R} + Cs + \frac{1}{Ls}\right] - Cv(0^-) + \frac{i(0^-)}{s}$$

or

$$I_{in}(s) = V_{out}(s)\left[\frac{RLCs^2 + Ls + R}{RLs}\right] - Cv(0^-) + \frac{i(0^-)}{s}$$

Finally we get

$$V_{out}(s) = \underbrace{\left[\frac{I_{in}(s)RLs}{RLCs^2 + Ls + R}\right]}_{ZSR} + \underbrace{\left[\frac{-RLi(0^-) + RLCsv(0^-)}{RLCs^2 + Ls + R}\right]}_{ZIR}$$

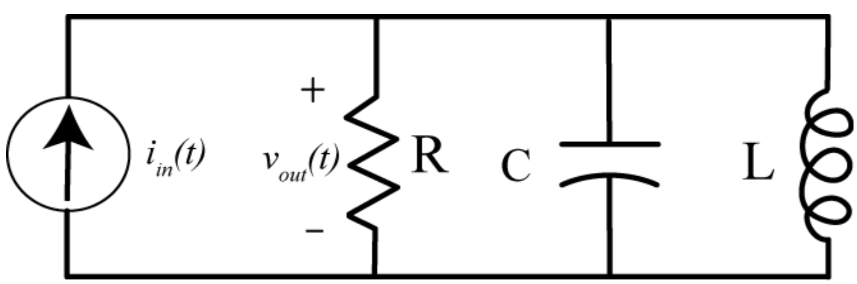

Figure 6.8. Circuit for Example 6.3.3.

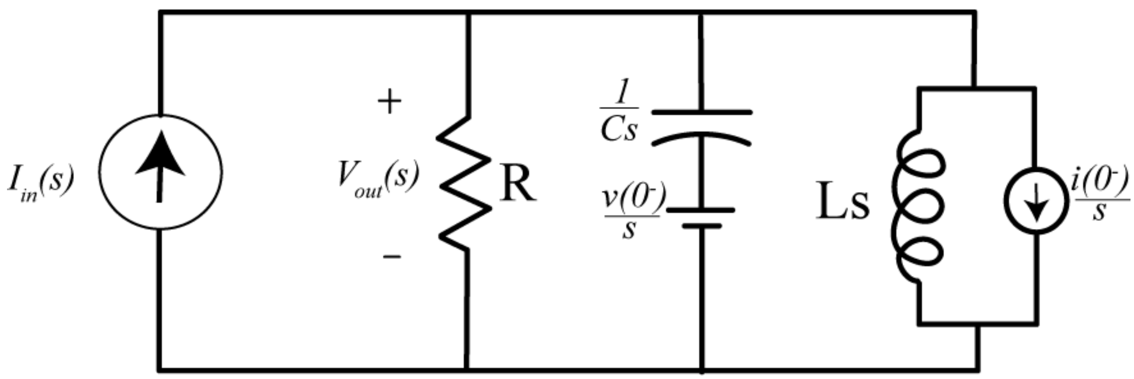

Figure 6.9. Circuit from Example 6.3.3 in the Laplace domain.

6.4 Transfer Functions and the Impulse Response

Assume we have and LTI system that is initially at rest (all initial conditions are zero). We define the *transfer function* as the ratio of the Laplace transform of the output divided by the Laplace transform of the input, or

$$H(s) = \frac{\mathcal{L}\{output(t)\}}{\mathcal{L}\{input(t)\}}$$

Here we have denoted the transfer function as $H(s)$, though obviously other letters are possible. For example if we have and LTI system with input $x(t)$ and output and have the Laplace transform pairs $x(t) \leftrightarrow X(s)$ and $y(t) \leftrightarrow Y(s)$, then the transfer function is defined as

$$H(s) = \frac{Y(s)}{X(s)}$$

We can rearrange this relationship to be

$$Y(s) = H(s)X(s)$$

Example 6.4.1. In Example 6.3.1 the transfer function is

$$H(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{RCs}{RCs+1}$$

Example 6.4.2. In Example 6.3.2 the transfer function is

$$H(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{R}{R+Ls}$$

Example 6.4.3. In Example 6.3.3 the transfer function is

$$H(s) = \frac{V_{out}(s)}{I_{in}(s)} = \frac{RLs}{RLCs^2 + Ls + R}$$

Note that the transfer function comes from the Zero State Response (ZSR), since the Zero Input Response (ZIR) includes the initial conditions, and we assume all initial conditions are zero when determining the transfer function.

If we know a system is LTI, then we know that the output is the convolution of the input with the impulse response,

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\lambda)x(t-\lambda)d\lambda$$

Next, let's assume the input is causal and the system is causal, so both are zero for $t < 0$. Then we have

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\lambda)x(t-\lambda)d\lambda = \int_{0}^{\infty} h(\lambda)x(t-\lambda)d\lambda$$

Since this is an equality, we can take the Laplace transform of each side of the equation,

$$Y(s) = \mathcal{L}\{y(t)\} = Y(s) = \int_0^{\infty} y(t)e^{-st}dt = \mathcal{L}\{h(t) * \mathrm{x(t)}\} = \int_0^{\infty}\left[\int_0^{\infty} h(\lambda)x(t-\lambda)d\lambda\right]e^{-st}dt$$

Rearranging the order of integration we have

$$\int_0^{\infty}\left[\int_0^{\infty} h(\lambda)x(t-\lambda)d\lambda\right]e^{-st}dt = \int_0^{\infty} h(\lambda)\left[\int_0^{\infty} x(t-\lambda)e^{-st}dt\right]d\lambda$$

Next, let's let $\sigma = t - \lambda$ in the innermost integral. As far as this integral is concerned, λ is just a constant parameter so $d\sigma = dt$. The integral then becomes

$$\int_0^{\infty} h(\lambda)\left[\int_0^{\infty} x(t-\lambda)e^{-st}dt\right]d\lambda = \int_0^{\infty} h(\lambda)\left[\int_0^{\infty} x(\sigma)e^{-s(\sigma+\lambda)}\mathrm{d}\sigma\right]d\lambda = \int_0^{\infty} h(\lambda)e^{-s\lambda}d\lambda \int_0^{\infty} x(\sigma)e^{-s\sigma}d\sigma = H(s)X(s)$$

Hence we have derived the convolution property of Laplace transforms,

$$y(t) = h(t) * x(t)$$
$$Y(s) = H(s)X(s)$$

We have also derived an important relationship between the impulse response $h(t)$ and the transfer function $H(s)$. These are Laplace transform pairs,

$$h(t) \leftrightarrow H(s)$$

6.5 Poles, Zeros, and Pole-Zero Plots

In most instances, the transfer function is of the form

$$H(s) = \frac{N(s)}{D(s)} = \frac{A(s-z_1)(s-z_2)...(s-z_m)}{(s-p_1)(s-p_2)...(s-p_n)}$$

where $N(s)$ and $D(s)$ are polynomials in s. The *zeros* of the transfer function , $z_1, z_2, ..., z_m$, are the roots of values $N(s)$, i.e., the values of s that make $N(s)$ zero. The *poles* of the transfer function, $p_1, p_2, ..., p_n$, are the roots of $D(s)$, i.e., the values of s that make $D(s)$ zero. If the degree of the numerator is less than the degree of the denominator, or $m < n$, then the transfer function is *strictly proper*, while if the degree of the numerator is less than or equal to the degree of the denominator, or $m \le n$, the transfer function is *proper*. Clearly a transfer function that is strictly proper is also proper. Note also that if the transfer function has real valued coefficients, then *the poles or zeros of the transfer function must occur as complex conjugate pairs.*

Sometimes it is useful to plot the poles and zeros of a transfer function. This *pole-zero plot* is an alternative way of presenting the information contained in the algebraic representation of $H(s)$. As you will see later, the pole-zero plot allows us to easily visually determine the response of an LTI system to a sinusoid of different frequencies. The pole-zero plot is just a plot of poles (represented by **x**'s) and zeros (represented by **o**'s) in the complex plane, where the horizontal axis represents real values, the vertical axis represents imaginary values. Thus any complex number can be represented in this plane.

Example 6.5.1. Determine the pole-zero plot for

$$H(s) = \frac{10(s-1)(s+2+j)(s+2-j)}{s(s+1)(s+4+2j)(s+4-2j)}$$

Note that this is a strictly proper transfer function. The zeros of the transfer function are at 1, -2-j, and -2+j, while the poles are at 0, -1, -4-2j, and -4+2j. The pole-zero plot for this transfer function is shown in Figure 6.10.

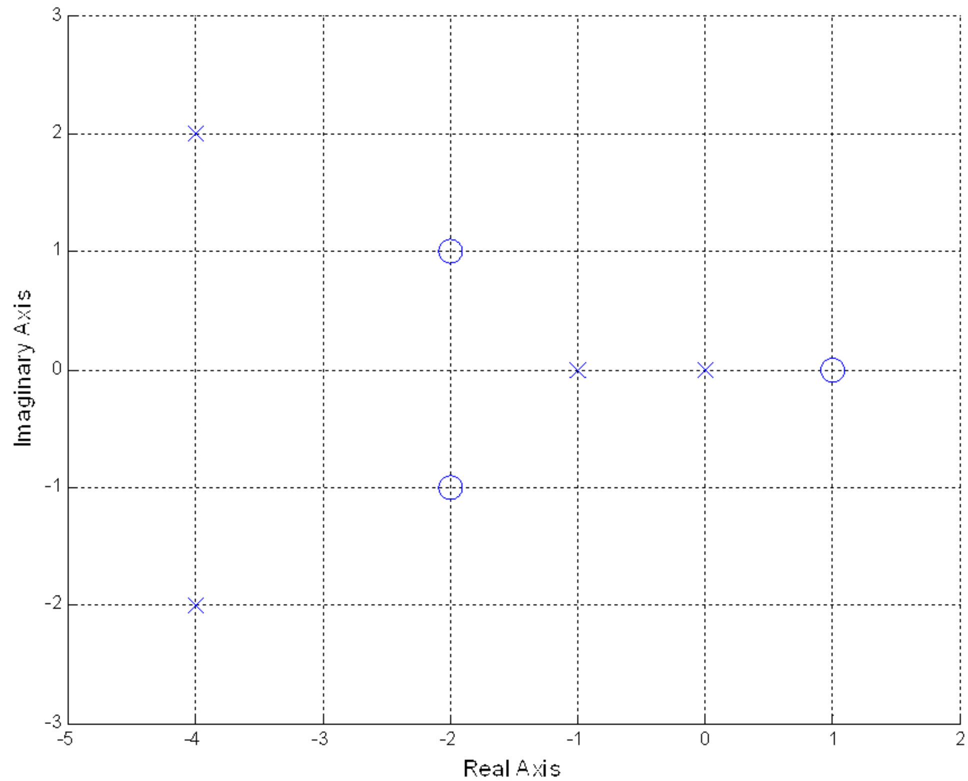

Figure 6.10. Pole-zero plot for transfer function of Example 6.5.1.

6.6 Partial Fractions for Computing Inverse Laplace Transforms

The first thing we need to do before using the partial fraction technique to find the inverse Laplace transform is to be sure the transfer function is a ratio of polynomials. If there is a time delay (e^{-st_0}) we remove this and account for it when we are done. In addition, we need a *strictly proper* ratio of polynomials. This means that the order of the numerator polynomial must be less than the order of the denominator polynomial. If this is not the case, then we use long division and use partial fractions on the remainder. Finally, we need to be sure the denominator polynomial is *monic*. This means the leading coefficient in the denominator polynomial is a one. This

Example 6.6.1. Prepare the transfer function

$$H(s) = \frac{e^{-2s}(s+1)}{(s+3)(s+4)}$$

so we can use partial fractions to determine the inverse Laplace transform. The first thing we need to do is to remove the time delay term, and write

$$H(s) = e^{-2s}G(s)$$

where

$$G(s) = \frac{s+1}{(s+3)(s+4)}$$

Since $G(s)$ is a strictly proper ratio of polynomials and the denominator polynomial is monic, we are ready for partial fractions now.

Example 6.6.2. Prepare the transfer function

$$H(s) = \frac{s^2+1}{(s+2)^2}$$

for partial fraction expansion. Here there is no time-delay, but the transfer function is not strictly proper so we need to do some long division. We have then

$$H(s) = \frac{s^2+1}{(s+2)^2} = \frac{s^2+1}{s^2+4s+4} = 1 - \frac{4s+3}{s^2+4s+4} = 1 - G(s)$$

where

$$G(s) = \frac{4s+3}{(s+2)^2}$$

Since $G(s)$ is a strictly proper ratio of polynomials and the denominator polynomial is monic, we are ready for partial fractions now.

Example 6.6.3. Prepare the transfer function

$$H(s) = \frac{e^{-s}(s^2+2s+2)}{(2s+1)(s+2)}$$

for partial fraction expansion. Here we have a time delay, the ratio of polynomials is not strictly proper, and the denominator is not monic.

First we pull out the time-delay part and do the long division, with the result

$$H(s) = \frac{e^{-s}(s^2+2s+2)}{(2s+1)(s+2)} = e^{-s}\frac{s^2+2s+2}{2s^2+5s+2} = e^{-s}\left[0.5 - \frac{0.5s-1}{2s^2+5s+2}\right]$$

Next we scale the denominator so it is monic

$$H(s) = e^{-s}\left[0.5 - \frac{0.25s-0.5}{s^2+2.5s+1}\right] = e^{-s}\left[0.5 - \frac{0.25s-0.5}{(s+0.5)(s+2)}\right]$$

Finally we have

$$H(s) = e^{-s}\left[0.5 - G(s)\right]$$

where

$$G(s) = \frac{0.25s-0.5}{(s+0.5)(s+2)}$$

<u>Partial Fractions with Distinct Poles</u>

Let's assume we have a strictly proper transfer function

$$H(s) = \frac{N(s)}{D(s)} = \frac{K(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

The poles of the system are at $p_1, p_2, ..., p_n$ and the zeros of the system are at. $z_1, z_2, ..., z_m$ Since we have distinct poles, we know that $p_i \neq p_j$ for $i \neq j$. We also assume that $N(s)$ and $D(s)$ have no common factors so there is no pole/zero cancellation. We would like to find the corresponding *impulse response* $h(t)$. To do this we assume

$$H(s) = a_1 \frac{1}{s-p_1} + a_2 \frac{1}{s-p_2} + ... + a_n \frac{1}{s-p_n}$$

If we can find the a_i then it will be easy to determine $h(t)$ since we know

$$\frac{1}{s-p_i} \leftrightarrow e^{p_i t} u(t)$$

To find a_1 we first multiply by $s - p_1$,

$$(s-p_1)H(s) = a_1 + a_2 \frac{(s-p_1)}{s-p_2} + ... + a_n \frac{(s-p_1)}{s-p_n}$$

Next take the limit as $s \to p_1$. Since all of the poles are distinct all of the terms on the right hand side of the equation are zero, except for the first term. Hence we have

$$a_1 = \lim_{s \to p_1} (s-p_1)H(s)$$

Similarly we get

$$a_2 = \lim_{s \to p_2} (s-p_2)H(s)$$

And in general

$$a_i = \lim_{s \to pi} (s-p_i)H(s)$$

Example 6.6.4. For the transfer function

$$H(s) = \frac{s+1}{(s+2)(s+3)}$$

determine the corresponding impulse response. We have

$$H(s) = a_1 \frac{1}{s+2} + a_2 \frac{1}{s+3}$$

Then

$$a_1 = \lim_{s \to -2} (s+2)H(s) = \lim_{s \to -2} \frac{(s+2)(s+1)}{(s+2)(s+3)} = \lim_{s \to -2} \frac{(s+1)}{(s+3)} = -1$$

$$a_2 = \lim_{s\to -3}(s+3)H(s) = \lim_{s\to -3}\frac{(s+3)(s+1)}{(s+2)(s+3)} = \lim_{s\to -3}\frac{(s+1)}{(s+2)} = 2$$

So

$$H(s) = \frac{-1}{s+2} + \frac{2}{s+3}$$

and

$$h(t) = e^{-2t}u(t) + 2e^{-3t}u(t)$$

It is often unnecessary to write out all of these steps, since we know in advance there will be a cancellation with one of the poles. In particular, when we want to find a_i we know we will have a cancellation between the $s - p_i$ term in the numerator and the $s - p_i$ term in the denominator. In fact, when we to find a_i, we can just ignore (or *cover up)* the $s - p_i$ term in the denominator. This is usually just called the "cover up" method. For our previous example, we have

$$a_1 = \lim_{s\to -2}\frac{(s+1)}{\cancel{(s+2)}\,(s+3)} = -1$$

$$a_2 = \lim_{s\to -3}\frac{(s+1)}{(s+2)\,\cancel{(s+3)}} = 2$$

This can often just be done in your head!

Example 6.6.5. Consider the transfer function from Example 6.6.1. We have

$$H(s) = \frac{e^{-2s}(s+1)}{(s+3)(s+4)} = e^{-2s}G(s)$$

where

$$G(s) = \frac{(s+1)}{(s+3)(s+4)}$$

We now do partial fractions for $G(s)$ and use the fact that the e^{-2s} terms corresponds to a time delay of 2 units. We have then

$$G(s) = \frac{s+1}{(s+3)(s+4)} = a_1\frac{1}{s+3} + a_2\frac{1}{s+4}$$

$$a_1 = \lim_{s\to -3}\frac{(s+1)}{\cancel{(s+3)}\,(s+4)} = -2$$

$$a_2 = \lim_{s\to -4}\frac{(s+1)}{(s+3)\,\cancel{(s+4)}} = 3$$

Then

$$g(t) = -2e^{-3t}u(t) + 3e^{-4t}u(t)$$

Finally we take the time delay into account to get the final answer

$$h(t) = g(t-2) = -2e^{-3(t-2)}u(t-2) + 3e^{-4(t-2)}u(t-2)$$

Example 6.6.6. Consider the transfer function from Example 6.6.3. We have

$$H(s) = \frac{e^{-s}(s^2+2s+2)}{(2s+1)(s+2)} = e^{-s}\left[0.5 - \frac{0.25s-0.5}{(s+0.5)(s+2)}\right] = e^{-s}\left[0.5 - G(s)\right]$$

where

$$G(s) = \frac{0.25s-0.5}{(s+0.5)(s+2)}$$

We then have

$$G(s) = \frac{0.25s-0.5}{(s+0.5)(s+2)} = a_1\frac{1}{s+0.5} + a_2\frac{1}{s+2}$$

$$a_1 = \lim_{s\to -0.5}\frac{0.25s-0.5}{\cancel{(s+0.5)}(s+2)} = -\frac{5}{12}$$

$$a_2 = \lim_{s\to -2}\frac{0.25s-0.5}{(s+0.5)\cancel{(s+2)}} = \frac{2}{3}$$

So

$$g(t) = -\frac{5}{12}e^{-0.5t}u(t) + \frac{2}{3}e^{-2t}u(t)$$

If we then define

$$F(s) = 0.5 - G(s)$$

then

$$f(t) = 0.5 - g(t) = 0.5 + \frac{5}{12}e^{-0.5t}u(t) - \frac{2}{3}e^{-2t}u(t)$$

Finally we take the time delay into account,

$$H(s) = e^{-s}F(s)$$

$$h(t) = f(t-1) = 0.5 + \frac{5}{12}e^{-0.5(t-1)}u(t-1) - \frac{2}{3}e^{-2(t-1)}u(t-1)$$

Example 6.6.7. Consider the circuit in Example 6.3.1. Here we have the relationship

$$V_{out}(s) = \frac{V_{in}(s)RCs}{RCs+1} - \frac{v(0^-)RC}{RCs+1}$$

Instead of determining the impulse response, let's determine the system output for a step input of amplitude A. Hence we have $v_{in}(t) = \mathrm{Au(t)}$, or $V_{in}(s) = \frac{A}{s}$. Then we can write the output as

$$V_{out}(s) = \frac{ARCs}{s(RCs+1)} - \frac{v(0^-)RC}{RCs+1}$$

Next we need to make the denominator polynomials monic, so we have

$$V_{out}(s) = \frac{As}{s\left(s+\frac{1}{RC}\right)} - \frac{v(0^-)}{s+\frac{1}{RC}} = \frac{A}{s+\frac{1}{RC}} - \frac{v(0^-)}{s+\frac{1}{RC}}$$

This problem is already in the correct for, and we have

$$v_{out}(t) = Ae^{-t/RC}u(t) - v(0^-)e^{-t/RC}u(t) = \left[A - v(0^-)\right]e^{-t/RC}u(t)$$

If we look at the circuit we have modeling (in Figure 6.4), this answer makes sense. The initial voltage across the resistor is the difference between the initial applied voltage and the initial voltage across the capacitor. In addition, the voltage across the resistor should approach zero as time increases, since eventually the voltage on the capacitor will reach the applied voltage. Finally, it is clear that the time constant for this simple circuit is $\tau = RC$ and this is the time constant of our results.

Example 6.6.8. Consider the circuit in Example 6.3.2. Here we have the relationship

$$V_{out}(s) = \frac{R}{R+Ls}V_{in}(s) + \frac{RL}{R+Ls}i(0^-)$$

Again let's determine the output when the input is a step of amplitude A. Then we have

$$V_{out}(s) = \frac{R}{R+Ls}\frac{A}{s} + \frac{RL}{R+Ls}i(0^-)$$

We need to make the denominators monic, so we rewrite this as

$$V_{out}(s) = \frac{\frac{AR}{L}}{s\left(s+\frac{R}{L}\right)} + \frac{R}{s+\frac{R}{L}}i(0^-)$$

Applying partial fraction to the first term we have

$$V_{out}(s) = \frac{A}{s} - \frac{A}{s+\frac{R}{L}} + \frac{R}{s+\frac{R}{L}}i(0^-)$$

In the time-domain we then have

$$v_{out}(t) = A(1-e^{-\frac{R}{L}t})u(t) + Ri(0^-)e^{-\frac{R}{L}t}u(t)$$

Partial Fractions with Repeated Poles

If there are repeated poles with no other poles, then the inverse Laplace transform is very straightforward, using the formula

$$\frac{t^{m-1}}{(m-1)!}e^{-pt}u(t) \leftrightarrow \frac{1}{(s+p)^m}$$

If instead of isolated repeated poles and nonrepeated poles, then we need to use a different form for the partial fractions for the repeated poles. If we assume we have a pole at $-p$ that is of order m, then for this pole we will use an expansion of the form

$$a_1\frac{1}{(s+p)} + a_2\frac{1}{(s+p)^2} + \cdots + a_m\frac{1}{(s+p)^m}$$

We will also need a new approach for determining some of the expansion coefficients for the repeated poles. There are two common approaches

- Multiply both sides of the equation by s and taking the limit as $s \to \infty$
- Select convenient values for s and evaluate both sides of the equation for these values of s

These techniques are probably most easily explained by the use of examples. In each of these examples you should note that *there are always as many unknowns as there are poles!*

Example 6.6.9. Find the impulse response that corresponds to the transfer function

$$H(s) = \frac{1}{(s+1)(s+2)^2}$$

To do this we look for a partial fraction expansion of the form

$$H(s) = a_1\frac{1}{s+1} + a_2\frac{1}{s+2} + a_3\frac{1}{(s+2)^2}$$

For the distinct pole we have

$$a_1 = \lim_{s\to -1}\frac{1}{\cancel{(s+1)}(s+2)^2} = 1$$

Next we determine the coefficient for the highest power of the repeated root. In this case we multiply both sides of our partial fraction expansion by $(s+2)^2$ and then take the limit as $s \to -2$

$$a_3 = \lim_{s\to -2}\frac{1}{(s+1)\cancel{(s+2)^2}} = -1$$

Finally, to get the value of a_2 we must resort to a different method. If we multiply both sides by s and take the limit as $s \to \infty$ we have

$$\lim_{s\to\infty} sH(s) = \lim_{s\to\infty}\frac{s}{(s+1)(s+2)^2} = \lim_{s\to\infty}\left[a_1\frac{s}{s+1} + a_2\frac{s}{s+2} + a_3\frac{s}{(s+2)^2}\right]$$

Simplifying this we have

$$0 = a_1 + a_2$$

or

$$a_2 = -a_1 = -1$$

Hence we have

$$H(s) = \frac{1}{s+1} - \frac{1}{s+2} - \frac{1}{(s+2)^2}$$

with corresponding impulse response

$$h(t) = e^{-t}u(t) - e^{-2t}u(t) - te^{-2t}u(t)$$

As an alternative method for determining a_2, we start with

$$H(s) = \frac{1}{(s+1)(s+2)^2} = \frac{1}{s+1} + a_2\frac{1}{s+2} - \frac{1}{(s+2)^2}$$

This expression must be true for all values of s ***as long as both sides remain finite.*** Let's choose a convenient value of s, like $s = 0$. Then we have

$$H(0) = \frac{1}{4} = 1 + a_2\frac{1}{2} - \frac{1}{4}$$

or

$$-\frac{1}{2} = a_2\frac{1}{2}$$

which again give us $a_2 = -1$.

Example 6.6.10. Determine the impulse response that corresponds to the transfer function

$$H(s) = \frac{s+1}{s^2(s+2)(s+3)}$$

The partial fraction expansion we need is of the form

$$H(s) = a_1\frac{1}{s} + a_2\frac{1}{s^2} + a_3\frac{1}{s+2} + a_4\frac{1}{s+3}$$

First we find the coefficients that correspond to the distinct poles and the coefficient that goes with the highest power of the repeated pole.

$$a_2 = \lim_{s\to 0}\frac{s+1}{\cancel{s^2}(s+2)(s+3)} = \frac{1}{6}$$

$$a_3 = \lim_{s\to -2}\frac{s+1}{s^2\cancel{(s+2)}(s+3)} = -\frac{1}{4}$$

$$a_4 = \lim_{s\to -3} \frac{s+1}{s^2(s+2)\cancel{(s+3)}} = \frac{2}{9}$$

At this point we have

$$H(s) = \frac{s+1}{s^2(s+2)(s+3)} = a_1\frac{1}{s} + \frac{1}{6}\frac{1}{s^2} - \frac{1}{4}\frac{1}{s+2} + \frac{2}{9}\frac{1}{s+3}$$

To find the unknown coefficient, we will multiply by s and let $s \to \infty$,

$$\lim_{s\to\infty} sH(s) = \lim_{s\to\infty} \frac{s(s+1)}{s^2(s+2)(s+3)} = \lim_{s\to\infty}\left[a_1\frac{s}{s} + \frac{1}{6}\frac{s}{s^2} - \frac{1}{4}\frac{s}{s+2} + \frac{2}{9}\frac{s}{s+3}\right]$$

or

$$0 = a_1 + 0 - \frac{1}{4} + \frac{2}{9}$$

This simplifies to

$$a_1 = \frac{1}{36}$$

Finally we have

$$H(s) = \frac{s+1}{s^2(s+2)(s+3)} = \frac{1}{36}\frac{1}{s} + \frac{1}{6}\frac{1}{s^2} - \frac{1}{4}\frac{1}{s+2} + \frac{2}{9}\frac{1}{s+3}$$

which corresponds to the impulse response

$$h(t) = \frac{1}{36}u(t) + \frac{1}{6}tu(t) - \frac{1}{4}e^{-2t}u(t) + \frac{2}{9}e^{-3t}u(t)$$

As an alternative to taking limits in the expression

$$H(s) = \frac{s+1}{s^2(s+2)(s+3)} = a_1\frac{1}{s} + \frac{1}{6}\frac{1}{s^2} - \frac{1}{4}\frac{1}{s+2} + \frac{2}{9}\frac{1}{s+3}$$

we can make the substitution $s = -1$ in both sides of the expression (note that we cannot use $s = 0, s = -2$, or $s = -3$, since these substitutions make the function infinite). The we have

$$0 = -a_1 + \frac{1}{6} - \frac{1}{4} + \frac{1}{9} = -a_1 + \frac{6-9+4}{36} = -a_1 + \frac{1}{36}$$

which again yields

$$a_1 = \frac{1}{36}$$

Example 6.6.11. Determine the impulse response for the transfer function

$$H(s) = \frac{s+3}{s(s+1)^2(s+2)^2}$$

The appropriate partial fraction expansion is of the form

$$H(s) = \frac{s+3}{s(s+1)^2(s+2)^2} = a_1\frac{1}{s} + a_2\frac{1}{s+1} + a_3\frac{1}{(s+1)^2} + a_4\frac{1}{s+2} + a_5\frac{1}{(s+2)^2}$$

Again we first find the easy coefficients, a_1, a_3 and a_5.

$$a_1 = \lim_{s\to 0}\frac{s+3}{\cancel{s}(s+1)^2(s+2)^2} = \frac{3}{4}$$

$$a_3 = \lim_{s\to -1}\frac{s+3}{s\,\cancel{(s+1)^2}\,(s+2)^2} = -2$$

$$a_5 = \lim_{s\to -2}\frac{s+3}{s(s+1)^2\,\cancel{(s+2)^2}} = -\frac{1}{2}$$

Next we use limits

$$\lim_{s\to\infty} sH(s) = \lim_{s\to\infty}\frac{s(s+3)}{s(s+1)^2(s+2)^2} = \lim_{s\to\infty}\left[\frac{3}{4}\frac{s}{s} + a_2\frac{s}{s+1} - 2\frac{s}{(s+1)^2} + a_4\frac{s}{s+2} - \frac{1}{2}\frac{1}{(s+2)^2}\right]$$

or

$$0 = \frac{3}{4} + a_2 + a_4$$

We need another equation, so let's let $s = -3$, in the expression

$$H(s) = \frac{s+3}{s(s+1)^2(s+2)^2} = \frac{3}{4}\frac{1}{s} + a_2\frac{1}{s+1} - 2\frac{1}{(s+1)^2} + a_4\frac{1}{s+2} - \frac{1}{2}\frac{1}{(s+2)^2}$$

which yields

$$0 = -\frac{1}{4} - a_2\frac{1}{2} - \frac{1}{2} - a_4 - \frac{1}{2}$$

or

$$a_2\frac{1}{2} - a_4 = \frac{5}{4}$$

Solving these two equations yields $a_2 = 1$ and $a_4 = -\frac{7}{4}$. Finally, we have

$$H(s) = \frac{s+3}{s(s+1)^2(s+2)^2} = \frac{3}{4}\frac{1}{s} + \frac{1}{s+1} - 2\frac{1}{(s+1)^2} - \frac{7}{4}\frac{1}{s+2} - \frac{1}{2}\frac{1}{(s+2)^2}$$

which corresponds to the impulse response

$$h(t) = \frac{3}{4}u(t) + e^{-t}u(t) - 2te^{-t}u(t) - \frac{7}{4}e^{-2t}u(t) - \frac{1}{2}te^{-2t}u(t)$$

Partial Fractions with Complex Conjugate Poles

The transform pairs we are primarily going to use with complex conjugate poles are

$$e^{-at}\cos(bt)u(t) \leftrightarrow \frac{s+a}{(s+a)^2 + b^2}$$

$$e^{-at}\sin(bt)u(t) \leftrightarrow \frac{b}{(s+a)^2 + b^2}$$

Note that complex conjugate poles always result in sines and cosines (or a single sine/cosine with a phase angle). We will try and make terms with complex conjugate poles look like these terms by completing the square in the denominator. That is, we need to be able to write the denominator as

$$D(s) = (s+a)^2 + b^2$$

Note that if we cannot write the denominator in this form, the poles are not complex conjugates! To determine the correct values of *a* and *b*, use the fact that the coefficient of *s* should be *2a*. Once we find *a*, the value for *b* is easy to find. A few examples will make this clear.

Example 6.6.12. Assume $D(s) = s^2 + s + 2$ and we want to write this in the correct form. First we recognize that the coefficient of *s* is *1*, so that *2a = 1*, or $a = \frac{1}{2}$. We then have

$$D(s) = s^2 + s + 2 = (s + \frac{1}{2})^2 + b^2 = s^2 + s + \frac{1}{4} + b^2$$

So then $2 = \frac{1}{4} + b^2$ and we can determine $b = \frac{\sqrt{7}}{2}$. Thus we have

$$D(s) = s^2 + s + 1 = (s + \frac{1}{2})^2 + \left(\frac{\sqrt{7}}{2}\right)$$

Example 6.6.13. Assume $D(s) = s^2 + 3s + 5$ and we want to write this in the correct form. The coefficient of *s* is *3*, so we have $2a = 3$ and $a = \frac{3}{2}$. We then have

$$D(s) = s^2 + 3s + 5 = (s + \frac{3}{2})^2 + b^2 = s^2 + 3s + \frac{9}{4} + b^2$$

Then $5 = \frac{9}{4} + b^2$ and we can determine $b = \frac{\sqrt{11}}{4}$.

Now that we know how to complete the square we will look at two simple examples of complex conjugate poles, then at more complicated examples.

Example 6.6.14. Assuming $H(s) = \frac{1}{s^2 + s + 2}$, determine the corresponding impulse response $h(t)$. From our previous example we know

$$H(s) = \frac{1}{(s + \frac{1}{2})^2 + \left(\frac{\sqrt{7}}{2}\right)^2}$$

This almost has the form we want, which is

$$\frac{b}{(s+a)^2 + b^2} \leftrightarrow e^{-at} sin(bt)u(t)$$

However, to use this form we need a *b* in the numerator. To achieve this we will multiply and divide by $b = \frac{\sqrt{7}}{2}$,

$$H(s) = \frac{1}{\frac{\sqrt{7}}{2}} \frac{\frac{\sqrt{7}}{2}}{(s + \frac{1}{2})^2 + \left(\frac{\sqrt{7}}{2}\right)^2} = \frac{2}{\sqrt{7}} \frac{\frac{\sqrt{7}}{2}}{(s + \frac{1}{2})^2 + \left(\frac{\sqrt{7}}{2}\right)^2}$$

and we can determine

$$h(t) = \frac{2}{\sqrt{7}} e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{7}}{2} t\right) u(t)$$

Example 6.6.15. Assuming $H(s) = \frac{s}{s^2 + 3s + 5}$, determine the corresponding impulse response $h(t)$. From our previous example we know

$$H(s) = \frac{s}{\left(s + \frac{3}{2}\right)^2 + \left(\frac{\sqrt{11}}{2}\right)^2}$$

This is almost the form we want, which is

$$H(s) = \frac{s+a}{(s+a)^2 + b^2}$$

However, to use this form, we will add and subtract 3/2 in our transfer function,

$$H(s)=\frac{s+\frac{3}{2}-\frac{3}{2}}{\left(s+\frac{3}{2}\right)^2+\left(\frac{\sqrt{11}}{2}\right)^2}=\frac{s+\frac{3}{2}}{\left(s+\frac{3}{2}\right)^2+\left(\frac{\sqrt{11}}{2}\right)^2}-\frac{\frac{3}{2}}{\left(s+\frac{3}{2}\right)^2+\left(\frac{\sqrt{11}}{2}\right)^2}$$

The first term is now what we want, but we need to scale the second term,

$$H(s)=\frac{s+\frac{3}{2}}{\left(s+\frac{3}{2}\right)^2+\left(\frac{\sqrt{11}}{2}\right)^2}-\frac{3}{2}\frac{2}{\sqrt{11}}\frac{\frac{\sqrt{11}}{2}}{\left(s+\frac{3}{2}\right)^2+\left(\frac{\sqrt{11}}{2}\right)^2}=\frac{s+\frac{3}{2}}{\left(s+\frac{3}{2}\right)^2+\left(\frac{\sqrt{11}}{2}\right)^2}-\frac{3}{\sqrt{11}}\frac{\frac{\sqrt{11}}{2}}{\left(s+\frac{3}{2}\right)^2+\left(\frac{\sqrt{11}}{2}\right)^2}$$

So we finally have the impulse response

$$h(t)=e^{-\frac{3}{2}t}\cos\left(\frac{\sqrt{11}}{2}t\right)u(t)-\frac{3}{\sqrt{11}}e^{-\frac{3}{2}t}\sin\left(\frac{\sqrt{11}}{2}t\right)u(t)$$

The next two examples are much more involved, but they have the same general approach. In general, if there are complex conjugate poles we will look for partial fractions of the form

$$\frac{c(s+a)}{(s+a)^2+b^2}+\frac{d(b)}{(s+a)^2+b^2}$$

where c and *d* are the parameters to be determined.

Example 6.6.16. Assume

$$Y(s)=\frac{1}{s+2}\frac{1}{s^2+s+1}$$

Use partial fractions to determine the corresponding time domain function. We will have

$$Y(s)=\frac{1}{s+2}\frac{1}{s^2+s+1}=\frac{1}{(s+2)\left[\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2\right]}=\frac{A}{s+2}+\frac{c(s+\frac{1}{2})}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}+\frac{d\,\frac{\sqrt{3}}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

We need to determine the three coefficients *A*, *c*, and *d*. To determine *A* we use the cover-up method as before

$$A=\lim_{s\to-2}\frac{1}{\cancel{s+2}}\frac{1}{s^2+s+1}=\frac{1}{4-2+1}=\frac{1}{3}$$

To determine *c*, multiply both sides by *s* and let $s\to\infty$,

$$\lim_{s\to\infty}\frac{s}{(s+2)(s^2+s+1)}=\lim_{s\to\infty}\left[\frac{sA}{s+2}+\frac{sc(s+\frac{1}{2})}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}+\frac{sd\,\frac{\sqrt{3}}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right]$$

or

$$0=A+c=\frac{1}{3}+c$$

So we have $c=-\frac{1}{3}$. Finally, we choose a convenient value for *s*, and evaluate both sides. Let's choose $s=0$, so we have

$$\lim_{s\to 0}Y(s)=\lim_{s\to 0}\frac{1}{(s+2)(s^2+s+1)}=\lim_{s\to 0}\left[\frac{A}{s+2}+\frac{c(s+\frac{1}{2})}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}+\frac{d\,\frac{\sqrt{3}}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right]$$

or

We can simplify this to

$$\frac{1}{2}=\frac{A}{2}+c(\frac{1}{2})+d(\frac{\sqrt{3}}{2})=\frac{1}{6}-\frac{1}{6}+d(\frac{\sqrt{3}}{2})$$

which yields $d=\frac{1}{\sqrt{3}}$. Finally the time-domain result is

$$y(t)=\frac{1}{3}e^{-2t}u(t)-\frac{1}{3}e^{-\frac{1}{2}t}\cos(\frac{\sqrt{3}}{2}t)u(t)+\frac{1}{\sqrt{3}}e^{-\frac{1}{2}t}\sin(\frac{\sqrt{3}}{2}t)u(t)$$

Example 6.6.17. Find the step response of the system with transfer function

$$H(s)=\frac{1}{s^2+2s+2}$$

The step response for this system will be given by

$$Y(s)=H(s)\frac{1}{s}=\frac{1}{s(s^2+2s+2)}=\frac{A}{s}+\frac{c(s+1)}{(s+1)^2+1}+\frac{d}{(s+1)^2+1}$$

Using the cover-up method we get $A=\frac{1}{2}$. To get *c*, multiply both sides by *s* and let $s\to\infty$,

$$\lim_{s\to\infty}sY(s)=\frac{s}{s(s^2+2s+2)}=\lim_{s\to\infty}\left[\frac{sA}{s}+\frac{sc(s+1)}{(s+1)^2+1}+\frac{sd}{(s+1)^2+1}\right]$$

or

$$0 = A + c = \frac{1}{2} + c$$

which gives us $c = -\frac{1}{2}$. Finally, we set s to a convenient value and equate both sides. In this case $s = -1$ is a good choice.

$$\lim_{s \to -1} Y(s) = \lim_{s \to -1} \frac{1}{s(s^2 + 2s + 2)} = \lim_{s \to -1} \left[\frac{A}{s} + \frac{c(s+1)}{(s+1)^2 + 1} + \frac{d}{(s+1)^2 + 1} \right]$$

$$-1 = -A + d = -\frac{1}{2} + d$$

So we can conclude that $d = -\frac{1}{2}$ and the complete answer is

$$y(t) = \frac{1}{2}u(t) - \frac{1}{2}e^{-t}cos(t)u(t) - \frac{1}{2}e^{-t}\sin(t)u(t)$$