5.0 Impulse Response, Step Response, and Convolution

In this chapter we confine ourselves to systems that can be modeled as linear and time-invariant, or LTI systems. For these types of systems, we can determine the output of the system to any input in a very systematic way. We can also determine a great deal about the system just by looking at how it responds to various types of inputs. The most fundamental of these inputs is the impulse response, or the response of a system at rest to an impulse. However, the response of a system to a step is much easier to determine and can be used to determine the impulse response of any LTI system.

5.1 Impulse or Delta Functions

An impulse, or delta function, $\delta(t)$, is defined as a function that is zero everywhere except at one point, and has an area of one. Mathematically, we can write this as

$$\delta(\lambda) = 0, \lambda \neq 0$$

$$\int_{-\mu}^{\mu} \delta(\lambda) d\lambda = 1, \quad \mu > 0$$

Note that we do not know the value of $\delta(0)$, it is undefined! We can think of, or model, delta functions as functions that exist in some type of limit. For example, the functions displayed in Figures 5.1, 5.2, and 5.3 can be thought of as different models for delta functions, since the meet our two (simplistic) requirements above.

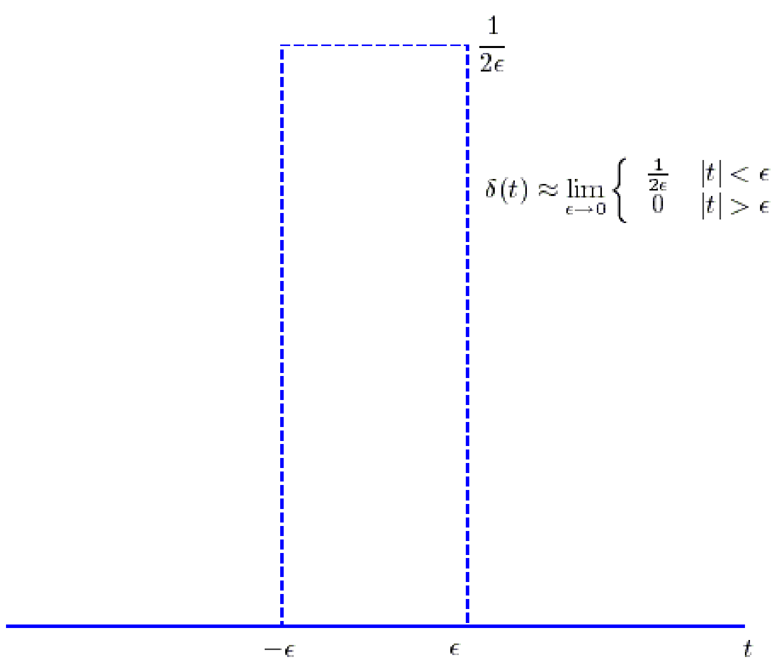

Figure 5.1. Rectangular model of an impulse (delta) function.

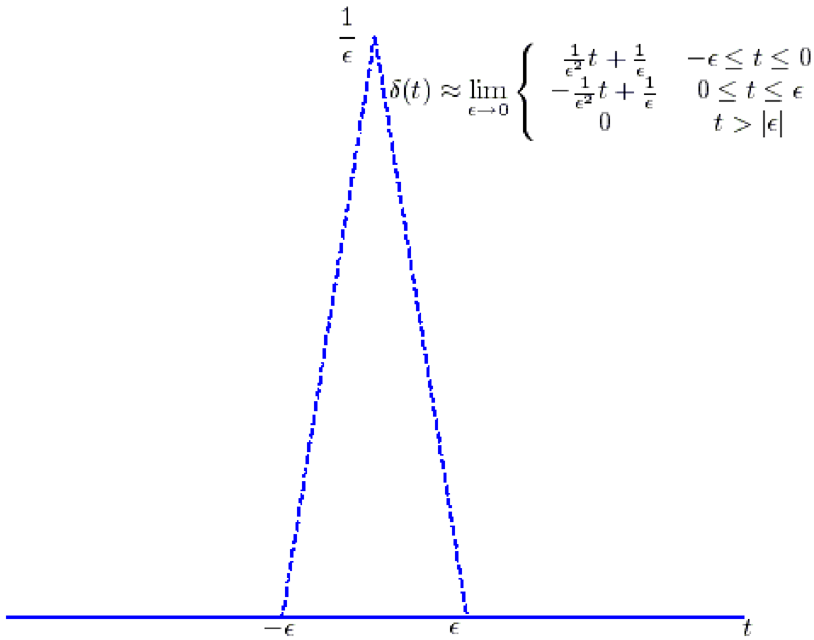

Figure 5.2. Triangular model of an impulse (delta) function.

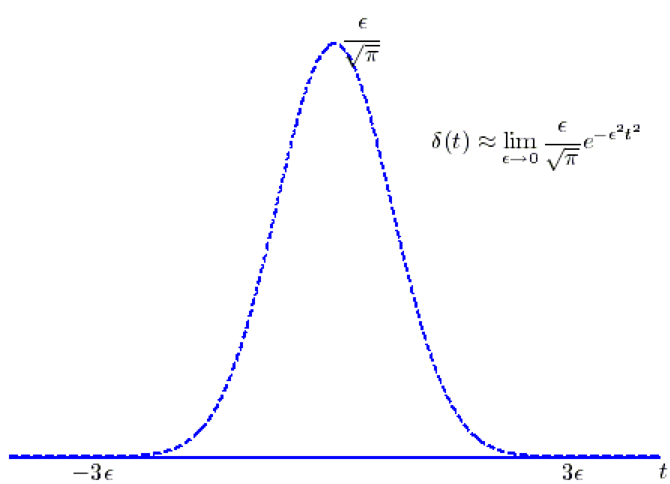

Figure 5.3. Gaussian model of an impulse (delta) function.

Although delta functions are really idealized functions, they form the basis for much or the study of systems. Knowing how a system will respond to an impulse (an impulse response) tells us a great deal about a system, and lets us determine how the system will response to any arbitrary input.

There following two very important properties of delta functions will be used extensively:

Property 1: $\phi(t)\delta(t-t_0)=\phi(t_0)\delta(t-t_0)$

Property 2 (Sifting Property): $\int_a^b \phi(t)\delta(t-t_0)dt=\phi(t_0) \quad a<t_0<b$

The first property is pretty easy to understand if we think about the definition of a delta function. A delta function is zero everywhere except when its argument is zero, so both sides of the equation are zero everywhere except at t_0, and then at t_0 both sides have the same value.

The second property follows directly from the first property as follows:

$$\int_a^b \phi(t)\delta(t-t_0)dt=\int_a^b \phi(t_0)\delta(t-t_0)dt=\phi(t_0)\int_a^b \delta(t-t_0)dt=\phi(t_0) \quad a<t_0<b$$

It is very important that the limits of the integral are such that the delta function is within the limits of the integral, or else the integral is zero.

Example 5.1.1. You should understand each of the following identities, and how to use the two properties to arrive at the correct solution.

$$e^t\delta(t-1)=e^1\delta(t-1)$$

$$t^2\delta(t-2)=4\delta(t-2)$$

$$\int_0^\infty t^2\delta(t-2)\mathrm{dt}=4$$

$$\int_0^{10} e^t\delta(t-1)dt=e^1$$

$$\int_{-10}^{10} e^t\delta(t-20)dt=0$$

$$\int_{-\infty}^{\infty} \delta(t-1)\delta(t-2)dt=0$$

5.2 Unit Step (Heaviside) Functions

We will define the unit step function as

$$u(\tau) = \begin{cases} 1 & \tau > 0 \\ 0 & \tau < 0 \end{cases}$$

We will not define $u(0)$, though some textbooks define $u(0) = \frac{1}{2}$. The argument of the unit step was deliberately not written as t, since this sometimes leads to some confusion when solving problems. It is generally better to remember that the unit step is one whenever the argument (τ) is positive, and then try and figure out what this might mean in terms of t.

Example. 5.2.1. The following are some simple examples with unit step functions:

a) $u(t-1)=1$ for $t-1>0$ or $t>1$
b) $u(2-t)=1$ for $2-t>0$ or $2>t$
c) $u\left(4-\frac{t}{3}\right)=1$ for $4-\frac{t}{3}>0$ or $12>t$

Unit step functions also show up in integrals, and it is useful to be able to deal with them in that context. The usual procedure is to determine when the unit step function (or functions) are one, and then do the integrals. If the unit step functions are not one, then the integral will be zero. When you are done with the integral, you may need to preserve the information indicating that the integral is zero unless the unit step functions are "on", and this is usually done by including unit step functions. A few examples will hopefully clear this up.

Example 5.2.1. Simplify $\int_{-\infty}^{\infty} u(t-\lambda)u(\lambda-1)d\lambda$ as much as possible. We need both unit step functions to be one, or the integral is zero. We need then

$$u(t-\lambda)=1 \text{ for } t-\lambda>0 \text{ or } t>\lambda$$
$$u(\lambda-1)=1 \text{ for } \lambda-1>0 \text{ or } \lambda>1$$

The integral then becomes $\int_1^t (1)(1)d\lambda = t-1$. However, we are not done yet. We need to be sure both of the unit step functions are 1, which means we need $t>\lambda>1$, or $t>1$. So the answer is zero for $t<1$ and $t-1$ for $t>1$. The way we can write this compactly is $(t-1)u(t-1)$, which is the final answer.

Example 5.2.2. Simplify $\int_{-\infty}^{t+2} e^{-t}u(\lambda-2)d\lambda$ as much as possible. We need the step function to be one, or the integral is zero. We need then

$$u(\lambda-2)=1 \text{ for } \lambda-2>0 \text{ or } \lambda>2$$

The integral becomes

$$\int_{2}^{t+2} e^{-\lambda}(1)d\lambda = e^{-2}-e^{-(t+2)} = e^{-2}(1-e^{-t})$$

However, the integral will be zero unless $t+2>\lambda>2$ or $t>0$. The final answer is then $e^{-2}(1-e^{-t})u(t)$.

Example 5.2.3. Simplify $\int_{-\infty}^{\infty} u(t-\lambda)\delta(\lambda+2)d\lambda$ as much as possible. This integral has both an impulse and a unit step function. While we might be tempted to use the unit step function to set the limits of the integral, the best (and easiest) thing to do is to just use the sifting property of impulse functions. This gives the result

$$\int_{-\infty}^{\infty} u(t-\lambda)\delta(\lambda+2)d\lambda = u(t+2)$$

Example 5.2.4. Simplify $\int_{-\infty}^{t} e^{-(t-\lambda)}\delta(\lambda-2)d\lambda$ as much as possible. We can again use the sifting property with this integral, but we must be careful. If we do not integrate past the impulse function, the integral will be zero. Hence we have

$$\int_{-\infty}^{t} e^{-(t-\lambda)}\delta(\lambda-2)d\lambda = \begin{cases} e^{-(t-2)} & t>2 \\ 0 & t<2 \end{cases}$$

which we can write in a more compressed form as $e^{-(t-2)}u(t-2)$.

Example 5.2.5. Simplify $\int_{t-1}^{\infty} e^{t+\lambda}\delta(\lambda+2)d\lambda$ as much as possible. Using the sifting property we have

$$\int_{t-1}^{\infty} e^{t+\lambda}\delta(\lambda+2)d\lambda = \begin{cases} e^{t-2} & t-1<-2 \\ 0 & t-1>-2 \end{cases}$$

which we can write more compactly as $e^{t-2}u(1-t)$.

Finally, if we consider integrating an impulse, $\int_{-\infty}^{t} \delta(\lambda)d\lambda$, we will either get a one (if we integrate past the impulse) or a zero (if we do not). Thus we have

$$\int_{-\infty}^{t} \delta(\lambda)d\lambda = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

or

$$\int_{-\infty}^{t} \delta(\lambda)d\lambda = u(t)$$

If we differentiate both sides of this we get

$$\frac{du(t)}{dt} = \delta(t)$$

This relationship is important to remember, but when doing integrals it is generally a better idea to remember what conditions you may need to impose in order to determine if and what unit step functions will be required.

5.3 Impulse Response

The *impulse response* of an LTI system is the response of the system initially at rest (no initial energy, all initial conditions are zero) to an impulse at time $t = 0$. The most common way to denote the impulse response is by lower case letters *h* and *g*, though others are used.

Example 5.3.1. Consider the circuit shown in Figure 5.4. Determine the impulse response of the system. The circuit is a simple voltage divider, so we have

$$y(t) = \left(\frac{R_b}{R_a + R_b} \right) x(t)$$

and the impulse response is

$$h(t) = \left(\frac{R_b}{R_a + R_b} \right) \delta(t)$$

Example 5.3.2. Consider the circuit shown in Figure 5.5. Determine the impulse response of the system. We have

$$\frac{x(t) - y(t)}{R} = C\frac{dy(t)}{dt} + \frac{y(t)}{R}$$

or

$$\frac{dy(t)}{dt} + \frac{2}{RC} y(t) = \frac{1}{RC} x(t)$$

Then

$$\frac{d}{dt}\left[y(t) e^{2t/RC} \right] = e^{2t/RC} \frac{1}{RC} x(t)$$

Integrating from $-\infty$ up to t, and assuming the system is initially at rest, we have

$$y(t) = \frac{1}{RC} \int_{-\infty}^{t} e^{-2(t-\lambda)/RC} x(\lambda) d\lambda$$

The impulse response is then given by

$$h(t) = \frac{1}{RC} \int_{-\infty}^{t} e^{-2(t-\lambda)/RC} \delta(\lambda) d\lambda = \frac{1}{RC} e^{-2t/RC} u(t)$$

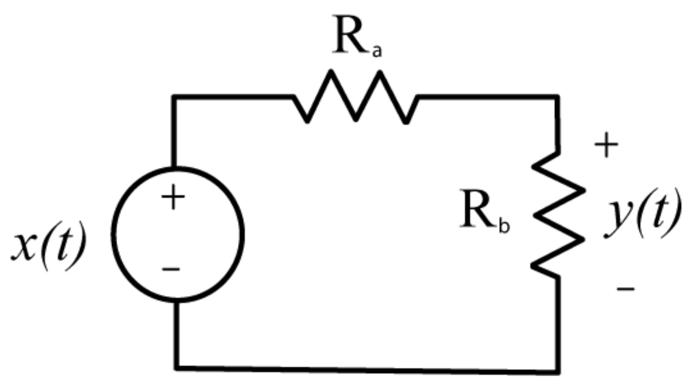

Figure 5.4. Circuit used for Example 5.3.1.

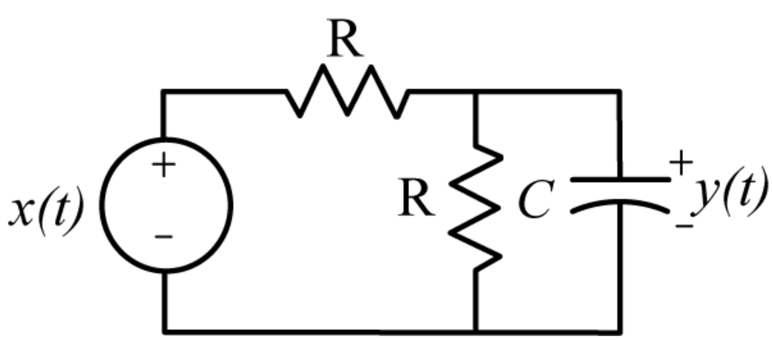

Figure 5.5. Circuit used for Example 5.3.2.

Example 5.3.3. Consider the system described by the mathematical model

$$y(t) = x(t-1) + \int_{-\infty}^{t} e^{-(t-\lambda)} x(\lambda + 2) d\lambda$$

The impulse response will be given by

$$h(t) = \delta(t-1) + \int_{-\infty}^{t} e^{-(t-\lambda)}\delta(\lambda+2)d\lambda = \delta(t-1) + e^{-(t+2)}u(t+2)$$

5.4 Step Response

The *step response* of an LTI system is the response of the system initially at rest (no initial energy, all initial conditions are zero) to a step at time $t=0$. There is no common method for denoting the step response, but we will sometimes denote the step response as $s(t)$. If we know the step response of an LTI system, we can determine the impulse response of the system using the relationship

$$h(t) = \frac{d}{dt}\left[s(t)\right]$$

Example 5.4.1. Determine the step response and then use it to determine the impulse response for the system in Example 5.3.2. From the example we have

$$y(t) = \frac{1}{RC}\int_{-\infty}^{t} e^{-2(t-\lambda)/RC}x(\lambda)d\lambda$$

The step response is then given by

$$s(t) = \frac{1}{RC}\int_{-\infty}^{t} e^{-2(t-\lambda)/RC}u(\lambda)d\lambda$$

or

$$s(t) = \frac{1}{RC}e^{-2t/RC}\int_{0}^{t} e^{2\lambda/RC}d\lambda = e^{-2t/RC}\,\frac{1}{2}[e^{2t/RC}-1]u(t) = \frac{1}{2}[1-e^{-2t/RC}]u(t)$$

The impulse response is then given by

$$h(t) = \frac{d}{dt}s(t) = \frac{d}{dt}\left\{[1-e^{-2t/RC}]\frac{1}{2}u(t)\right\} = \frac{1}{RC}e^{-2t/RC}u(t) + [1-e^{-2t/RC}]\frac{1}{2}\delta(t) = \frac{1}{RC}e^{-2t/RC}u(t)$$

which is the same answer we obtained before.

5.5. Convolution

Derivation of the Convolution Integral

We will derive the convolution integral using two different, though equivalent methods. Consider an LTI system with input $x(t)$. We can approximate $x(t)$ as a piecewise constant function over intervals of length ΔT , as shown in Figure 5.6. Thus we have the approximation

$$x(t) \approx \sum_{k=-\infty}^{k=\infty} x(k\Delta T)\left\{u\left[t-\left(k-\frac{1}{2}\right)\Delta T\right] - u\left[t-\left(k+\frac{1}{2}\right)\Delta T\right]\right\}$$

Next, we can write the step response of the system as $s(t)$. Because the system is time-invariant, the response of the system to the input $u(t-(k-\frac{1}{2})\Delta T)$ is $s(t-(k-\frac{1}{2})\Delta T)$, and the response of the system to $u(t-(k+\frac{1}{2})\Delta T)$ is $s(t-(k+\frac{1}{2})\Delta T)$. Because the system is both linear and time-invariant, the response of the system to input $x(t)$ can then be approximated as

$$y(t) \approx \sum_{k=-\infty}^{k=\infty} x(k\Delta T)\left\{ s\left[t-\left(k-\frac{1}{2}\right)\Delta T \right] - s\left[t-\left(k+\frac{1}{2}\right)\Delta T \right]\right\}$$

Now we can approximate the derivative of the step response as

$$h(t-k\Delta t) \approx \frac{s\left[t-\left(k-\frac{1}{2}\right)\Delta T \right] - s\left[t-\left(k+\frac{1}{2}\right)\Delta T \right]}{\Delta T}$$

or

$$h(t-k\Delta t)\Delta T \approx s\left[t-\left(k-\frac{1}{2}\right)\Delta T \right] - s\left[t-\left(k+\frac{1}{2}\right)\Delta T \right]$$

Thus the output can be approximated as

$$y(t) \approx \sum_{k=-\infty}^{k=\infty} x(k\Delta T) h(t-k\Delta T)\Delta T$$

If we define $\lambda = k\Delta T$, then as $\Delta T \to 0$ the sum becomes an integral and we have

$$y(t) = \int_{-\infty}^{\infty} x(\lambda) h(t-\lambda) d\lambda$$

This is one form of the *convolution integral*, which tells us how to determine the output of an LTI system if we know the impulse response of the system and the system input. We would write this as $y(t) = x(t) \star h(t)$, where $\star$ represents the *convolution operator*.

An alternative derivation for convolution would be to start with the same approximation

$$x(t) \approx \sum_{k=-\infty}^{k=\infty} x(k\Delta T)\left\{ u\left[t-\left(k-\frac{1}{2}\right)\Delta T \right] - u\left[t-\left(k+\frac{1}{2}\right)\Delta T \right]\right\}$$

We can then approximate the impulse response as

$$\delta(t-k\Delta T) \approx \frac{u\left[t-\left(k-\frac{1}{2}\right)\Delta T \right] - u\left[t-\left(k+\frac{1}{2}\right)\Delta T \right]}{\Delta T}$$

or

$$\delta(t-k\Delta T)\Delta T \approx u\left[t-\left(k-\frac{1}{2}\right)\Delta T\right]-u\left[t-\left(k+\frac{1}{2}\right)\Delta T\right]$$

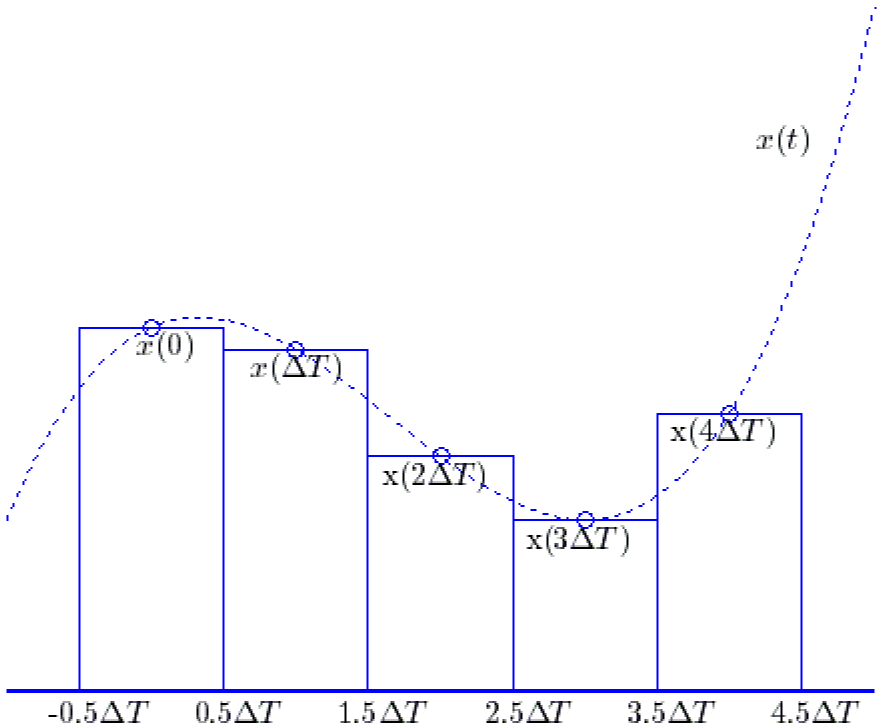

Figure 5.6. Approximating the continuous function $x(t)$ as a piecewise constant function for the derivation of the convolution integral.

We then have

$$x(t) \approx \sum_{k=-\infty}^{k=\infty} x(k\Delta T)\delta(t-k\Delta T)\Delta T$$

If the input to the LTI system is $\delta(t-k\Delta T)$ the output will be $h(t-k\Delta T)$, so we can write the output as

$$y(t) \approx \sum_{k=-\infty}^{k=\infty} x(k\Delta T)h(t-k\Delta T)\Delta T$$

If we define $\lambda = k\Delta T$, then as $\Delta T \to 0$ the sum again becomes an integral and we have

$$y(t) = \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)d\lambda$$

By a change of variable we can show that convolution has *the commutative property*, or

$$y(t) = h(t) \star x(t) = x(t) \star h(t)$$

This means that we can compute the output in one of two equivalent ways:

$$y(t) = \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)d\lambda = \int_{-\infty}^{\infty} x(t-\lambda)h(\lambda)d\lambda$$

Now that we know how to determine the output of a system given the impulse response and the input, we need to determine the best way to compute this. There are two general ways for computing the convolution, analytical and graphical. Both methods give the same results, but usually the answers initially look different. Analytical convolution is generally used for very simple problems, and becomes unwieldy for very complicated problems. Graphical convolution is usually used for more complicated problems and is also useful for visualizing what is happening to the signals during convolution.

Analytical Convolution

In analytical convolution, we basically just evaluate the integral. It is necessary to utilize any step function in the impulse response or system input to change the limits of the integral. In addition, it is important to remember to include any necessary unit step functions on the output, since the output of one system may be the input to another system.

Example 5.5.1. Determine the output of an LTI system with impulse response $h(t) = Ae^{-t/\tau}u(t)$ to input $x(t) = Bu(t-1) - Bu(t-2)$. To solve this problem we must first choose the way we are going to perform the convolution. We will use the form

$$y(t) = \int_{-\infty}^{\infty} h(t-\lambda)x(\lambda)d\lambda$$

Substituting in our functions we have

$$y(t) = \int_{-\infty}^{\infty} Ae^{-(t-\lambda)/\tau}u(t-\lambda)[Bu(\lambda-1) - Bu(\lambda-2)]d\lambda$$

or

$$y(t) = \int_{-\infty}^{\infty} ABe^{-(t-\lambda)/\tau}u(t-\lambda)u(\lambda-1)d\lambda - \int_{-\infty}^{\infty} ABe^{-(t-\lambda)/\tau}u(t-\lambda)u(\lambda-2)d\lambda$$

Using the step functions to change the limits on the integrals we have

$$y(t)=\int_1^t ABe^{-(t-\lambda)/\tau}d\lambda-\int_2^t ABe^{-(t-\lambda)/\tau}d\lambda = ABe^{-t/\tau}\int_1^t e^{\lambda/\tau}d\lambda - ABe^{-t/\tau}\int_2^t e^{\lambda/\tau}d\lambda$$

Finally we have

$$\begin{aligned} y(t) &= ABe^{-t/\tau}\tau[e^{t/\tau}-e^{1/\tau}]u(t-1)-ABe^{-t/\tau}\tau[e^{t/\tau}-e^{2/\tau}]u(t-2) \\ &= \tau AB[1-e^{-(t-1)/\tau}]u(t-1)-\tau AB[1-e^{-(t-2)/\tau}]u(t-2) \end{aligned}$$

Graphical Convolution

As with analytical convolution, the first thing to do is decide which of the two forms of the convolution integral to use. Let's assume that we are going to use the form

$$y(t)=\int_{-\infty}^{\infty} h(t-\lambda)x(\lambda)d\lambda$$

We need to keep in mind that we want the area under the product of two functions, $h(t-\lambda)$ and $x(\lambda)$. In addition, we need to remember that we are integrating with respect to the dummy variable λ, not t. This is important to understand, since the function $h(t-\lambda)$ will be at different places along the λ axis as the variable t varies. In fact, the whole point of doing graphical convolution is to sketch the function $h(t-\lambda)$ as a function of t and $x(\lambda)$, determine the overlap, and then perform the integration.

One simple method for being able to locate $h(t-\lambda)$ as a function of t and λ is to look at $h(t)$ and find suitable "marker" points. Let's call two such points t_1 and t_2. The we can find where these marker points are on the λ axis as follows

$$\begin{aligned} h(t_1)=h(t-\lambda)\rightarrow\lambda=t-t_1 \\ h(t_2)=h(t-\lambda)\rightarrow\lambda=t-t_2 \end{aligned}$$

This will all make more sense with a few examples.

Example 5.5.2. Determine the output of an LTI system with impulse response $h(t)=Ae^{-t/\tau}u(t)$ to input $x(t)=Bu(t-1)-Bu(t-2)$. (This is the same problem as Example 5.5.1.) Let's use the integral form

$$y(t)=\int_{-\infty}^{\infty} h(\lambda)x(t-\lambda)d\lambda$$

The top panel of Figure 5.7 displays the input signal. For this signal the most convenient markers are $x(1)$ and $x(2)$. If we can figure out how these points move we can determine the location of $x(t-\lambda)$ on the λ axis as t. For these points we have

$$\begin{aligned} x(1)=x(t-\lambda)\rightarrow\lambda=t-1 \\ x(2)=x(t-\lambda)\rightarrow\lambda=t-2 \end{aligned}$$

The bottom panel of Figure 5.7 displays $x(t-\lambda)$ as a function of λ. From this figure we can determine where this function is as t varies.

We next need to graph pictures of $h(\lambda)$ and $x(t-\lambda)$ and look for times when the product of the functions is not zero. For these functions there are three different pictures, corresponding to $t<1$, $1\le t\le 2$, and $t\ge 2$, as shown in Figure 5.8.

For $t<1$ there is no place the product of the functions is not zero, so the output $y(t)=0$.

For $1\le t\le 2$ we have

$$y(t)=\int_0^{t-1} Ae^{-\lambda/\tau}Bd\lambda = AB\tau\left[1-e^{-(t-1)/\tau}\right]$$

which agrees with our previous answer.

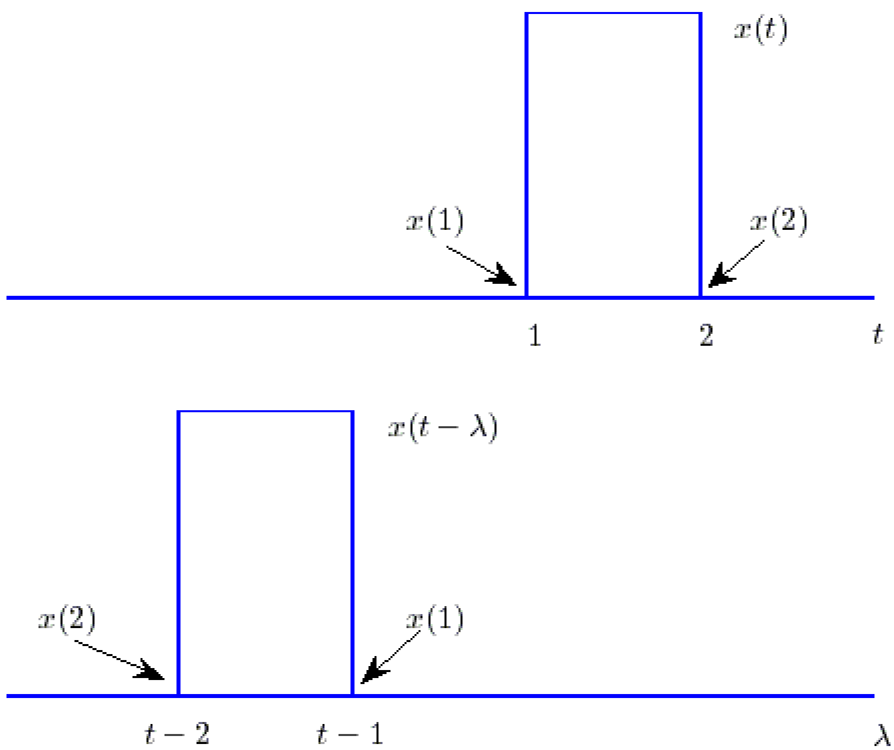

Figure 5.7. Input signal for Example 5.5.2. The original signal $x(t)$ is shown in the top panel, with the two convenient "marker" points. The lower panel shows $x(t-\lambda)$ as a function of λ, and shows how these marker points move as the parameter t is varied. Note that the function $x(t)$ has been flipped (reversed) from its original orientation.

For $t \ge 2$ we have

$$y(t) = \int_{t-2}^{t-1} Ae^{-\lambda/\tau} B d\lambda = AB\tau\left[e^{-(t-2)/\tau} - e^{-(t-1)/\tau}\right]$$

Our solution is then $y(t) = \begin{cases} 0 & t \le 1 \\ AB\tau\left[1-e^{-(t-1)/\tau}\right] & 1 \le t \le 2 \\ AB\tau\left[e^{-(t-2)/\tau} - e^{-(t-1)/\tau}\right] & t \ge 2 \end{cases}$

Our solution should be continuous, so we need to check the values at the boundaries. We have

$$y(1) = 0$$
$$y(2) = AB\tau[1-e^{-1/\tau}]$$

While this looks different than our previous answer, it is really the same thing for this range of t. To see this note that from before we had

$$y(t) = \tau AB[1-e^{-(t-1)/\tau}]u(t-1) - \tau AB[1-e^{-(t-2)/\tau}]u(t-2)$$

If $t \ge 2$ both unit step functions are one and we have

$$y(t) = \tau AB[1-e^{-(t-1)/\tau}] - \tau AB[1-e^{-(t-2)/\tau}] = AB\tau[e^{-(t-2)/\tau} - e^{-(t-1)/\tau}]$$

The solution is plotted in Figure 5.9.

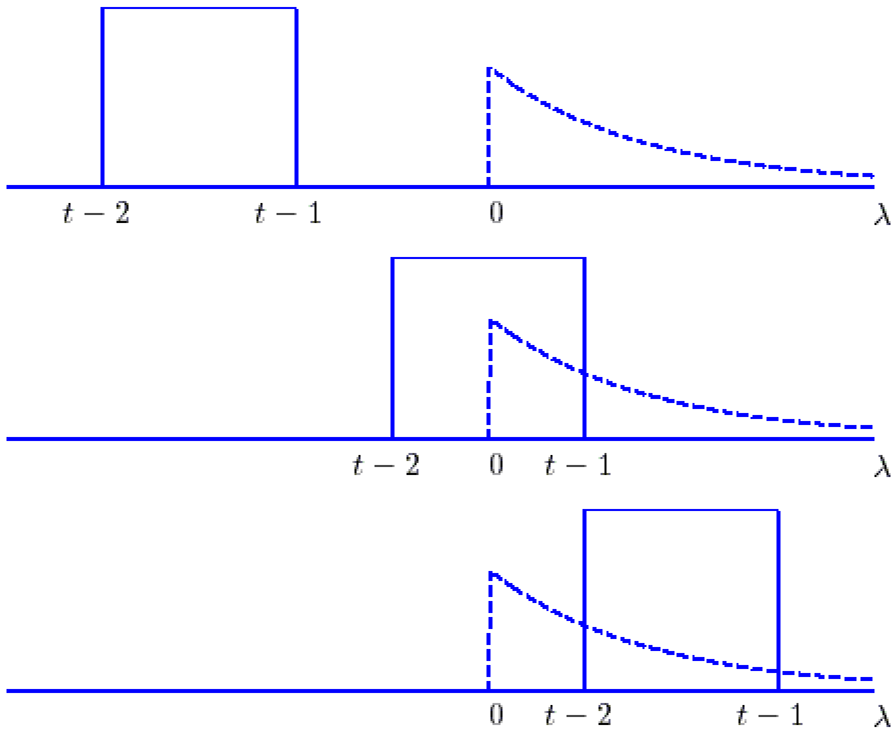

Figure 5.8. Plots of $h(\lambda)$ and $x(t-\lambda)$ (dashed line) for Example 5.5.2 for representative values of t. For these graphs *A=2*, *B=3*, and $\tau = 0.8$.

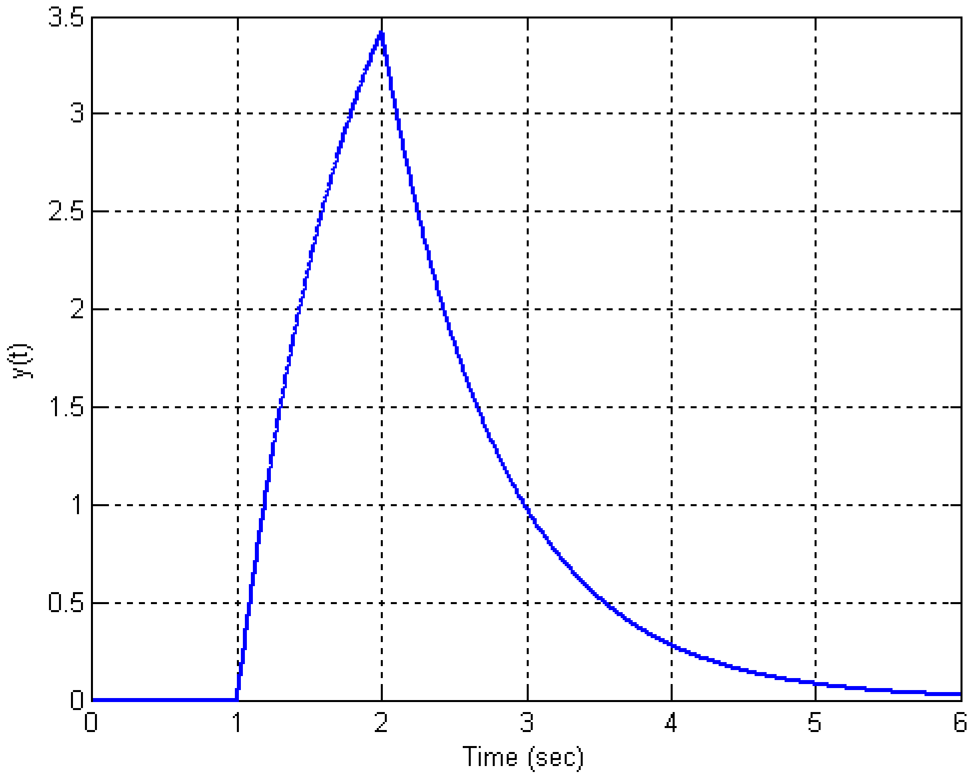

Figure 5.9. Result (output $y(t)$) for Example 5.5.2 assuming A=2, B=3, and $\tau = 0.8$.

Example 5.5.3. Determine the output of an LTI system with impulse response $h(t) = Ae^{-(t-1)/\tau}u(t-1)$ to input $x(t) = 2u(t) - 2u(t-2) - 3u(t-3)$. Let's use the integral form

$$y(t) = \int_{-\infty}^{\infty} h(t-\lambda)x(\lambda)d\lambda$$

There is really only one marker point of note, that of $h(1)$, which gives $h(1) = h(t-\lambda) \rightarrow \lambda = t-1$.

There are four different graphs we need for this example, $t \le 1$, $1 \le t \le 3$, $3 \le t \le 4$, and $t \ge 4$. For $t \le 1$ the product of the functions $h(t-\lambda)$ and $x(\lambda)$ is zero, so $y(t) = 0$.

For $1 \le t \le 3$ we have the situation shown in Figure 5.10. Evaluating the integrals we have

$$y(t) = \int_0^{t-1} e^{-(t-\lambda-1)/\tau}(2)d\lambda = 2e^{-(t-1)/\tau}\int_0^{t-1} e^{\lambda/\tau}d\lambda = 2\tau e^{-(t-1)/\tau}\left[e^{(t-1)/\tau} - 1\right] = 2\tau\left[1 - e^{-(t-1)/\tau}\right]$$

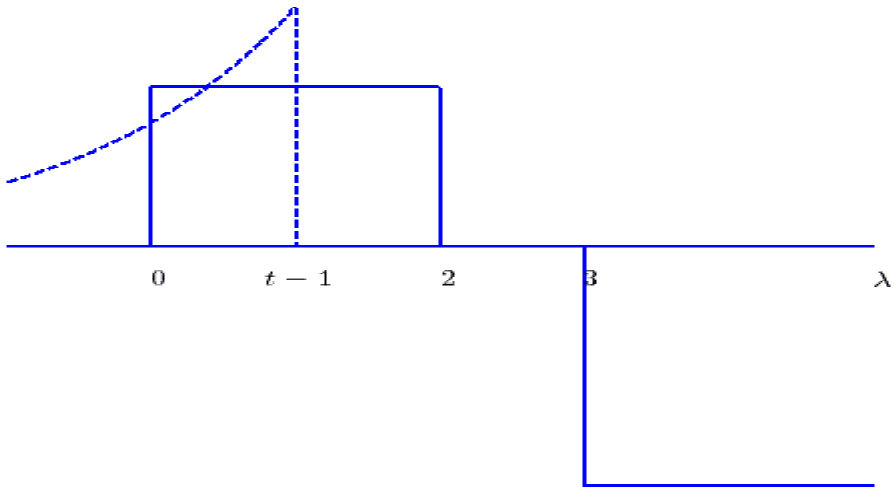

Figure 5.10. First overlapping region for Example 5.5.3 assuming $\tau = 1.5$. This figure is valid for $1 \le t \le 3$.

For $3 \le t \le 4$ we have the situation shown in Figure 5.11. Evaluating the integral we have

$$y(t) = \int_0^2 e^{-(t-\lambda-1)/\tau}(2)d\lambda = 2e^{-(t-1)/\tau}\int_0^2 e^{\lambda/\tau}d\lambda = 2\tau e^{-(t-1)/\tau}\left[e^{2/\tau} - 1\right] = 2\tau\left[e^{-(t-3)/\tau} - e^{-(t-1)/\tau}\right]$$

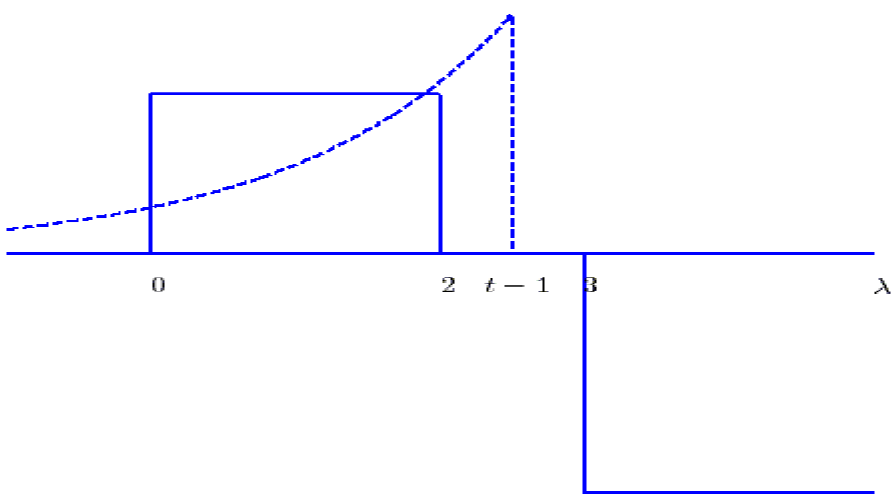

Figure 5.11. Second overlapping region for Example 5.5.3 assuming $\tau = 1.5$. This figure is valid for $3 \le t \le 4$.

For $t \geq 4$ we have the situation shown in Figure 5.12. For this situation we will need two integrals,

$$
\begin{aligned}
y(t) &= \int_0^2 e^{-(t-\lambda-1)/\tau}(2)d\lambda + \int_3^{t-1} e^{-(t-\lambda-1)/\tau}(-3)d\lambda = \\
&= 2e^{-(t-1)/\tau}\int_0^2 e^{\lambda/\tau}d\lambda - 3e^{-(t-1)/\tau}\int_3^{t-1} e^{\lambda/\tau}d\lambda \\
&= 2\tau e^{-(t-1)/\tau}\left[e^{2/\tau}-1\right] - 3\tau e^{-(t-1)/\tau}\left[e^{(t-1)/\tau}-e^{3/\tau}\right] \\
&= 2\tau\left[e^{-(t-3)/\tau}-e^{-(t-1)/\tau}\right] - 3\tau\left[1-e^{-(t-4)/\tau}\right]
\end{aligned}
$$

In summary we have

$$
y(t) = \begin{cases}
0 & t \leq 1 \\
2\tau\left[1-e^{-(t-1)/\tau}\right] & 1 \leq t \leq 3 \\
2\tau\left[e^{-(t-3)/\tau}-e^{-(t-1)/\tau}\right] & 3 \leq t \leq 4 \\
2\tau\left[e^{-(t-3)/\tau}-e^{-(t-1)/\tau}\right] - 3\tau\left[1-e^{-(t-4)/\tau}\right] & t \geq 4
\end{cases}
$$

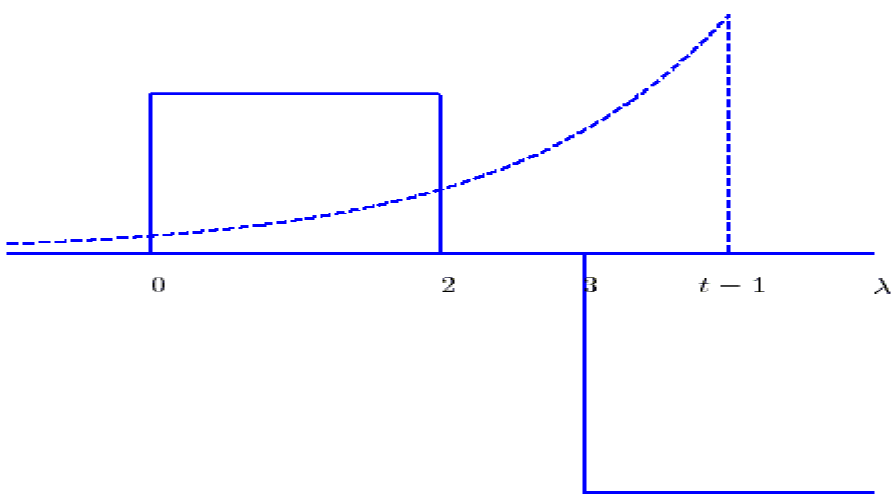

Figure 5.12. Third overlapping region for Example 5.5.3 assuming $\tau = 1.5$. This figure is valid for $t \geq 4$.

Checking the values of $y(t)$ at each boundaries we have

$$
\begin{aligned}
y(1) &= 0 \\
y(3) &= 2\tau[1-e^{-2/\tau}] \\
y(4) &= 2\tau[e^{1/\tau}-e^{-3/\tau}]
\end{aligned}
$$

The final solution is plotted in Figure 5.13.

Example. 5.5.4. Determine the output of an LTI system with impulse response $h(t) = t[u(t+1) - u(t-1)]$ to input $x(t) = u(t) - u(t-1) + 2u(t-2)$. Let's use the integral form

$$y(t) = \int_{-\infty}^{\infty} h(t-\lambda)x(\lambda)d\lambda$$

There are two marker points of note, that of, $h(-1)$ which gives

$$h(-1) = h(t-\lambda) \rightarrow \lambda = t+1$$
$$h(1) = h(t-\lambda) \rightarrow \lambda = t-1$$

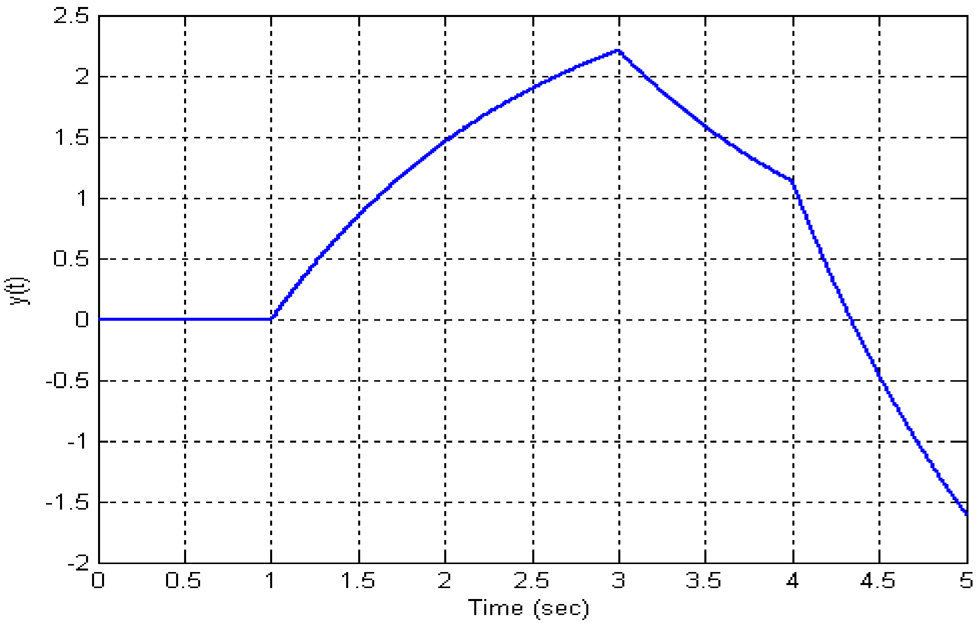

Figure 5.13. Result (output $y(t)$) for Example 5.5.3 assuming $\tau = 1.5$.

There are six different graphs we need for this example, $t \le -1$, $-1 \le t \le 0$, $0 \le t \le 1$, $1 \le t \le 2$, $2 \le t \le 3$ and $t \ge 3$.

For $t \le -1$ the product of the functions $h(t-\lambda)$ and $x(\lambda)$ is zero, so $y(t) = 0$.

For $-1 \le t \le 0$ we have the situation depicted in Figure 5.14, and

$$y(t) = \int_{0}^{t+1} (t-\lambda)(1)d\lambda = \frac{1}{2}(t^2 - 1)$$

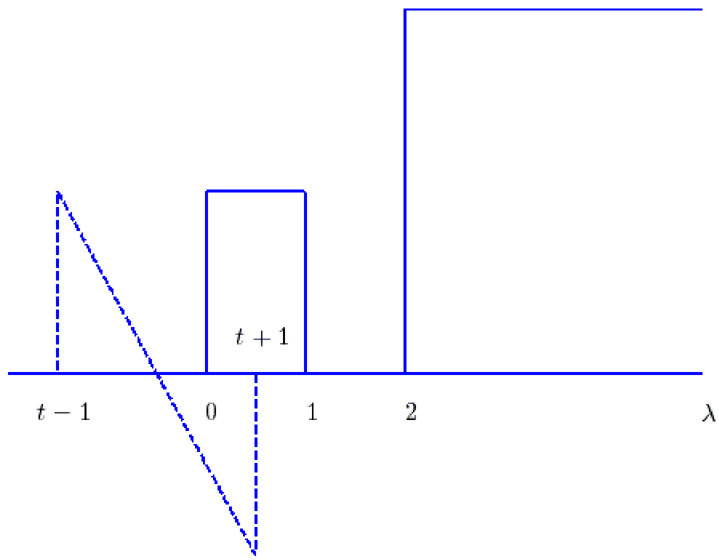

Figure 5.14. Initial overlapping region for Example 5.5.4. This figure is valid for $-1 \le t \le 0$.

For $0 \le t \le 1$ we have the situation depicted in Figure 5.15, and

$$y(t) = \int_0^1 (t-\lambda)(1)d\lambda = t - \frac{1}{2}$$

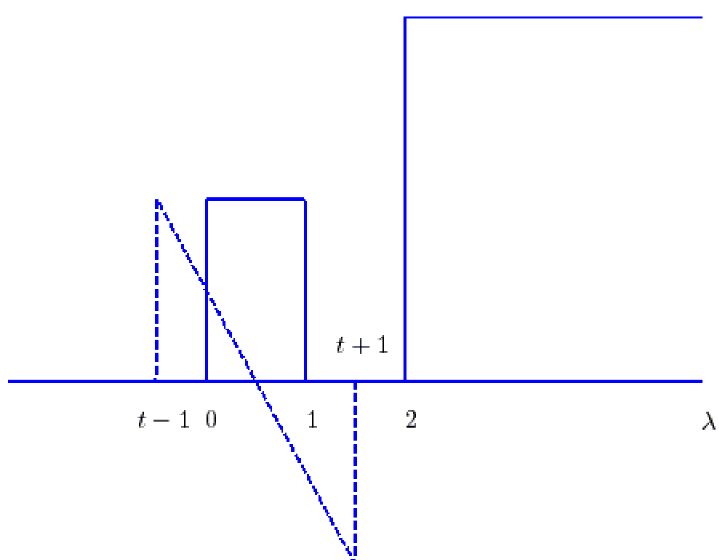

Figure 5.15. Second overlapping region for Example 5.5.4. This figure is valid for $0 \le t \le 1$.

For $1 \le t \le 2$ we have the situation depicted in Figure 5.16, and

$$y(t) = \int_{t-1}^{1} (t-\lambda)(1)d\lambda + \int_{2}^{t+1} (t-\lambda)(2)d\lambda = \frac{1}{2}(t^2 - 6t + 6)$$

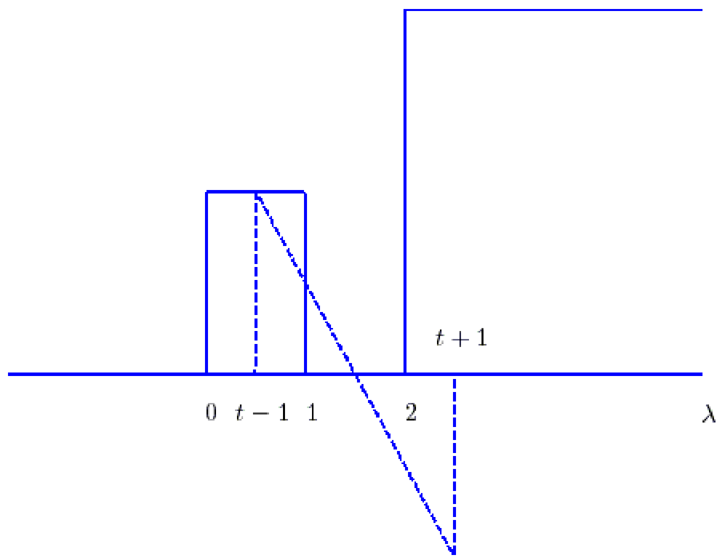

Figure 5.16. Third overlapping region for Example 5.5.4. This figure is valid for $1 \le t \le 2$.

For $2 \le t \le 3$ we have the situation depicted in Figure 5.17, and

$$y(t) = \int_{2}^{t+1} (t-\lambda)(2)d\lambda = t^2 - 4t + 3$$

For $t \ge 3$ we have the situation depicted in Figure 5.18, and

$$y(t) = \int_{t-1}^{t+1} (t-\lambda)(2)d\lambda = 0$$

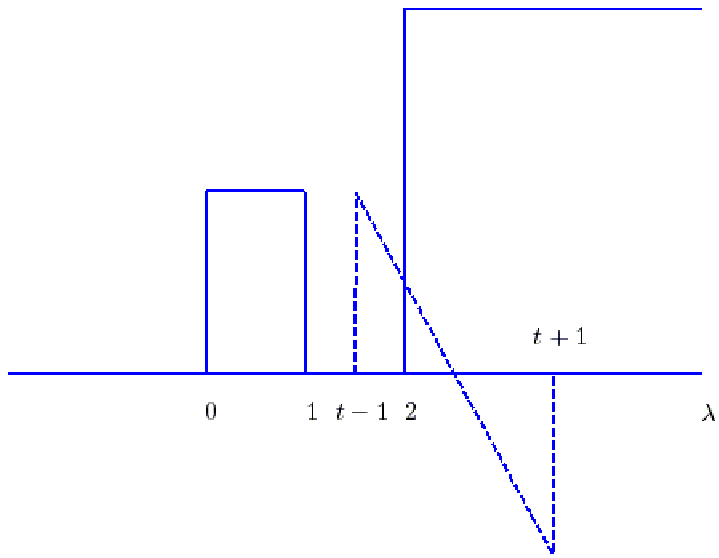

Figure 5.17. Fourth overlapping region for Example 5.5.4. This figure is valid for $2 \le t \le 3$.

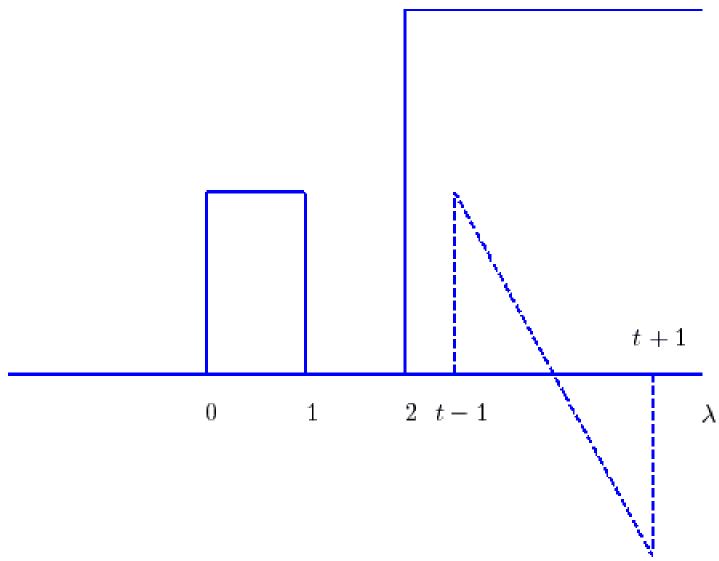

Figure 5.18. Final overlapping region for Example 5.5.4. This figure is valid for $t \ge 3$.

In summary we have

$$y(t)=\begin{cases} 0 & t\le -1 \\ \frac{1}{2}(t^2-1) & -1\le t\le 0 \\ t-\frac{1}{2} & 0\le t\le 1 \\ \frac{1}{2}(t^2-6t+6) & 1\le t\le 2 \\ t^2-4t+3 & 2\le t\le 3 \\ 0 & t\ge 3 \end{cases}$$

Note that although the input to the system starts at time $t=0$, the output starts at time $t=-1$. Thus the system is noncausal. The system output is graphed in Figure 5.19.

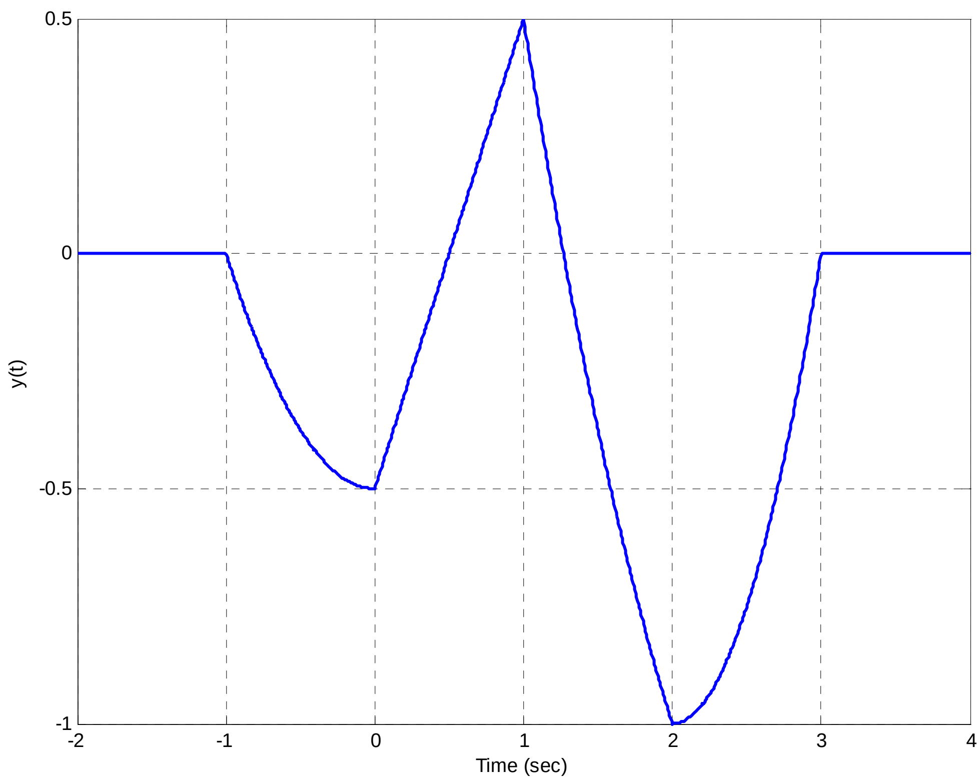

Figure 5.19 Result (output $y(t)$) for Example 5.5.4. Note that the system is not causal since the input starts at $t=0$ but the output starts at $t=-1$

5.6 Causality and BIBO Stability

Now that we can write the output of an LTI system in terms of the convolution of the input with the impulse response, we can also determine some fairly simple tests to determine if an LTI system is BIBO stable or causal. We have

$$y(t) = \int_{-\infty}^{\infty} h(\lambda)x(t-\lambda)d\lambda$$

If we know the input is bounded, $|x(t)| \le N$, then we know

$$|y(t)| \le \left| \int_{-\infty}^{\infty} h(\lambda)x(t-\lambda)d\lambda \right| \le \int_{-\infty}^{\infty} |h(\lambda)||x(t-\lambda)|d\lambda \le \int_{-\infty}^{\infty} |h(\lambda)|Nd\lambda = N\int_{-\infty}^{\infty} |h(\lambda)|d\lambda$$

Thus an LTI system will be BIBO stable if

$$\int_{-\infty}^{\infty} |h(\lambda)|d\lambda < \infty$$

Next, let's assume we want to find the output of an LTI system at the time t_0 , so we have

$$y(t_0) = \int_{-\infty}^{\infty} h(t_0-\lambda)x(\lambda)d\lambda$$

We can then break the integral into two parts,

$$y(t_0) = \int_{-\infty}^{t_0} h(t_0-\lambda)x(\lambda)d\lambda + \int_{t_0}^{\infty} h(t_0-\lambda)x(\lambda)d\lambda$$

If the system is causal, then the second integral must be zero, since it depends on future values of the input. In order of the second integral to be zero we need

$$h(t_0-\lambda) = 0 \quad \text{for } \lambda \in (t_0, \infty)$$

Let's assume $\lambda = t_0 + \epsilon, \epsilon > 0$. Substituting this into our expression for the impulse response we have

$$h(t_0-\lambda) = h(t_0-[t_0+\epsilon]) = h(-\epsilon) = 0, \epsilon > 0$$

or

$$h(t) = 0, t < 0$$

This means the impulse response must be zero for any time less than zero in order for the LTI system to be causal.

In summary, an LTI system is BIBO stable if

$$\int_{-\infty}^{\infty} |h(\lambda)|d\lambda < \infty$$

and is causal if

$$h(t) = 0, t < 0$$

Note that these are independent properties, a system can be stable and not causal, or causal and not stable.

5.7 Convolution Properties and Interconnected Systems

There are a number of useful and important properties of convolution. Among the most useful are the following:

Commutative Property: $y(t) = h(t) \star x(t) = x(t) \star h(t)$

Associative Property: $h_2(t) \star [h_1(t) \star x(t)] = [h_2(t) \star h_1(t)] \star x(t)$

Distributive Property: $h(t) \star [x_1(t) + x_2(t)] = h(t) \star x_1(t) + h(t) \star x_2(t)$

The commutative property means that

$$y(t) = \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)d\lambda = \int_{-\infty}^{\infty} x(t-\lambda)h(\lambda)d\lambda$$

so that we have two equivalent ways of determining the system output, $y(t)$. A convenient method of presenting the relationship between the input, output, and impulse response of a system is depicted in Figure 5.7.

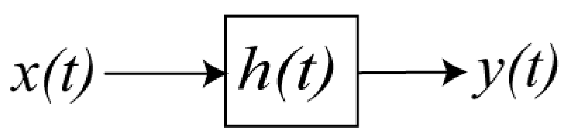

Figure 5.20. Input, impulse response, and output for an LTI system.

If we have two LTI systems in *series*, as shown in Figure 5.21, then we can relate the input to the output as follows:

$$v(t) = x(t) \star h_1(t)$$

Then

$$y(t) = v(t) \star h_2(t) = \left[x(t) \star h_1(t)\right] \star h_2(t)$$

Using the commutative property we can write this as

$$y(t) = [h_1(t) \star h_2(t)] \star x(t)$$

The impulse response between the input and output is then

$$h(t) = h_1(t) \star h_2(t)$$

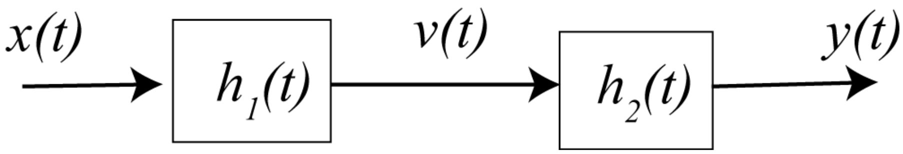

Figure 5.21. Two LTI systems connected in series.

If we have to LTI systems in *parallel*, as shown in Figure 5.22, then we can relate the input to the output as follows

$$v(t) = x(t) \star h_1(t) \text{ and } w(t) = x(t) \star h_2(t)$$

Combining we have

$$y(t) = v(t) + w(t) = x(t) \star h_1(t) = x(t) \star h_2(t)$$

Using the associative and distributive properties we then have

$$y(t) = [h_1(t) + h_2(t)] \star x(t)$$

Hence the system transfer function is then

$$h(t) = h_1(t) + h_2(t)$$

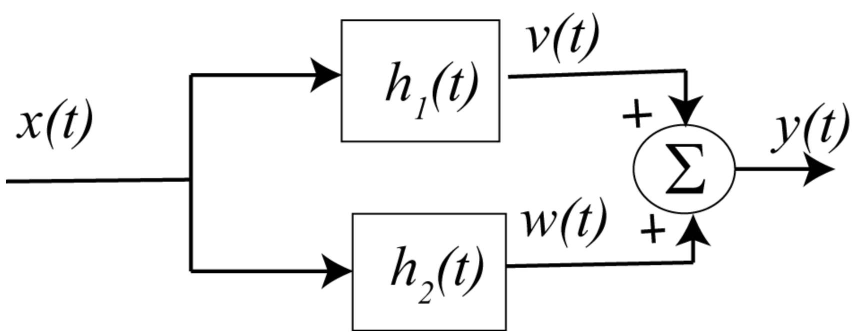

Figure 5.22. Two LTI systems connected in parallel.

Example 5.7.1. Consider the system shown in Figure 5.23. In this figure we have *negative feedback.* For this system we have

$$e(t) = x(t) - v(t) \text{ and } v(t) = y(t) \star h_2(t)$$

Combining we have

$$y(t) = e(t) \star h_1(t) = [x(t) - v(t)] \star h_1(t) =]x(t) - y(t) \star h_2(t)] \star h_1(t)$$

We can rearrange this as

$$y(t) + y(t) \star h_1(t) \star h_2(t) = x(t) \star h_1(t)$$

or

$$y(t) \star [\delta(t) + h_1(t) \star h_2(t)] = x(t) \star [h_1(t)]$$

This is an awkward expression, but at this point we cannot simplify it anymore.

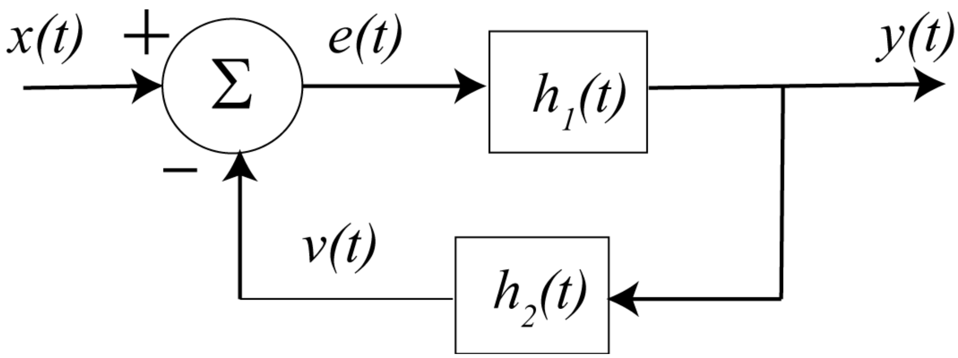

Figure 5-23. System for Example 5.7.1.

Example 5.7.2. Consider the system shown in Figure 5.24. Again we have negative feedback in this system. Here we have

$$e(t) = x(t) \star h_1(t) - v(t)$$
$$v(t) = y(t) \star h_4(t)$$
$$w(t) = e(t) \star h_2(t) + x(t) \star h_5(t)$$
$$y(t) = w(t) \star h_3(t)$$

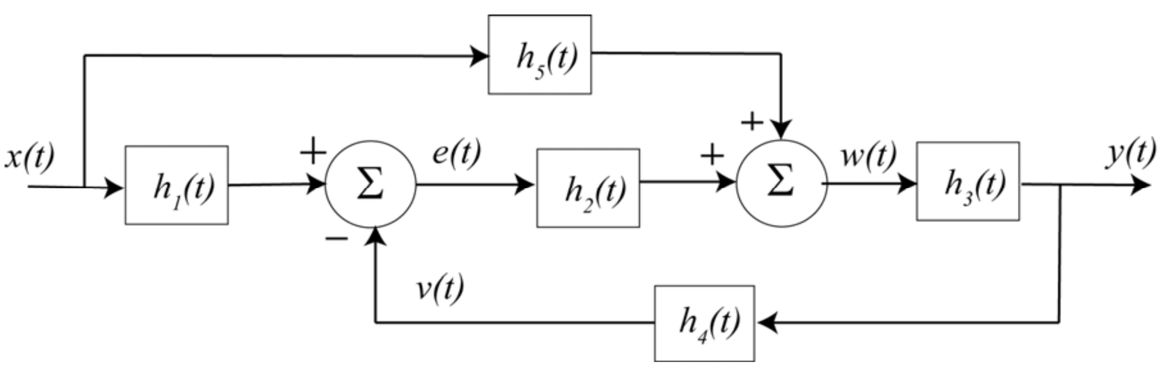

Figure 5-24. System for Example 5.7.2.

Starting with our expression for $y(t)$ and working backwards we have

$$y(t) = w(t) \star h_3(t)$$
$$y(t) = [e(t) \star h_2(t) + x(t) \star h_5(t)] \star h_3(t)$$
$$y(t) = [\{x(t) \star h_1(t) - v(t)\} \star h_2(t) + x(t) \star h_5(t)] \star h_3(t)$$
$$y(t) = [\{x(t) \star h_1(t) - (y(t) \star h_4(t))\} \star h_2(t) + x(t) \star h_5(t)] \star h_3(t)$$

Simplifying we get

$$y(t) = x(t) \star h_1(t) \star h_2(t) \star h_3(t) - y(t) \star h_4(t) \star h_2(t) \star h_3(t) + x(t) \star h_5(t) \star h_3(t)$$

Finally

$$y(t) \star [\delta(t) + h_2(t) \star h_3(t) \star h_4(t)] = x(t) \star [h_1(t) \star h_2(t) \star h_3(t) + h_3(t) \star h_5(t)]$$

Trying to determine the system impulse response in this way is very difficult, if not impossible. In addition, if we are trying to modify one of the subsystems system to change the behavior of the overall system, this method of relating the input to the output does not lend itself to any intuition or easy analysis. This is one of the primary reasons Laplace and Fourier transforms were developed. Laplace transforms are hence our next topic.