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ECE-205

Exam 1

Spring 2013

Calculators can only be used for simple calculations. Solving integrals, differential equations, systems of equations, etc. does not count as a simple calculation.

You must show your work to receive credit.

Problem 1	**________/16**
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Problem 7-10	**________/16**

Total ____________

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1) (16 points) Assume we have a first order system with the governing differential equation

$$0.2\dot{y}(t) + y(t) = 3x(t)$$

The system has the initial value of 0.4, so $y(0) = 0.4$. The input to this system is

$$x(t) = \begin{cases} 0 & t < 0 \\ 0.2 & 0 \le t < 0.6 \\ 0 & 0.6 \le t \end{cases}$$

a) Determine the output of the system in each of the above time intervals. *Simplify your final answer as much as possible and box it. Be sure to include the correct initial value in the first interval!*

b) Make a sketch of $y(t)$. (I am looking mostly at the shape of $y(t)$ from 0 to 0.6 and from 0.6 on.)

$$y(t) = [y(t_0) - KA]e^{-(t-t_0)/\tau} + KA$$

a) $0 \le t < 0.6 \quad y(0) = 0.4 \quad KA = (3)(0.2) = 0.6 \quad \tau = 0.2$

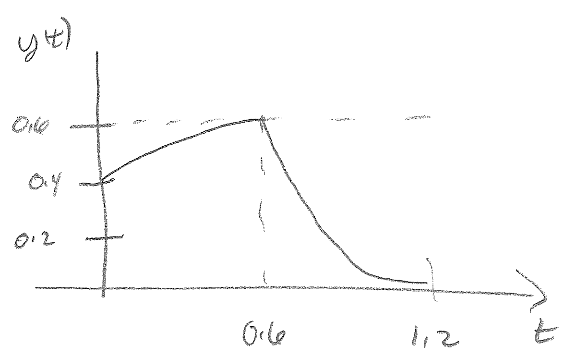
$$y(t) = [0.4 - 0.6]\,e^{-t/0.2} + 0.6 = \boxed{-0.2e^{-t/0.2} + 0.6 = y(t)}$$

$0.6 \le t \quad y(0.6) = -0.2e^{-3} + 0.6 = 0.59$

$KA = (3)(0) = 0$

$$y(t) = [0.59 - 0]\,e^{-(t-0.6)/0.2} + 0 = \boxed{0.59\,e^{-(t-0.6)/0.2} = y(t)}$$

b)



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2) (16 points) For the following three differential equations, assume the input is $x(t) = 3u(t)$ (the input is equal to one for time greater than zero), and the initial conditions are $y(0) = \dot{y}(0) = 0$

Determine the solution to each of the following differential equations and put your final answer in a box. Be sure to use the initial conditions to solve for all unknowns. You must show all your work to receive credit, ***you cannot simply use the formula from problem 3.***

a) $\ddot{y}(t) + 6\dot{y}(t) + 9y(t) = 9x(t)$

$r^2 + 6r + 9 = 0$ $\quad 9y_f = 9 \cdot 3$

$(r+3)^2 = 0$ $\quad y_f = 3$

$y(t) = 3 + c_1 e^{-3t} + c_2 t e^{-3t}$ $\quad y(0) = 0 = 3 + c_1$ $\quad \boxed{c_1 = -3}$

$\dot{y}(t) = 0 - 3c_1 e^{-3t} + c_2 e^{-3t} - 3c_2 t e^{-3t}$ $\quad \dot{y}(0) = -3c_1 + c_2 = 0$ $\quad c_2 = 3c_1 = \boxed{-9 = c_2}$

$$\boxed{y(t) = 3 - 3e^{-3t} - 9te^{-3t}}$$

b) $\ddot{y}(t) + 4\dot{y}(t) + 13y(t) = 26x(t)$

$r^2 + 4r + 13 = 0$ $\quad 13y_f = 26 \cdot 3$

$(r+2)^2 + 3^2 = 0$ $\quad y_f = 6$

$r = -2 \pm j3$

$y(t) = 6 + ce^{-2t}\sin(3t + \theta)$ $\quad y(0) = 0 = 6 + c\sin(\theta)$

$\dot{y}(t) = 0 - 2ce^{-2t}\sin(3t+\theta) + 3ce^{-2t}\cos(3t+\theta)$

$\dot{y}(0) = 0 = -2\sin(\theta) + 3\cos(\theta)$ $\quad \tan(\theta) = 3/2$ $\quad \boxed{\theta = 56.31^\circ}$

$c = \dfrac{-6}{\sin(56.31^\circ)} = \boxed{-7.21 = c}$

$$\boxed{y(t) = 6 - 7.21e^{-2t}\sin(3t + 56.31^\circ)}$$

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3) (14 points) The form of the under damped ($0 < \zeta < 1$) solution to the second order differential equation

$$\ddot{y}(t) + 2\zeta\omega_n \dot{y}(t) + \omega_n^2 y(t) = \mathrm{K}\ \omega_n^2 x(t)$$

for a step input $x(t) = Au(t)$ is

$$y(t) = KA + ce^{-\zeta\omega_n t}\sin(\omega_d t + \phi)$$

where c and ϕ are constants to be determined and the damped frequency $\omega_d = \omega_n\sqrt{1-\zeta^2}$

a) Using the initial condition $\dot{y}(0) = 0$ show that $\tan(\phi) = \dfrac{\sqrt{1-\zeta^2}}{\zeta}$

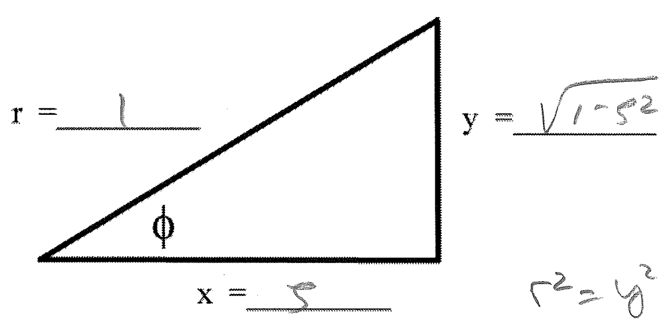
$$\dot{y}(t) = 0 - \zeta\omega_n c e^{-\zeta\omega_n t}\sin(\omega_d t+\phi) + \omega_d c e^{-\zeta\omega_n t}\cos(\omega_d t+\phi)$$

$$\dot{y}(0) = 0 = -\zeta\omega_n \sin(\phi) + \omega_d \cos(\phi)$$

$$\tan(\phi) = \frac{\omega_d}{\zeta\omega_n} = \frac{\omega_n\sqrt{1-\zeta^2}}{\omega_n \zeta}$$

$$\boxed{\tan(\phi) = \frac{\sqrt{1-\zeta^2}}{\zeta}}$$

b) We can express the relationship in part a using the following triangle. Fill in the blanks and then use this triangle determine an expression for $\sin(\phi)$.



$$r^2 = y^2 + x^2 = 1 - \zeta^2 + \zeta^2 = 1$$

$$\boxed{\sin(\phi) = \sqrt{1-\zeta^2}}$$

c) Use your answer to part b, and the initial condition $y(0) = 0$ to determine the remaining unknown constant, and write out the complete solution for $y(t)$.

$$y(0) = 0 = KA + c\sin(\phi) = KA + c\sqrt{1-\zeta^2} \qquad c = \frac{-KA}{\sqrt{1-\zeta^2}}$$

$$\boxed{y(t) = KA\left[1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \phi)\right]}$$

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4) (10 points) For the following first order differential equation,

$$3\dot{y}(t) + y(t) = x^2(t)$$

determine an expression for the output assuming $t_0 = 0$ and $y(t_0) = y(0) = 1$.

$$\dot{y}(t) + \frac{1}{3} y(t) = \frac{1}{3} x^2(t)$$

$$\frac{d}{dt}\left[y(t) e^{t/3}\right] = \frac{1}{3} e^{t/3} x^2(t)$$

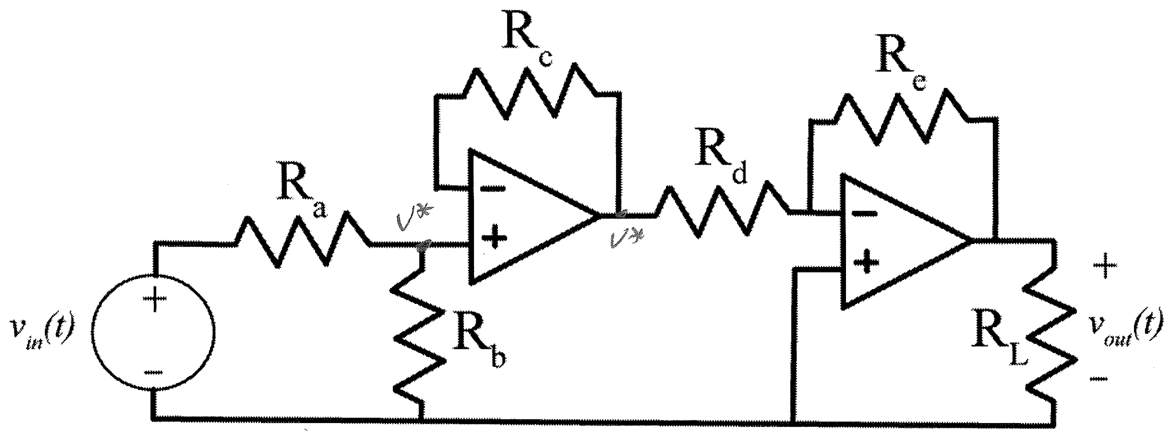
$$y(t) e^{t/3} - y(t_0) e^{t_0/3} = \frac{1}{3} \int_{t_0}^{t} e^{\lambda/3} x^2(\lambda) \, d\lambda$$

$$y(t) e^{t/3} - 1 = \frac{1}{3} \int_{0}^{t} e^{\lambda/3} x^2(\lambda) \, d\lambda$$

$$y(t) e^{t/3} = 1 + \frac{1}{3} \int_{0}^{t} e^{\lambda/3} x^2(\lambda) \, d\lambda$$

$$\boxed{y(t) = e^{-t/3} + \frac{1}{3} e^{-t/3} \int_{0}^{t} e^{\lambda/3} x^2(\lambda) \, d\lambda}$$

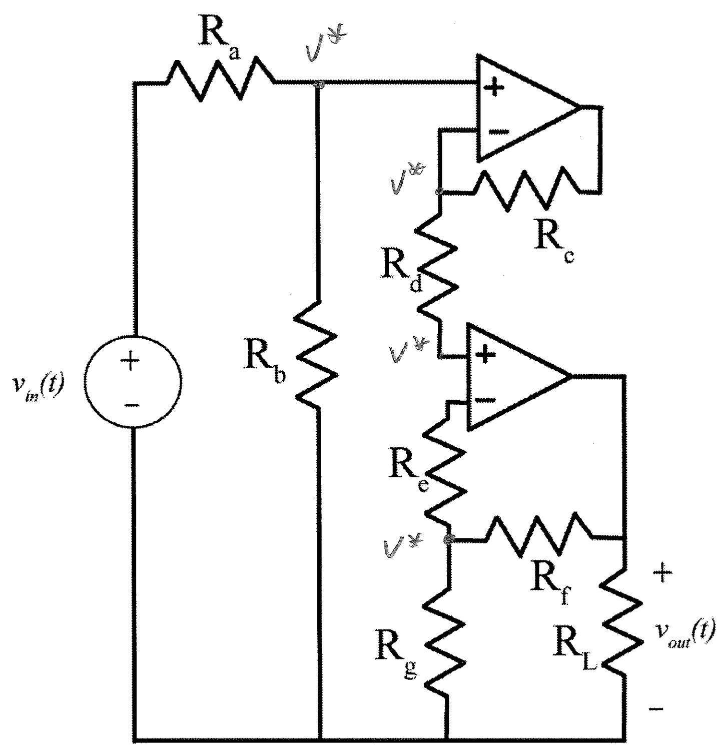
5) (12 points) For the following two op-amps circuits, we can write $v_{out}(t) = G\, v_{in}(t)$. Determine the value of G for each circuit.



$$V^* = \frac{V_{in} R_b}{R_a + R_b} \qquad \frac{V^*}{R_d} + \frac{V_{out}}{R_e} = 0 \qquad V_{out} = \frac{-R_e}{R_d} V^*$$

$$= \left(-\frac{R_e}{R_d} \frac{R_b}{R_a + R_b}\right) V_{in}$$

$$\boxed{G = -\frac{R_e}{R_d} \frac{R_b}{R_a + R_b}}$$



$$V^* = \frac{V_{in} R_b}{R_a + R_b}$$

$$\frac{V_{out} R_g}{R_g + R_f} = V^* = \frac{R_b}{R_a + R_b} V_{in}$$

$$V_{out} = \left(\frac{R_g + R_f}{R_g} \frac{R_b}{R_a + R_b}\right) V_{in}$$

$$\boxed{G = \frac{R_g + R_f}{R_g} \frac{R_b}{R_a + R_b}}$$

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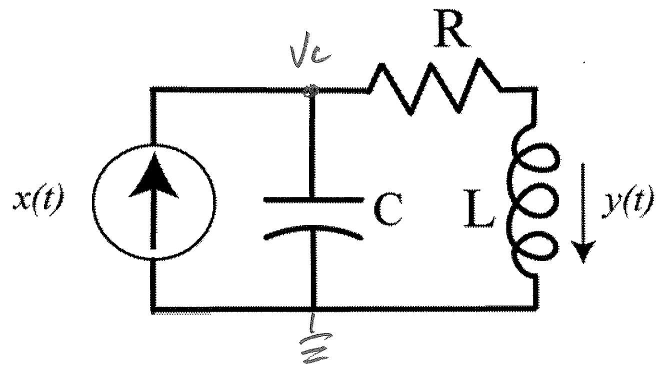
6) (16 points) For the second order circuit below,



Derive the governing second order differential equation for the output $y(t)$ and input $x(t)$. You do not need to put the equation in standard form.

$$x(t) = C\frac{dV_c(t)}{dt} + y(t)$$

$$V_c(t) - y(t)R - L\dot{y}(t) = 0$$

$$V_c(t) = y(t)R + L\dot{y}(t)$$

$$x(t) = C\left[\dot{y}(t)R + L\ddot{y}(t)\right] + y(t)$$

$$\boxed{x(t) = LC\ddot{y}(t) + RC\dot{y}(t) + y(t)}$$

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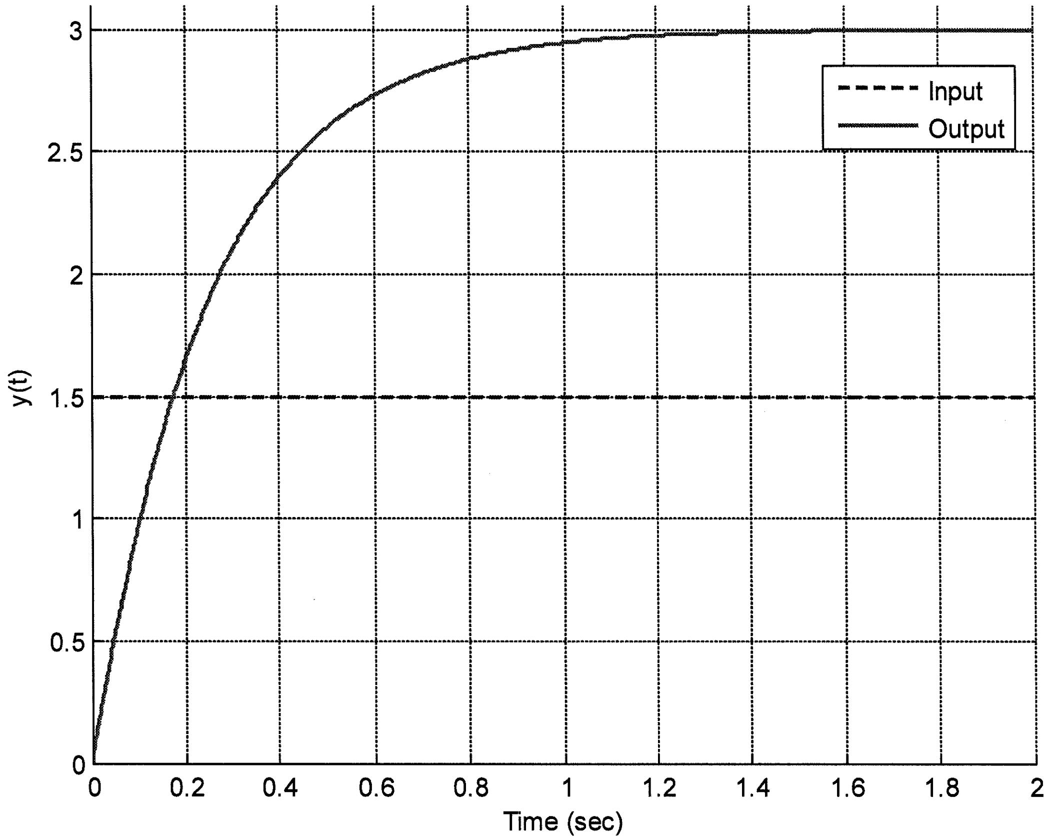
Problems 7-10, 4 points each (16 points)

For problems 7 and 8, refer to the following graph showing the input and output of a first order system. For this system the input is a step of amplitude 1.5.



7) What is the static gain? $K(1.5) = 3$ $\boxed{K = 2}$

8) What is the time constant? $T_s = 4\tau = 1\,sec$ $\boxed{\tau = 0.25\,sec}$

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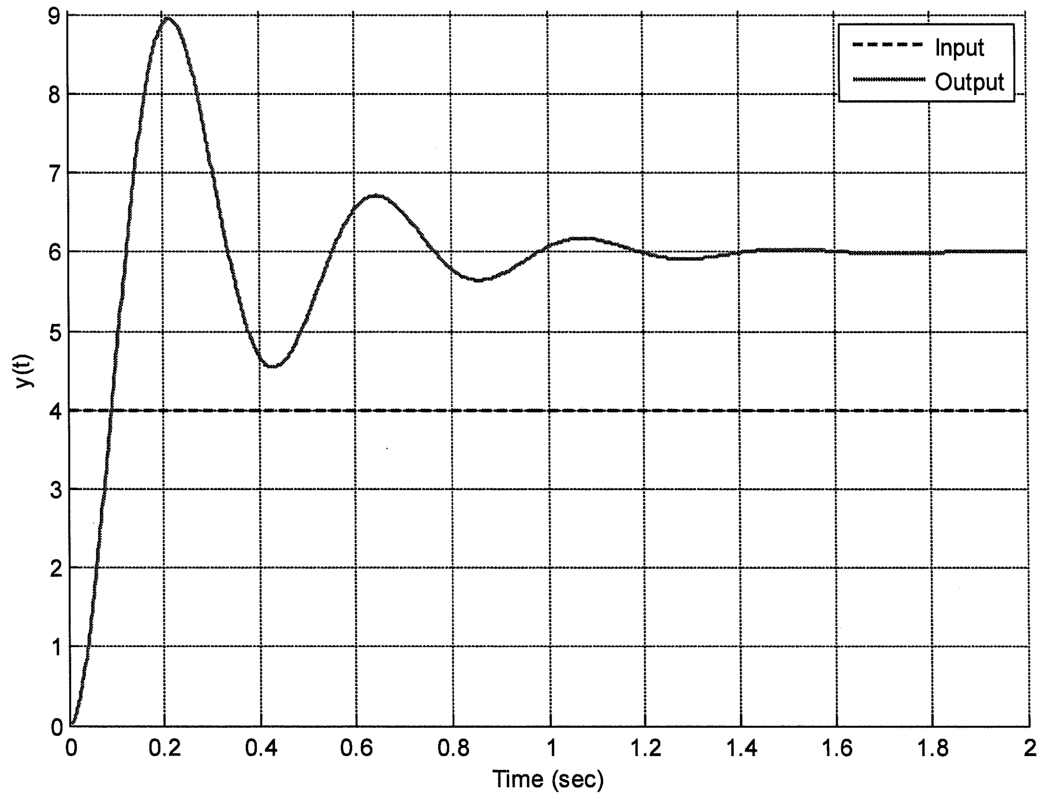
For problems 9 and 10, refer to the following graph showing the input and output of a second order system. For this system the input is a step of amplitude 4.



9) What is the static gain of the system? $K(4) = 6 \quad K = \frac{6}{4} \Rightarrow \boxed{1.5 = K}$

10) What is the percent overshoot? $\frac{9-6}{6} \times 100\% = \boxed{50\% = PO}$