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# ECE-205

# Exam 1

# Fall 2011

**Calculators can only be used for simple calculations. Solving integrals, differential equations, systems of equations, etc. does not count as a simple calculation.**

**You must show your work to receive credit.**

| | | | |
|---|---|---|---|
| **Problem 1** | ________/20 | 100 | 5 |
| **Problem 2** | ________/30 | 90-99 | 14 |
| **Problem 3** | ________/18 | 80-89 | 8 |
| **Problem 4-11** | ________/32 | 70-79 | 5 |
| | | 60-69 | 5 |
| | | <60 | 2 |
| | | | 39 |

**Total** ____________

median = 89

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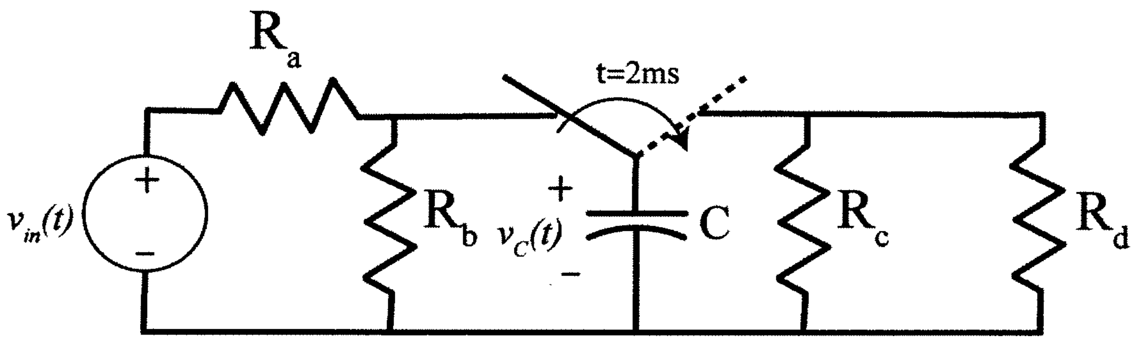
1) **(20 points)** Consider the circuit shown in the figure below:




a) Determine an expression for the time constant of the circuit for the time when the capacitor is charging ($t < 0.002$ seconds) and discharging ($t > 0.002$ seconds) in terms of the parameters $C, R_a, R_b, R_c$ and $R_d$. (Do not use numbers).

b) Determine an expression for the static gain of the circuit for $t < 0.002$ seconds in terms of the parameters $C, R_a, R_b, R_c$ and $R_d$. (Do not use numbers).

**c)** For $R_a = 1k\Omega, R_b = 1k\Omega, R_c = 2k\Omega, R_d = 10k\Omega, C = 2\mu F, V_{in} = 6V$ accurately *sketch* the voltage across the capacitor from 0 to 12 ms. You need to specifically lablel the voltages at $t = 0.002$ seconds and $t = 0.012$ seconds. You need to primarily determine the appropriate time constants and steady state values, and use the following table as a guide.

| Time ($t$) | $t/\tau$ | $y(t)$ |
|---|---|---|
| 0 | 0 | $0\, y_{ss}$ |
| $\tau$ | 1 | $0.632\, y_{ss}$ |
| $2\tau$ | 2 | $0.865\, y_{ss}$ |
| $3\tau$ | 3 | $0.950\, y_{ss}$ |
| $4\tau$ | 4 | $0.982\, y_{ss}$ |
| $5\tau$ | 5 | $0.993\, y_{ss}$ |

a) $\tau_{charging} = C \dfrac{R_a R_b}{R_a + R_b} = 1ms$

$\tau_{discharging} = C \dfrac{R_c R_d}{R_c + R_d} = 3.33ms$

b) $K = \dfrac{R_b}{R_a + R_b} = 0.5$

at $t = 0.002 = 2\tau$ $V_c = (6)(\frac{1}{2})(0.865) = 2.6$

$y(\infty)$ $K$ $(0.865\, y_{ss})$

at $t = 0.012 = 3\tau$

$V_c = (2.6)(0.05) = 0.13$

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**2) (30 points)** For the following three differential equations, assume the input is $x(t) = 4u(t)$ (the input is equal to four for time greater than zero), and the initial conditions are $y(0) = \dot{y}(0) = 0$

Determine the solution to each of the following differential equations and put your final answer in a box. Be sure to use the initial conditions to solve for all unknowns. You must show all your work to receive credit.

**a)** $\ddot{y}(t) + 2\dot{y}(t) + y(t) = 3x(t)$ $\quad y_F = 3\cdot 4 = 12 \quad r^2 + 2r + 1 = 0 \quad (r+1)^2 = 0$

$y(t) = c_1 e^{-t} + c_2 t e^{-t} + 12 \qquad y(0) = c_1 + 0 + 12 = 0$

$\dot{y}(t) = -c_1 e^{-t} - c_2 t e^{-t} + c_2 e^{-t} \qquad \dot{y}(0) = -c_1 + c_2 = 0$

adding $c_2 + 12 = 0 \quad \boxed{c_2 = -12}$

so $c_1 = -12$

$$\boxed{y(t) = -12e^{-t} - 12te^{-t} + 12}$$

**b)** $\ddot{y}(t) + 5\dot{y}(t) + 4y(t) = 2x(t)$ $\quad 4y_F = 2\cdot 4 \quad y_F = 2 \quad r^2 + 5r + 4 = 0 \quad (r+4)(r+1) = 0$

$y(t) = c_1 e^{-t} + c_2 e^{-4t} + 2 \qquad y(0) = c_1 + c_2 + 2 = 0$

$\dot{y}(t) = -c_1 e^{-t} - 4c_2 e^{-4t} \qquad \dot{y}(0) = -c_1 - 4c_2 = 0$

adding $-3c_2 + 2 = 0$

$c_2 = 2/3$

$c_1 = -8/3$

$$\boxed{y(t) = -\frac{8}{3}e^{-t} + \frac{2}{3}e^{-4t} + 2}$$

**c)** $\ddot{y}(t) + 4\dot{y}(t) + 16y(t) = 4x(t)$ $\quad 16y_F = 4\cdot 4 = 16 \quad y_F = 1 \quad r^2 + 4r + 16 = 0$

$$r = \frac{-4 \pm \sqrt{16-64}}{2} = -2 \pm 2\sqrt{3}$$

$y(t) = ce^{-2t}\sin(2\sqrt{3}t + \phi) + 1 \qquad y(0) = 0 = c\sin(\phi) + 1 \qquad c = \frac{-1}{\sin(\phi)}$

$\dot{y}(t) = -2ce^{-t}\sin(2\sqrt{3}t + \phi) + 2\sqrt{3}ce^{-2t}\cos(2\sqrt{3}t + \phi)$

$\dot{y}(0) = -2\sin(\phi) + 2\sqrt{3}\cos(\phi) = 0$

$\tan(\phi) = \sqrt{3} \quad \phi = 60° \qquad c = \frac{-1}{\sin(\phi)} = -1.1547$

$$\boxed{y(t) = 1 - 1.1547\, e^{-2t}\sin(2\sqrt{3}t + 60°)}$$

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**3) (18 points)** The form of the under damped ( $0<\zeta<1$ ) solution to the second order differential equation

$$\ddot{y}(t)+2\zeta\omega_n\dot{y}(t)+\omega_n^2 y(t)=\mathrm{K}\ \omega_n^2 x(t)$$

for a step input $x(t)=Au(t)$ is

$$y(t)=KA+ce^{-\zeta\omega_n t}\sin(\omega_d t+\phi)$$

where $c$ and $\phi$ are constants to be determined and the damped frequency $\omega_d=\omega_n\sqrt{1-\zeta^2}$

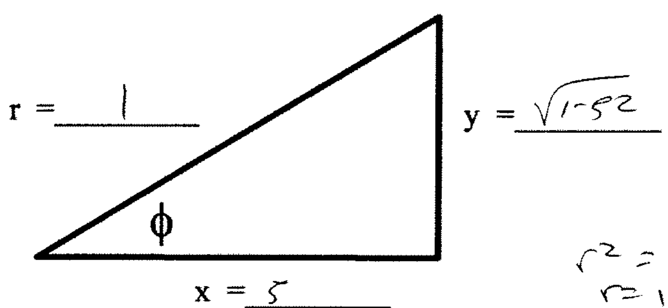
a) Using the initial condition $\dot{y}(0)=0$ show that $\tan(\phi)=\dfrac{\sqrt{1-\zeta^2}}{\zeta}$

$$\dot{y}(t)=-\zeta\omega_n ce^{-\zeta\omega_n t}\sin(\omega_d t+\phi)+\omega_d ce^{-\zeta\omega_n t}\cos(\omega_d t+\phi)$$

$$\dot{y}(0)=0=-\zeta\omega_n\sin(\phi)+\omega_d\cos(\phi)$$

$$\tan(\phi)=\frac{\omega_d}{\zeta\omega_n}=\frac{\omega_n\sqrt{1-\zeta^2}}{\omega_n\zeta}\quad\boxed{\tan(\phi)=\frac{\sqrt{1-\zeta^2}}{\zeta}}$$

b) We can express the relationship in part a using the following triangle. Fill in the blanks and then use this triangle determine an expression for $\sin(\phi)$.




$$r^2=x^2+y^2=\zeta^2+1-\zeta^2=1$$
$$r=1$$

$$\sin(\phi)=\frac{\sqrt{1-\zeta^2}}{1}=\boxed{\sqrt{1-\zeta^2}=\sin(\phi)}$$

c) Use your answer to part b, and the initial condition $y(0)=0$ to determine the remaining unknown constant, and write out the complete solution for $y(t)$.

$$y(0)=0=c\sin(\phi)+KA\quad c=\frac{-KA}{\sin(\phi)}=\frac{-KA}{\sqrt{1-\zeta^2}}$$

$$\boxed{y(t)=KA-\frac{KA}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin(\omega_d t+\phi)}$$

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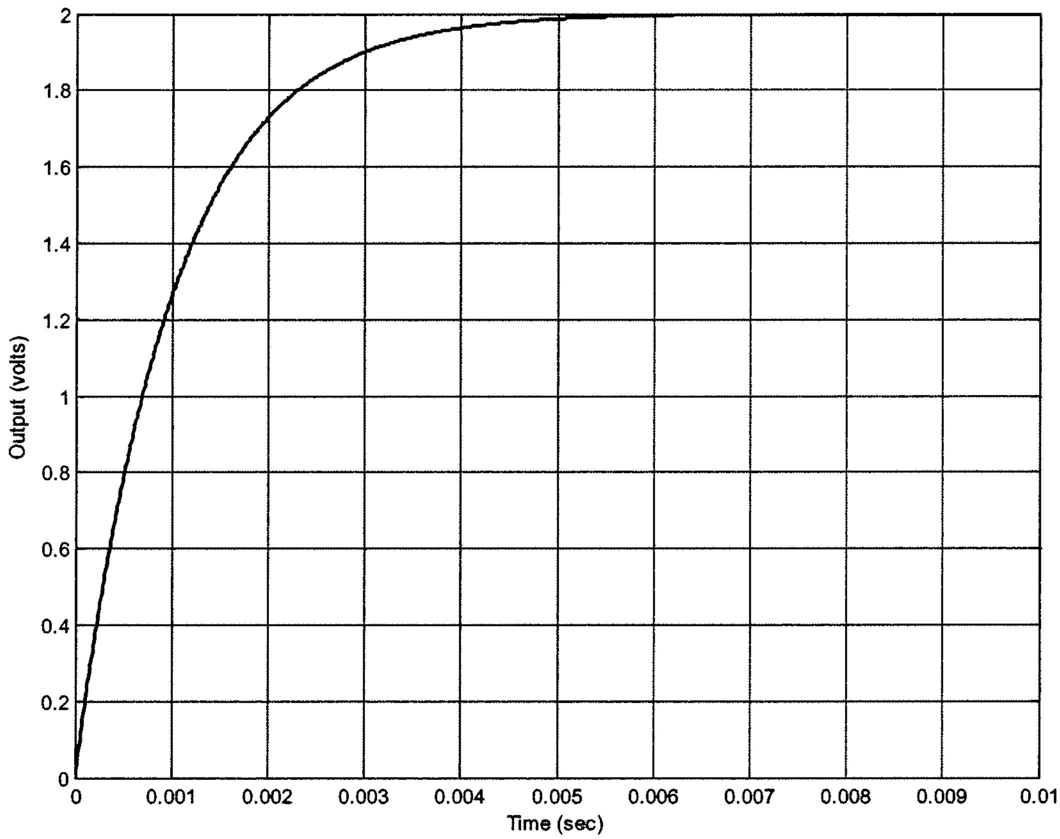
**Problems 4-11, 4 points each, no partial credit (32 points)**

**4)** Consider the response of a first order circuit shown below. Of the following, which is the best estimate of the time constant?

a) 0.001 sec b) 0.002 sec c) 0.003 sec d) 0.004 sec e) 0.005 sec f) 0.006 sec




at $4\tau$ $V = (0.98)(2) = 1.96$ $4\tau \approx 0.004$

$\tau \approx 0.001$

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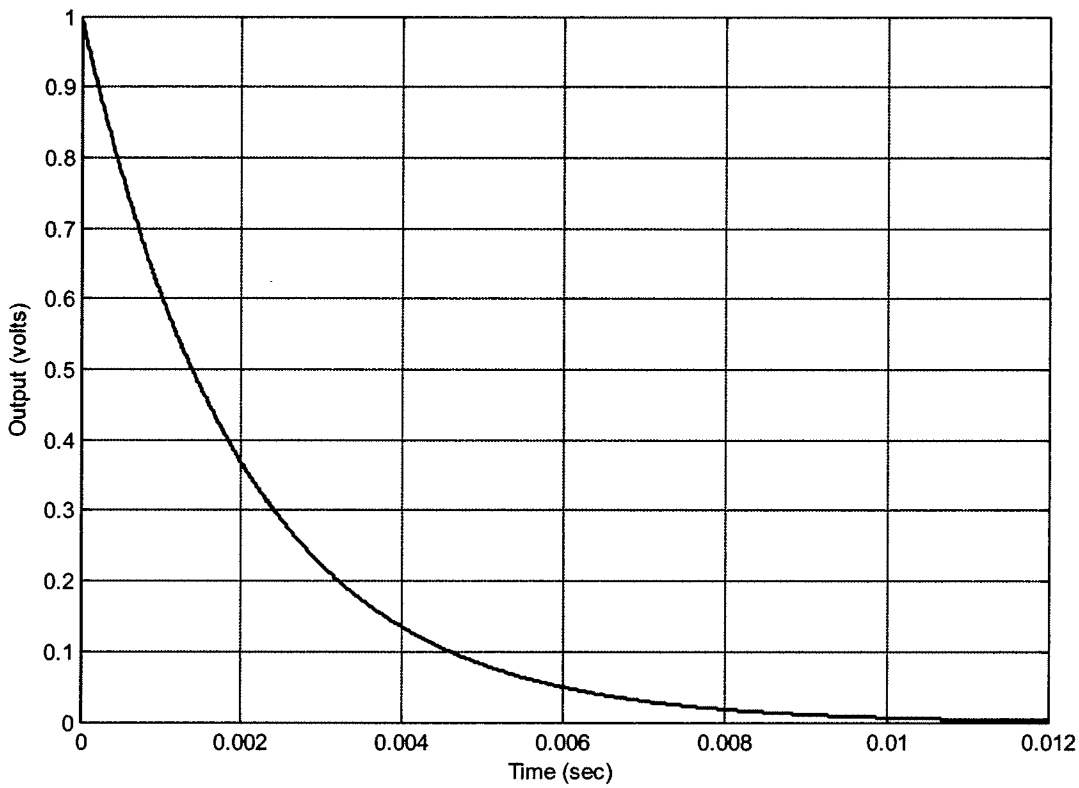
**5)** Consider the response of a first order circuit shown below. Of the following, which is the best estimate of the time constant?

a) 0.001 sec b) 0.002 sec c) 0.003 sec d) 0.004 sec e) 0.005 sec f) 0.006 sec




at $4\tau$ $V = 0.02$ $4\tau \approx 0.008$ $\tau = 0.002$

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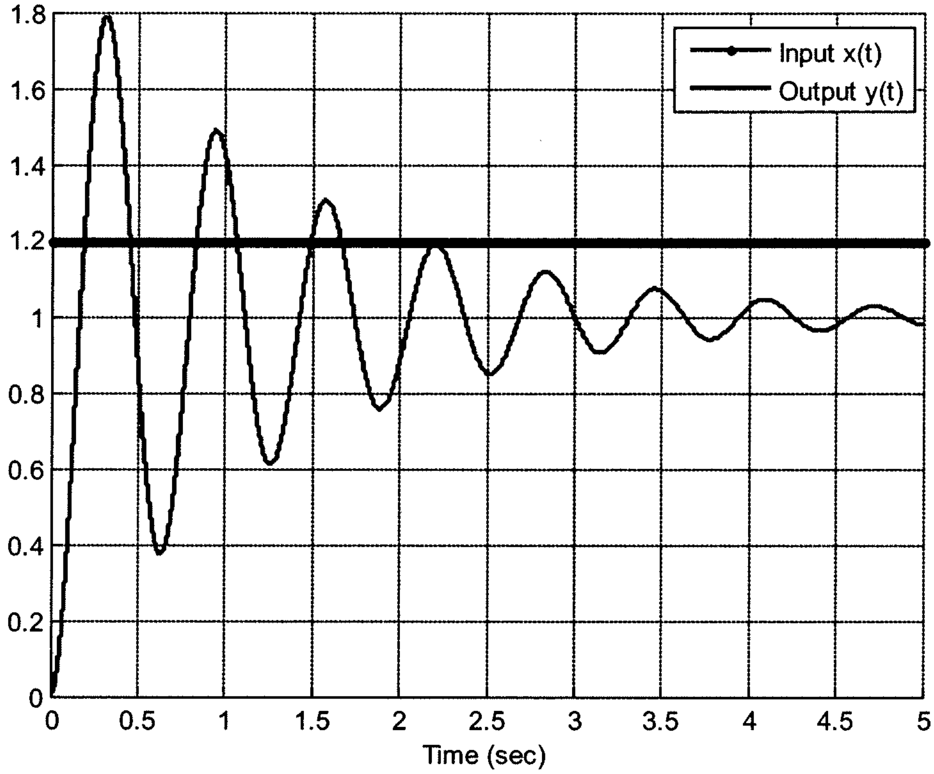
Problems 6 and 7 refer the following graph showing the response of a second order system to a step input.




**6)** The percent overshoot for this system is best estimated as

$PO = \frac{1.8-1}{1} \times 100\% = 80\%$

a) 180% b) 150 % c) 100% d) 80 % e) 60% f) 50%

**7)** The static gain for this system is best estimated as

a) 1.8 b) 1.2 c) 1.00 d) 0.83 e) 0.5

$KA = y(\infty)$

$K(1.2) = 1$

$K = 1/1.2 = 0.83$

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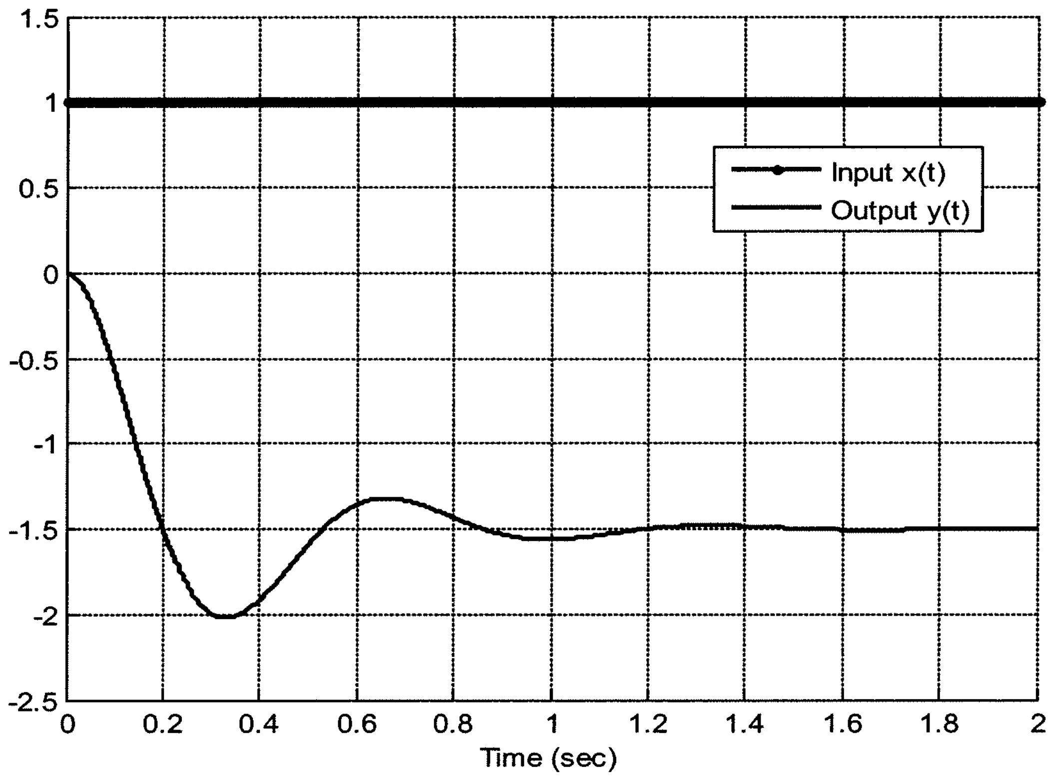
Problems 8-9 refer the following graph showing the response of a second order system to a step input.




**8)** The percent overshoot for this system is best estimated as

a) 300% b) -300 % c) 200% d) -200 % e) 33% f) -33%

$$P.O. = \frac{-2-(-1.5)}{-1.5} \times 100\%$$

$$= \frac{1}{3} \times 100\%$$

$$= 33\%$$

**9)** The static gain for this system is best estimated as

a) -3 b) 3 c) 2.5 d) -2.5 e) 1.5 f) -1.5

$KA = y(\infty)$

$K(1) = -1.5$

$K = -1.5$

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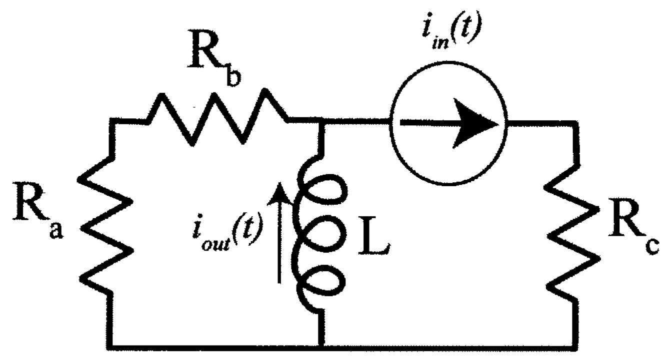
Problems 10 and 11 refer to the following circuit




**10)** The Thevenin resistance seen from the ports of the inductor is

a) $R_{th} = R_c \,||\, (R_a + R_b)$ b) $R_{th} = R_c$ c) $R_{th} = R_a + R_b$ d) $R_{th} = R_a + R_b + R_c$ e) none of these

**11)** The static gain for the system is

a) $K = 1$ b) $K = \dfrac{R_a + R_b}{R_a + R_b + R_c}$ c) $K = \dfrac{R_c}{R_a + R_b + R_c}$ d) $K = \dfrac{R_c}{R_a + R_b}$ e) none of these

$R_{th} = R_a + R_b$ (current source an open circuit)

$i_{out}(\infty) = i_n(\infty) \Rightarrow K = 1$