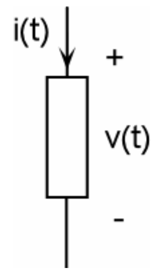
Power in AC steady-state (power in phasor circuits)

For a circuit with sinusoidal sources, all voltages and currents (in steady-state) have the same form. All are cosines with various amplitudes and phases.

$$i(t) = I \cos(\omega t + \theta_i)$$

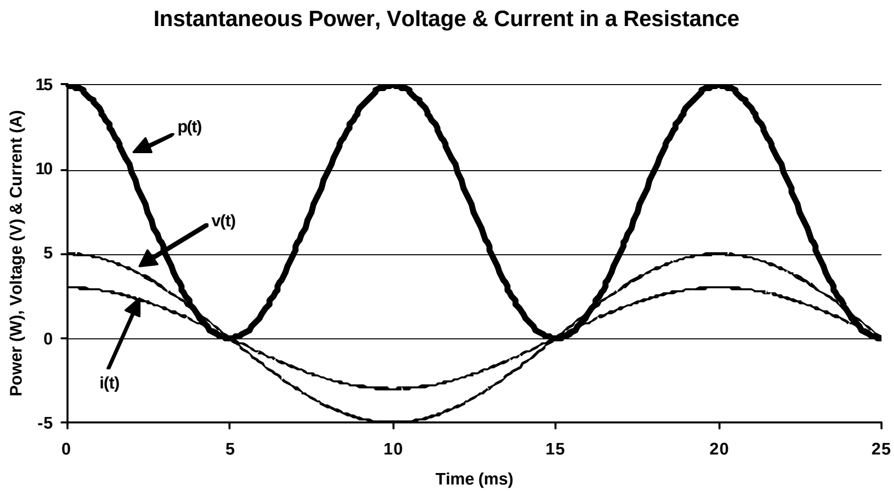
$$v(t) = V \cos(\omega t + \theta_v)$$



The power absorbed by the above element at any instant of time is, as always, the product of the voltage and current when the voltage and current are labeled with the passive sign convention (PSC).

$$p(t) = v(t)\, i(t) = V \cos(\omega t + \theta_v)\, I \cos(\omega t + \theta_i)$$

Consider the voltage and current associated with a 5/3Ω resistance. The amplitude of the voltage is 5V, the amplitude of the current is 3A, f = 50Hz (ω=314 rad/s).



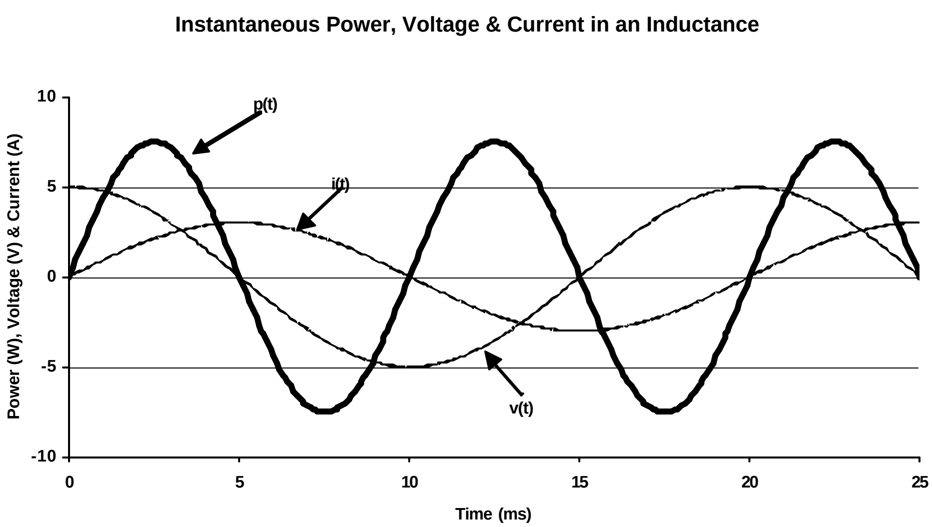
The voltage and current are in phase so their product is always positive. As a consequence, the power dissipated is never negative—power is always absorbed by the resistance, resistance never supplies power. The average power dissipated is half the maximum value.

Consider now an inductor. The current lags the voltage by 90°. Taking the voltage as phase reference:

$$v(t) = V \cos \omega t$$
$$i(t) = I \cos (\omega t - 90°)$$

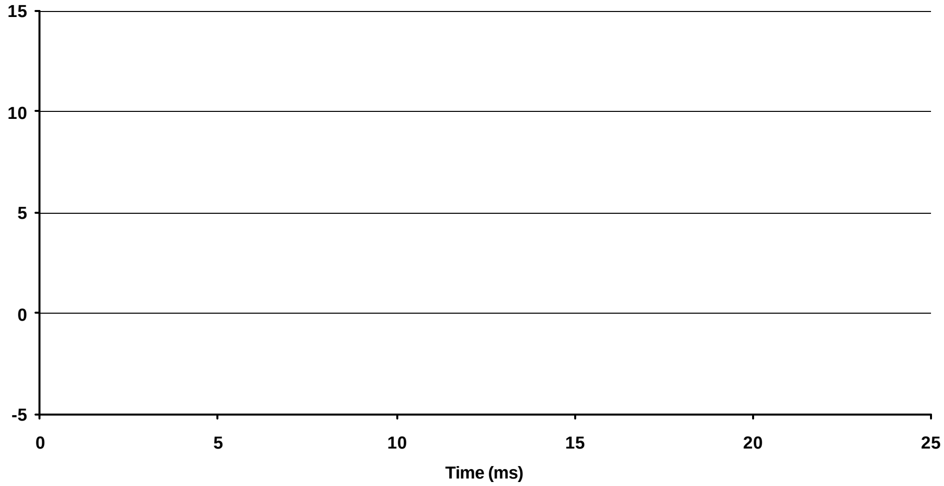
$$p(t) = v(t)\, i(t) = V \cos \omega t \; I \cos (\omega t - 90°)$$

With v(t) and i(t) having the same amplitudes and frequency as previously, the power absorbed by the inductance will vary with time as shown below



Notice that the average power dissipated is zero. This is because the inductor stores energy (takes it from the source) in the 1st half cycle shown and then returns all of the stored energy to the source in the 2nd half cycle. As in the resistor case, the power oscillates at twice the line frequency. The maximum amplitude now is $\pm V_m I_m/2$

How about a capacitance?



Consider now a more general impedance, one that has a phase, θ, between zero and $\pm 90^\circ$.

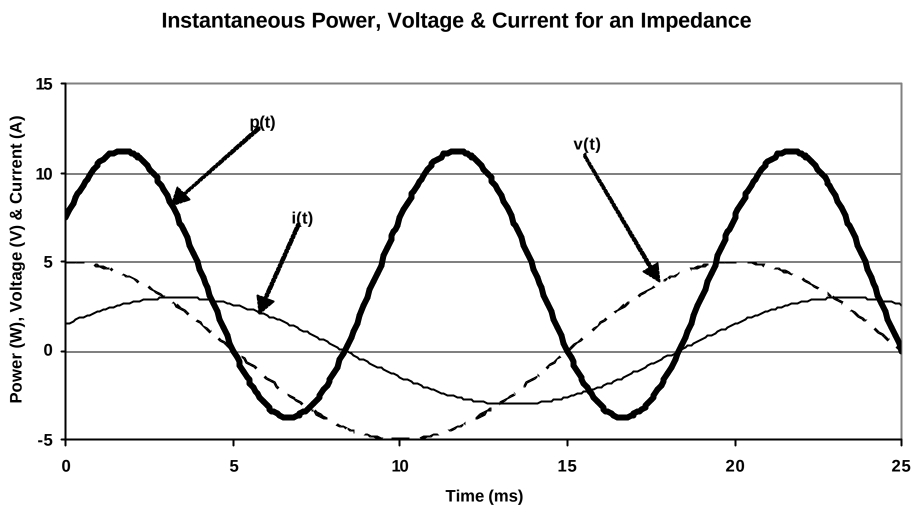
Taking voltage as reference, in sinusoidal steady-state, the current and votlage are:

$$v(t) = V \cos \omega t$$

$$i(t) = I \cos (\omega t - \theta)$$

$$p(t) = v(t)\, i(t) = V \cos \omega t \; I \cos (\omega t - \theta)$$

Taking V and I to have the same amplitudes and frequency as previously and $\theta = -60^0$ ($\boldsymbol{Z} = \frac{5}{3}\angle 60^\circ \; \Omega$), then power will vary with time as shown below



Power oscillates at twice the line frequency and passes through zero if either voltage or current is zero and, in general, can be off-set from the time axis. The average power dissipated over one cycle is the mean of the maximum (~11.25 W) and the minimum (~ -3.75 W), which is 3.75 W in the example.

In cases where p(t) varies many times each second, the quantity of interest is often not the value of p(t) at each instant of time, but rather the average value of power, P_{av}. Since in this case p(t) is periodic, the average can be found by finding the average over a period.

$$P_{av} = \frac{1}{T}\int_T V \cos(wt + q_v)\, I \cos(wt + q_i)\, dt$$

where T is the period.

Using the identity

$$\cos a \cos b = \frac{1}{2}\cos(a + b) + \frac{1}{2}\cos(a - b)$$

with $a = wt + q_v$ and $b = wt + q_i$

$P_{av} = ½ V I \cos(\theta_v - \theta_i)$ can also be expressed in terms of phasors, without integration.

v(t) = V cos (ωt + θ_v) $\rightarrow$ $\boldsymbol{V} = V\angle\theta_v$ *(phasor in peak, **not** RMS)*

i(t) = I cos (ωt + θ_i) $\rightarrow$ $\boldsymbol{I} = I\angle\theta_I$ *(phasor in peak, **not** RMS)*

$P_{av} = \text{Re}(½\, \boldsymbol{V}\boldsymbol{I}^*) = \text{Re}[½ V I \cos(\theta_v - \theta_i) + j\, ½ V I \sin(\theta_v - \theta_i)]$

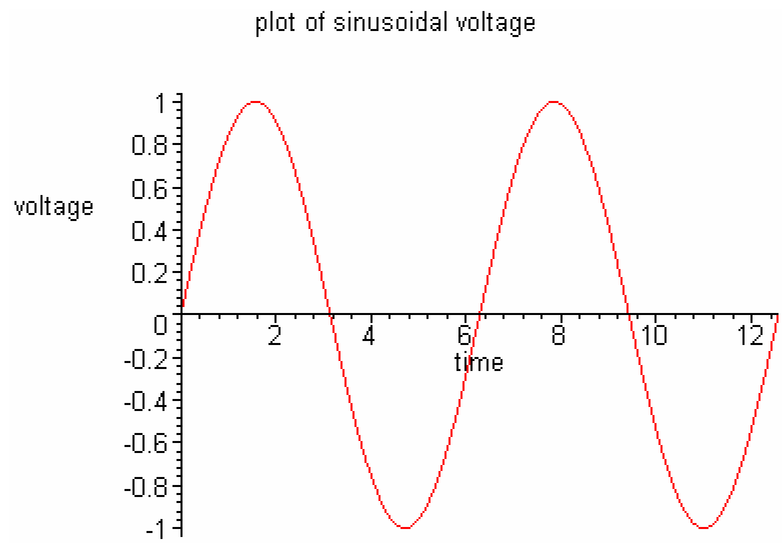
$P_{av} = ½ V I \cos(\theta_v - \theta_i)$ *Note*: $\boldsymbol{I}^*$ denotes the complex conjugate of $\boldsymbol{I}$.

This is an important result since it allows average powers to be readily calculated using phasor analysis.

The real part of the complex quantity, ½VI^* *(phasors **not** in RMS)* is the average power dissipated in the load. Dissipated power is electromagnetic energy that is changed to another form—thermal energy or mechanical energy, for example.

The imaginary part is also meaningful. The imaginary part of the quantity ½VI^* *(again, with the factor of ½, phasors are **not** in RMS)* is related to the flow of energy that is not being dissipated. It is related to energy that is swapped between the in the load and the source. Or, as we shall see, stored at different points in the power cycle by different elements at the load.

A physical look at energy storage



Consider the voltage waveform shown above, and suppose that the voltage is across a capacitance. From previous work, the energy stored by a capacitance is

$$E(t) = ½\, C\, v^2(t)$$

Start at t=0. Here v(t) = 0 and so E(t) = 0.

For t > 0, v(t) begins to grow until v(t) reaches a maximum at something less that 2 seconds ($\pi/2$ to be exact). During this time interval energy is being delivered to the capacitance from the rest of the circuit—the capacitance is building up its electric field.

Then, for time greater than π /2 seconds, v(t) grows smaller until it reaches zero for t = π. For this time, the energy stored by the capacitance is again zero so that for the time interval $\pi /2 < t < \pi$ the capacitance gave back the energy that is had stored during $\pi < t < \pi /2$.

The same process repeats itself again and again.

For the time interval $\pi < t < 3\pi/2$, energy is delivered to the capacitance, which then gives it back in the time interval $3\pi/2 < t < 2\pi$.

This process then, storing energy and giving it back, occurs twice in each period of the voltage waveform. Thus, in a 60 Hz power system, this process occurs 120 times each second.

Doing the same thing for an inductance which stores energy in the form of a magnetic field would result in.

$$E(t) = \frac{1}{2} L\, i^2(t)$$

Phasor Analysis in Power

Now, getting back to the main story. The imaginary part of $\frac{1}{2}\boldsymbol{VI}^*$ is related to this flow of energy to and from energy storage devices in circuits.

We can even tell what type of device dominates energy storage by the sign of the imaginary part—inductance dominates if it is positive and capacitance dominates if negative.

Since both parts of $\frac{1}{2}\boldsymbol{VI}^*$, real and imaginary, turn out to be useful, the parts of this quantity have been given names and so here we must define several terms.

The quantity $\frac{1}{2}\boldsymbol{VI}^*$ is called the *complex power* $\boldsymbol{S}$.

$\boldsymbol{S} = \frac{1}{2}\boldsymbol{VI}^*$ [$\boldsymbol{S}$ is complex power with units of volt-ampere (VA)]

The complex power can be split into real and imaginary parts.

$$\boldsymbol{S} = P_{av} + jQ$$

P_{av} is *average power* with units of watts (W)

Q is *reactive power* with units of volt-ampere reactive (VAR)

The complex power can also be express in polar form.

$$\boldsymbol{S} = ½\boldsymbol{VI}^* = ½\, V\, I\, \angle(\theta_v - \theta_i) = S\angle\theta_s$$

S, the magnitude of ***S***, is called the *apparent power* with units of volt-ampere (VA).

For cases where P_{av} is positive, for loads that do dissipate average power rather than supplying average power, ***S*** occupies the 1st and 4th quadrants in the complex plane so that $-90° < \theta_s < 90°$. In this case, $\cos\theta_s = \cos(\theta_v - \theta_i)$ is called the power factor (pf).

The pf is further characterized as to whether θ_s is positive or not. If θ_s is negative, then $\theta_i > \theta_v$ so that the current *leads* the voltage in phase. If $\theta_s < 0$, we say the power factor $\cos\theta_s$ is leading. *Note*: for a leading pf, Q is negative.

If θ_s is positive, then $\theta_i < \theta_v$ so that the current *lags* the voltage in phase. If $\theta_s > 0$, we say the power factor $\cos\theta_s$ is lagging. *Note*: for a lagging pf, Q is positive.

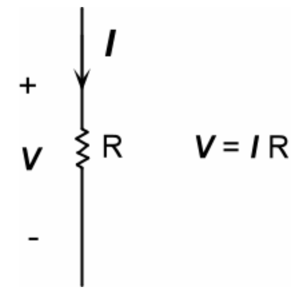
Let's explore these relationships by looking at specific examples.
Let's start with the complex power absorbed by a resistance

Resistance

$\boldsymbol{S} = \frac{1}{2}\boldsymbol{VI^*} = \frac{1}{2}\boldsymbol{I}\,R\,\boldsymbol{I^*} = \frac{1}{2}R\,I\angle\theta_i\,(I\angle{-\theta_i})$

$\quad = \frac{1}{2}R\,I^2\angle(\theta_i - \theta_i) = \frac{1}{2}R\,I^2\angle 0°$

which is a real number.



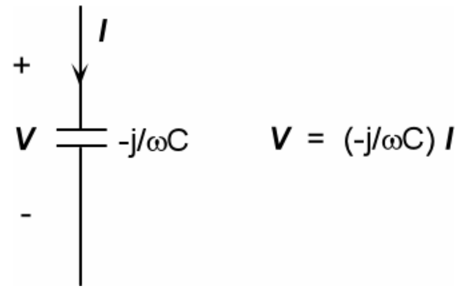
The complex power absorbed by a resistance is all real, which makes sense if we consider that the imaginary part of ***S*** is associated with energy storage that the resistance does not do.

For a resistance,

$$\boldsymbol{S} = \frac{1}{2}\boldsymbol{VI^*} = \frac{1}{2}V\,I = \frac{1}{2}I^2 R = \frac{1}{2}(V^2 / R)$$

where I and V are the amplitude of i(t) and v(t).

Capacitance



$$\boldsymbol{S} = \tfrac{1}{2}\, \boldsymbol{VI^*} = \tfrac{1}{2}\,(-j/\omega C)\, \boldsymbol{I}\ \ \boldsymbol{I^*} = \tfrac{1}{2}\,(I/\omega C)\ I \angle(\theta_i - 90°)\ \ I\angle{-\theta_i}$$
$$= \tfrac{1}{2}\, I^2/\omega C \angle(-90°) = \tfrac{1}{2}\,(-j\ I^2/\omega C)$$

which is an imaginary number.

The complex power absorbed by a capacitance is purely imaginary which makes sense if we consider that the real part of ***S*** is associated with energy dissipation and that the imaginary part of ***S*** is associated with energy storage.

Since the capacitance does not dissipate energy but only store it, the complex power associated with the capacitance must be purely imaginary. In power engineering, the imaginary component of complex power is called reactive since it is associated with the reactive elements, L and C.

For a capacitance,

$$\boldsymbol{S} = \tfrac{1}{2}\, \boldsymbol{VI^*} = \tfrac{1}{2}\, {-j}\ V\ I\ \ = \tfrac{1}{2}\,(-j\ I^2/\omega C)\ \ =\ \ \tfrac{1}{2}\,(-j\omega C V^2)$$

where I and V are the amplitude of i(t) and v(t).

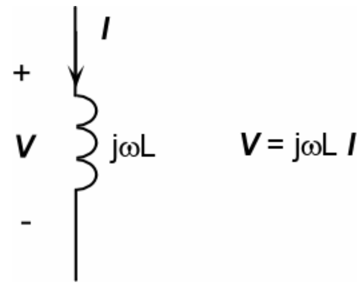
Note: The reactive power absorbed by the capacitance is negative and the current's phase—that is, the phase of i(t)—leads that of v(t).

Be careful here. Absorbed reactive power is describes stored energy being swapped between the element and other parts of the circuit. Reactive power does not refer to average power being absorbed. Work is not done with reactive power.

Inductance

$\mathbf{S} = \frac{1}{2}\,\mathbf{VI^*} = \frac{1}{2}\, j\omega L\, \mathbf{I}\ \mathbf{I^*} = \frac{1}{2}\,\omega L\ I\ \angle(\theta_i + 90°)\ \ I\angle\text{-}\theta_i$
$\quad = \frac{1}{2}\,\omega L\ I^2\angle\text{-}90° = \frac{1}{2}\, j\ \omega L\ I^2$

which is an imaginary number.



The complex power absorbed by an inductance is purely imaginary which makes sense if we consider that the real part of S is associated with energy dissipation and that the imaginary part of S is associated with energy storage.

Since the inductance does not dissipate energy but only store it, the complex power associated with the inductance is purely imaginary, purely reactive.

For an inductance,

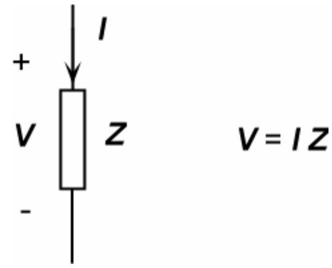
$$\mathbf{S} = \frac{1}{2}\,\mathbf{VI^*} = \frac{1}{2}\, j\ V\ I\ \ = \frac{1}{2}\,(j\omega L\ I^2)\ =\ \frac{1}{2}\,(j\ V^2 / \omega L)$$

where I and V are the amplitude of i(t) and v(t).

Note: The reactive power "absorbed" by the inductance is positive and the current's phase—that is, the phase of i(t)—lags that of v(t).

In particular, notice how the reactive power absorbed by the inductance is 180° out-of-phase with respect to the reactive power absorbed by the capacitance.   This merely means that when the inductance is ready to store energy, the capacitance is ready to give up stored energy. And when the capacitance is ready to store energy, the inductance is ready to give up stored energy.

General Impedance



$\boldsymbol{S}$ = 1/2 $\boldsymbol{VI^*}$ = 1/2 $\boldsymbol{I\,Z\;\;I^*}$ = 1/2 Z I $\angle(\theta_i + \theta_z)$ I$\angle\text{-}\theta_i$
= 1/2 Z $I^2\angle\theta_z$

which is a complex number.

In general the complex power absorbed by an impedance has nonzero real and nonzero imaginary parts. That is, in general both energy dissipation and energy storage are involved.

If capacitance dominates energy storage,
- θ_z and Q, the reactive power, will be negative
- current will lead the voltage in phase. This is why the term "leading" or "lead" is used for the power factor, cos θ_z.

If inductance dominates energy storage,
- θ_z and Q, the reactive power, will be positive
- current will lag the voltage in phase. This is why the term "lagging" or "lag" is used for the power factor, cos θ_z.

For a general impedance,

$$\boldsymbol{S} = ½\,\boldsymbol{VI^*} = ½\, V\, I \angle\theta_z \quad = ½\, I^2\, Z \angle\theta_z \quad = ½\,(V^2 / Z)\angle\theta_z$$

where I and V are the amplitude of i(t) and v(t) and Z is the magnitude of the impedance.

Note: It is assumed here that the real part of ***Z*** is positive. This causes the real part of ***S*** (aborbed complex power) to be positive. That is, the load ***Z*** is assumed not to be a source for average power.

Using RMS values of current and voltage

When performing sinusoidal steady-state analysis in circuits, especially in power, the convention is to use RMS values for voltages and currents.

For sinusoids, the root-mean-square (RMS) value is the amplitude divided by the square root of two. In the study of power systems, RMS values for the current and voltages are usually used rather that peak values.

When RMS are used, the ½ in the power relation is eliminated.

$$\boldsymbol{S} = \frac{1}{2}\,\boldsymbol{V}\,\boldsymbol{I^*} = \frac{\boldsymbol{V}}{\sqrt{2}}\,\frac{\boldsymbol{I^*}}{\sqrt{2}} = \boldsymbol{V}_{rms}\,\boldsymbol{I^*}_{rms}$$

Ohm's Law is unchanged (just divide both sides by √2).

When performing circuit analysis with sources given as RMS values, the current-voltage relationships are unchanged so that circuit analysis proceeds as usual. The voltages and currents obtained will be RMS values so that, when calculating powers, the ½ should be left out of the power relationship.

When moving into the time domain, remember to replace the √2.

That is, suppose we find a phasor current in RMS.

$$\boldsymbol{I} = I\angle\theta_i\ A_{rms}$$

The corresponding i(t) will then be

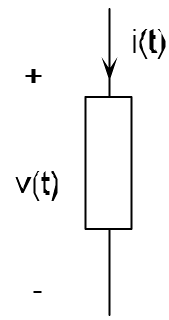
$$i(t) = \sqrt{2}\ I\ \cos(\omega t + \theta_i)\ \ A$$

Since the amplitude is √2 times the rms value.

General formulas, nomenclature, units, and conventions in power

Time domain

$v(t) = V_m \cos(\omega t + \theta_v) = V_p \cos(\omega t + \theta_v)$
$i(t) = I_m \cos(\omega t + \theta_i) = I_p \cos(\omega t + \theta_i)$



Phasors with magnitude in peak value

$\boldsymbol{V}_p = V_p \angle\, \theta_v$
$\boldsymbol{I}_p = I_p \angle\, \theta_i$

Phasors in rms

$\boldsymbol{V} = V\angle\theta_v = (V_p/\sqrt{2})\angle\theta_v$
$\boldsymbol{I} = I\angle\theta_i = (I_p/\sqrt{2})\angle\theta_i$

> ***If a phasor current or voltage is given, without other qualification, it's assumed to be in rms.***

Power

$\boldsymbol{S} = S\angle(\theta_v - \theta_i) \equiv S\angle\theta \equiv P_{av} + jQ$

$\boldsymbol{S}$ ~ complex power, in VA
S ~ apparent power, in VA
P_{av} ~ average power, in W
Q ~ reactive power, in VAR
$\theta \equiv \theta_v - \theta_i$

> ***If power is referred to, without any other qualification, it's assumed to be average power***

$\boldsymbol{S} = \boldsymbol{VI}^* = V\angle\theta_v \; I\angle{-\theta_i} = VI\angle(\theta_v - \theta_i) = VI\cos(\theta_v - \theta_i) + j\,VI\sin(\theta_v - \theta_i)$

$S = VI$
$P_{av} = VI\cos(\theta_v - \theta_i) = VI\cos\theta$
$Q = VI\sin(\theta_v - \theta_i) = VI\sin\theta$

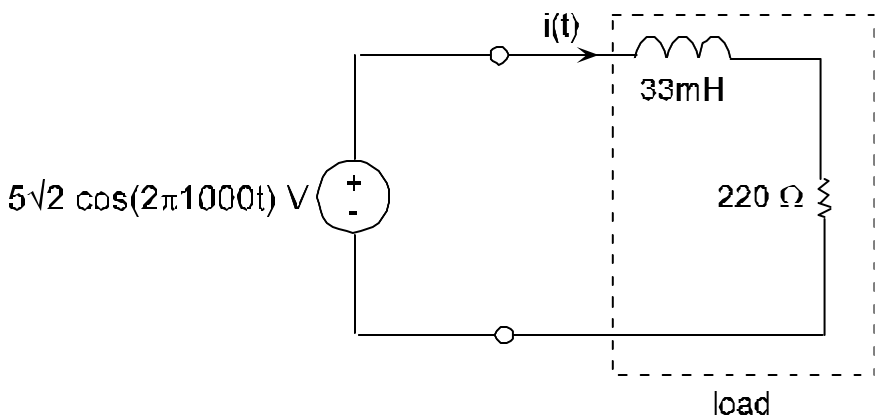
power factor = $\cos(\theta_v - \theta_i)$
lagging for $\theta_v > \theta_I$ ($\theta > 0$, $Q > 0$, inductive loads)
leading for $\theta_v < \theta_I$ ($\theta < 0$, $Q < 0$, capacitive loads)

Example 1

1. Find the phasor circuit for the circuit shown below (use rms value for source)

Use phasor analysis to find:

2. ***I*** and i(t)
3. ***S***, S, P, and Q delivered to RL load
4. load power factor
5. compare P with I^2R



Example 2

1. Find the phasor circuit for the circuit shown below (use the rms value for source)

Use phasor analysis to find:

2. ***I*** and i(t)
3. ***S***, S, P, and Q delivered to RC load
4. load power factor
5. compare P with V^2/R