Power in the Sinusoidal Steady-State

Power is the rate at which work is done by an electrical component. It tells us how much heat will be produced by an electric furnace, or how much light will be generated by a bank of fluorescent tubes. It is also important to know the power rating of a device so that we can ensure that its rating is not exceeded.

5.1 Instantaneous Power in the Sinusoidal Steady-State

When voltage and current vary with time their product is instantaneous power. We have seen that when an electrical component is modeled by lumped resistance, inductance and capacitance it is represented by figure 5.1 and the power dissipated is:

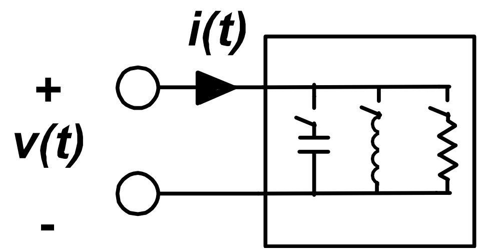

Figure 5.1 General circuit elements

$$\boxed{p(t) = \pm v(t)i(t)} \qquad \textbf{(5.1)}$$

Equation 5.1 refers to instantaneous power which is positive when power is consumed (load) and is negative when power is produced (source). If we assume that the element in figure 5.1 is a resistor then voltage and current are in phase:

$$v(t) = V_P \cos \omega t$$

$$i(t) = I_P \cos \omega t$$

and equation 5.1 becomes: $$p(t) = V_P \cos\omega t \, I_P \cos \omega t \qquad \textbf{(5.2)}$$

Equation 5.2 describes a sinusoid that oscillates at twice the line frequency, is fully off-set, and has a maximum value of $V_P I_P$. It can be evaluated by a spreadsheet as shown in figure 5.2, where: V_P=5V, I_P=3A, f = 50 Hz, and ω = 314 rad/s.

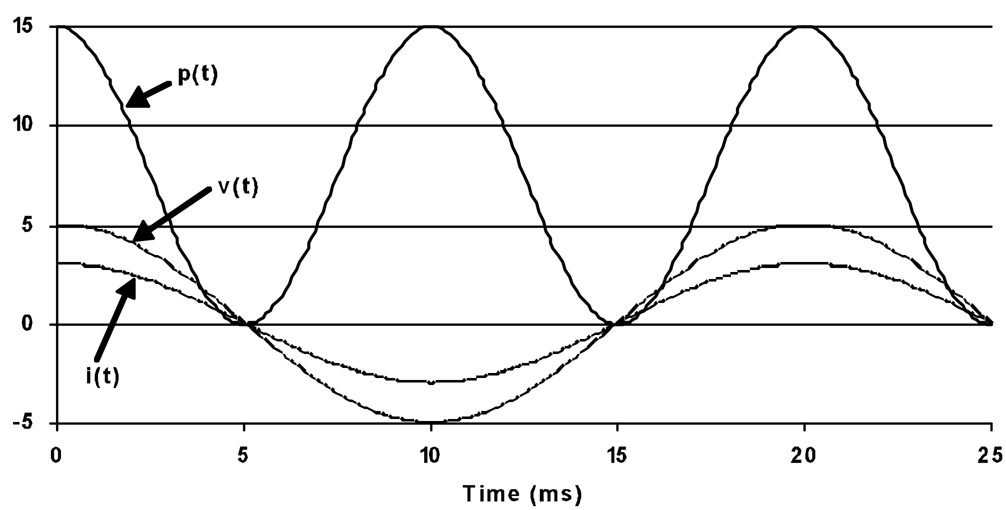

Figure 5.2 Instantaneous Power, Voltage & Current in a Resistor

It can be seen that the voltage and current are in phase and consequently their product is always positive. In other words, the power dissipated is never negative. This is because a resistor always takes power from the source and turns it into heat (conversely, if you heat a resistor, you do not produce any electricity). Since the resistor cannot store energy it can never return power to the source. Notice that the average power dissipated is half the maximum value.

Assume now that the element in figure 5.1 is an inductor in which case the current lags the voltage by 90^0. If voltage is taken as reference, then the steady-state voltage and current are:

$$v(t) = V_P \cos \omega t$$
$$i(t) = I_P \cos(\omega t - 90^0)$$

and equation 5.1 becomes:

$$p(t) = V_P \cos\omega t \, I_P \cos(\omega t - 90^0)$$

This can also be plotted by a spreadsheet. If v(t) and i(t) have the same amplitudes and frequency as previously, then power will vary with time as shown in figure 5.3:

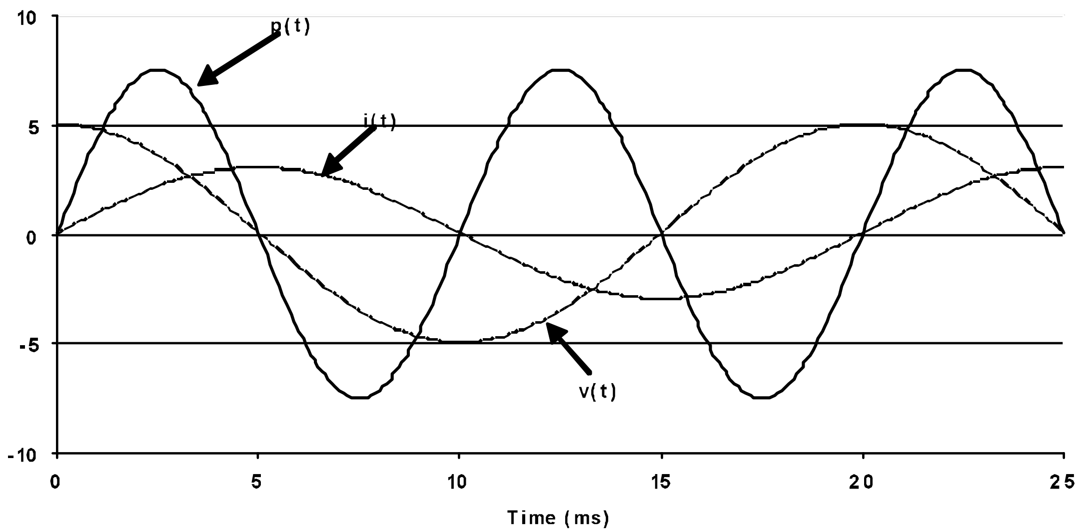

Figure 5.3 Instantaneous Power, Voltage & Current in an Inductor

Notice now that the average power dissipated is zero. This is because the inductor stores energy (takes it from the source) in the 1^{st} half cycle shown and then returns all of the stored energy to the source in the 2^{nd} half cycle. As in the resistor case, the power oscillates at twice the line frequency. The maximum amplitude now is: $\pm \frac{V_P I_P}{2}$.

It is left as an exercise for the student to show that a similar result could be produced for a capacitive element in figure 5.1. (HINT: the current will now be leading the voltage by 90^0.)

If we assume next that the element in figure 5.1 is a generic load, which is made up of R, L, and C elements (in any series/parallel combination) then the voltage, and current will have an angular displacement (θ) between zero and $\pm 90^0$. Taking voltage as reference, in sinusoidal steady-state they are:

$$v(t) = V_P \cos \omega t \qquad \textbf{(5.3)}$$

$$i(t) = I_P \cos(\omega t + \theta) \qquad \textbf{(5.4)}$$

Substituting 5.3 & 5.4 into 5.1 gives:

$$P = V_P \cos\omega t \, I_P \cos(\omega t + \theta)$$

Once again, this can be plotted by a spreadsheet. If V and I have the same amplitudes and frequency as previously and θ = 60^0, then power will vary with time as shown in figure 5.4:

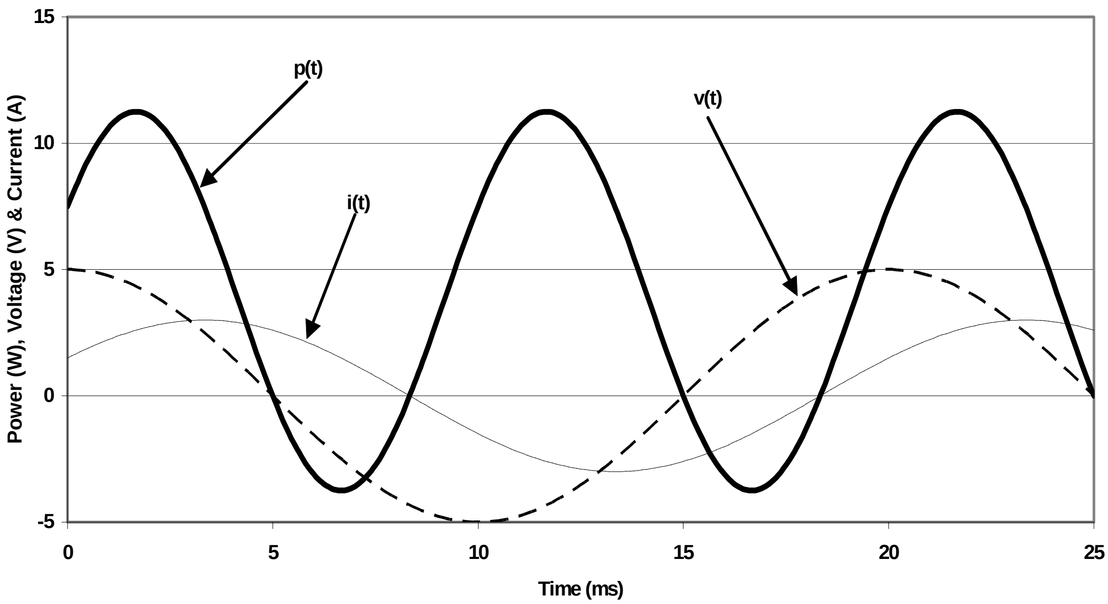

Figure 5.4 Instantaneous Power, Voltage & Current in a Generic R, L, C Circuit

We now see that power still oscillates at twice the line frequency, passes through zero if either voltage or current is zero and is partly off-set from the time axis. The average power dissipated over one cycle is the mean of the maximum (~11.25 W) and the minimum (~ -3.75 W), which is 3.75 W in the example.

Evaluating the product of sinusoids and then averaging is unnecessary if we are only interested in calculating the average power dissipated by a device. This is nearly always the case when determining the power rating of equipment.

5.2 Average Power in the Sinusoidal Steady-State

Our objective in this case is to determine the average power dissipated over one cycle, as this is what determines the amount of work done in a specified time period. It is important to note that all AC equipment has its power rating specified in terms of the average power dissipated over one cycle. In other words, by default, the power rating on all AC equipment is the average power dissipated over one cycle. From now on the term "power" will mean "average power over one cycle" unless stated otherwise.

The most straightforward way to determine the average power dissipated over one cycle is to integrate p(t) over one cycle and divide the result by the periodic time. Once again let us assume that the element in figure 5.1 is a resistor then this becomes:

$$P = \frac{1}{T}\int_0^T v(t)i(t)dt \qquad \textbf{(5.5)}$$

In sinusoidal steady-state:

$$v(t) = V_P \cos\omega t \qquad \textbf{(5.6)}$$

$$i(t) = I_P \cos\omega t \qquad \textbf{(5.7)}$$

$$T = \frac{2\pi}{\omega} \qquad \textbf{(5.8)}$$

Substituting 5.6, 5.7 & 5.8 into 5.5 gives:

$$P = \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} V_P \cos\omega t \, I_P \cos\omega t dt$$

Applying the trig identity, $\cos^2(\omega t) = 0.5[1 + \cos(2\omega t)]$ allows us to write:

$$P = \frac{\omega}{2\pi}\left[\frac{V_P I_P}{2}\left\{t + \frac{\sin(2\omega t)}{2\omega}\right\}\right]_0^{\frac{2\pi}{\omega}} \qquad \textbf{(5.9)}$$

The sin term is zero at both limits and so equation 5.9 reduces to:

$$\boxed{P = \frac{V_P I_P}{2}}$$

Which is the result obtained by observation of figure 5.2. Also, since the maximum value of voltage (V_m) is equal to the maximum value of current (I_m) times the resistance we can write:

$$P = \frac{I_m^2 R}{2} = I_{EFF}^2 R \qquad \textbf{(5.10)}$$

Where I_{EFF} is the "effective" value of the sinusoidal current and is a constant. Equation 5.10 leads to the conclusion that:

$$\boxed{I_{EFF} = \frac{I_P}{\sqrt{2}}}$$

This is the rms or "Root Mean Squared" value of the current. A similar reasoning would have lead to:

$$\boxed{V_{EFF} = \frac{V_P}{\sqrt{2}}}$$

The rms values of voltage and current are the appropriate quantities to use when calculating power usage. *When reading meters and power specifications, all voltages and currents are rms unless stated otherwise.*

If we assume that the element in figure 5.1 is a generic load, which is made up of R, L, and C elements then the voltage and current will have an angular displacement (θ) between zero and $\pm 90^0$. Taking voltage as reference, in sinusoidal steady-state the sinusoids are:

$$v(t) = V_P \cos\omega t = \sqrt{2}V\cos\omega t \quad \textbf{(5.11)}$$

$$i(t) = I_P \cos(\omega t + \theta) = \sqrt{2}I\cos(\omega t + \theta) \quad \textbf{(5.12)}$$

Substituting 5.8, 5.11 & 5.12 into 5.5 gives:

$$P = \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} \sqrt{2}V\cos\omega t\sqrt{2}\, I\cos(\omega t + \theta)dt$$

Where V and I are the RMS values of voltage and current. Applying the trig identity:

$$\cos x \cos y = 0.5[\cos(x - y) + \cos(x + y)]$$

where: x = $\omega t + \theta$, and y = ωt, allows us to write:

$$P = \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} VI\,[\cos\theta + \cos(2\omega t + \theta)]dt$$

The second term is a function of time and is integrated over two complete cycles which causes it to go to zero on both top and bottom limits. Thus the first term is the only one that has to be integrated and since V, I and θ are all constants the result is:

$$P = \frac{\omega}{2\pi} VI\cos\theta[t]_0^{\frac{2\pi}{\omega}}$$

This produces the result:

$$\boxed{P = VI\cos\theta} \quad \textbf{(5.13)}$$

If you think of voltage and current as being rms vector quantities (phasors have magnitude and direction) then power dissipated is their "dot product".

Recall that θ is the "phase-angle" between the voltage and current. The term $\cos\theta$ is called the "power factor" (pf) and is a measure of how effective the circuit is in turning voltage and current into power. Notice that:

$$\boxed{pf = \cos\theta = \frac{P}{VI}}$$

A number of important points have been introduced in this chapter and it is worth summarizing them before we go any further. The following is a list that highlights the main items.

- Instantaneous power oscillates at twice the applied frequency.
- It has a mean value that determines the amount of work done over a specified period of time.
- When calculating the mean value of power, always use the rms values of voltage and current.
- The rms values are the peak values divided by √2. All values quoted in product specifications are rms unless stated otherwise.
- Average power is the product of voltage (rms magnitude), current (rms magnitude) and power factor, where power factor is the cosine of the angle between voltage and current.

EXAMPLE 5.1

A trace from an oscilloscope is shown in figure 5.5. The voltage channel was set on 5 V/cm, while the current channel was set on 100 mA/cm. Determine the power associated with the circuit.

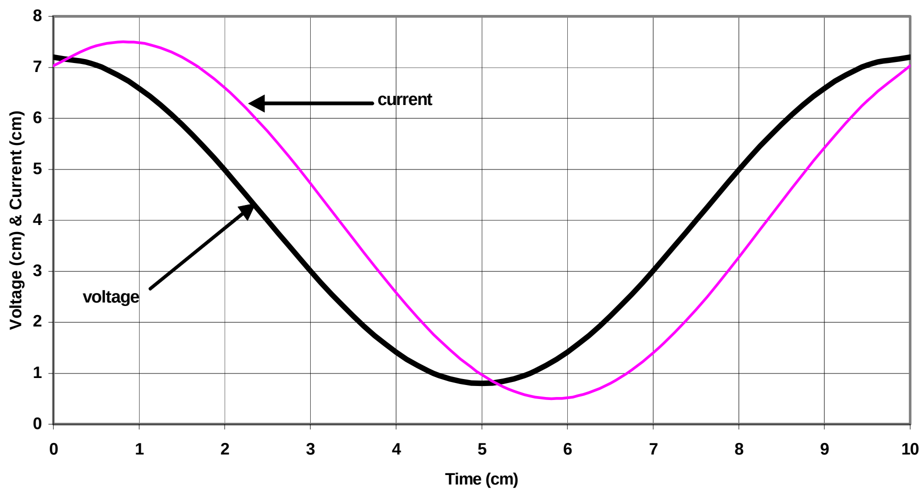

Figure 5.5 Oscilloscope Trace of Voltage & Current

Solution

The voltage trace has max & min values of 7.2 & 0.8 cm respectively. This means that the peak-to-peak voltage is:

$$V_{pp} = (7.2 - 0.8)x5 = 32 \text{ V} \text{ and } V_{peak} = 16 \text{ V}$$

$$V_{RMS} = \frac{V_{peak}}{\sqrt{2}} = 11.31 \text{ V}$$

Following the same procedure with current, whose max and min are 7.5 & 0.5 cm respectively:

$$I_{pp} = (7.5 - 0.5)x0.1 = 0.7 \text{ A} \text{ and } I_{peak} = 0.35 \text{ A}$$

$$I_{RMS} = \frac{I_{peak}}{\sqrt{2}} = 0.2475 \text{ A}$$

The phase shift (θ) is determined by noticing that the current trace is 0.8 cm behind the voltage trace. Since the voltage trace takes 10 cm to cover one cycle of 360^0 this corresponds to 36^0/cm.

$$\theta = 36x0.8 = 28.8^o$$

$$\boxed{P = VI\cos\theta = 11.3 \text{ x } 0.2475 \text{ x } \cos(28.8) = 2.45 \text{ W.}}$$