# Sinusoidal Steady-State Analysis

Almost all electrical systems, whether signal or power, operate with alternating currents and voltages. We have seen that when any circuit is disturbed (switched on or off etc.) it undergoes a *transient* period when voltages and currents can vary wildly and are characterized by their values being enclosed within decaying exponential envelopes. After the transient comes the *steady-state* period; and the term "Sinusoidal Steady-State" refers to how ac circuits are modeled once the transient has passed. It may seem that we are being overly restrictive to confine ourselves to ac waveforms, but we know that any periodic waveform can be represented by a series of sinusoidal waveforms, so studying a single sinusoidal waveform at this stage will make subsequent analysis easier.

It is important for all engineers to be able to analyze steady-state conditions, because it is the steady-state that determines continuous ratings of the equipment. Failure to correctly size equipment results in either:

- excessive expenditure of capital when the plant is oversized, or
- damage when the plant is undersized.

In ac systems the steady-state voltages and currents are modeled by sinusoidal waveforms.

## 4.1 Sinusoidal Sources

An ac voltage is modeled as: $v(t) = V_P\cos(\omega t + \theta)$ **(4.1)**

When equation 4.1 is plotted, the result is shown in figure 4.1.

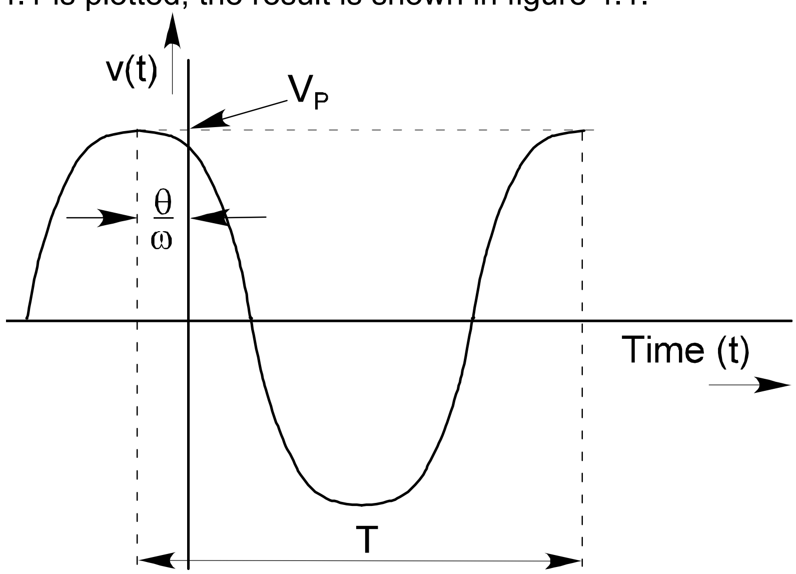


**Figure 4.1 A sinusoidal voltage in steady-state**

The quantities being modeled are:

- $V_P$ is the *peak voltage*.
- $\omega$ is the *frequency* in radians per second (rad/s), $\omega = 2\pi f$.
- f is the frequency in hertz (Hz), which are cycles per second, $f = 1/T$.
- T is the *periodic time* of the oscillations, $T = \dfrac{2\pi}{\omega}$
- $\theta$ is the *phase angle* and allows for the fact that the waveform may not be zero or maximum when t = 0. The phase angle is shown in figure 4.1 for a waveform modeled by a cosine wave, since it is measured from the maximum.

For a given frequency, $\omega$ is constant and if all waveforms are modeled as cosine waves then the only information that has to be conveyed from equation 4.1 is $V_P$ and $\theta$, which can be put into vector form as: $\boldsymbol{V_P} = V_P\underline{/\theta}$. This vector form is known as a *phasor*. The full mathematical derivation of phasors is presented in chapter 6; for now we will simply state that the only information you need to work with sinusoids is their magnitudes and angles.

Using the peak value to represent voltage (or current) does not seem reasonable since this value only exists for an infinitesimal time and using the average value is also not appropriate since it is zero. The appropriate value to use is one that gives the same power dissipation in a resistor in an ac circuit that would result if dc were applied. This dc *equivalent* quantity is the rms value, which for a sinusoid is obtained by dividing the peak value by $\sqrt{2}$. The mathematical basis for the $\sqrt{2}$ factor is presented in chapter 5 and for now we are being asked to "accept it on faith." The result is:

$$V = \frac{V_P}{\sqrt{2}} \quad \text{and} \quad I = \frac{I_P}{\sqrt{2}}$$

Observe that the capital letter symbols V and I in ac, used without subscript, and the words *voltage* and *current* unmodified, always refer to the rms magnitude of a waveform. In other words, rms quantities are the defaults.

## EXAMPLE 4.1

An electrical appliance is a pure resistor and is rated as: 120 V, 8 A, 60 Hz. Determine: a) Peak voltage and current, b) Time between voltage zeros, c) Radian frequency, d) Resistance, e) Power dissipated.

### *Solution*

**Strategy:** We will use the definitions resulting from figure 4.1 to determine items a), b), and c), and then apply the methods of chapter 1 to determine items d) and e).

**Assumptions:** The resistor is linear and rms quantities are defaults.

**Analysis:**

a) The stated voltage and current are rms because this is the default and we are not told otherwise. Then: $V_P = \sqrt{2}V_{rms} = \sqrt{2} \times 120 = 169.7$ V.
And: $I_P = \sqrt{2}I_{rms} = \sqrt{2} \times 8 = 11.3$ A.

b) The time between voltage zeros is the periodic time, $T = \frac{1}{f} = \frac{1}{60} = 16.7 \text{ m sec}$.

c) The radian frequency is, $\omega = 2\pi f = 2 \times \pi \times 60 = 377$ rad/s.

d) Resistance is given by Ohm's Law as: $R = \frac{120}{8} = 15\ \Omega$

e) In ac circuits: $P = I_{rms}^2 R = 8^2 \times 15 = 960$ W.

## 4.3 Phasor Analysis

When dealing with ac circuits, Ohm's Law must be re-stated in terms of phasors. Since both voltage and current are complex numbers, their quotient will also be a complex number and this quantity is called *impedance*, denoted by the symbol ***Z***. By definition:

$$Z = \frac{V}{I} \qquad \textbf{4.3}$$

Since it is possible for voltage and current to be out of phase with each other, the impedance will have real and imaginary parts. The units of impedance are ohms, just the same as resistance, in fact, the real part of ***Z*** is the resistance, while the imaginary part is associated with inductance and/or capacitance.

### 4.3.1 Impedance of a Resistor

We have seen that for an ideal resistor, voltage and current are directly proportional, with *Ohm's Law* stating that resistance is the constant of proportionality. For time-varying voltages and currents, as shown in figure 4.4, we get:

$$v_R(t) = R.i_R(t)$$

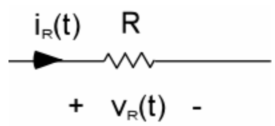


**Figure 4.4**

If the waveforms are switched on at peak values ($\theta = 0$, and the peak is $\sqrt{2}I$, where "I" is the rms current) we can say:

$i_R(t) = \sqrt{2}I\cos\omega t$, from which we get: $v_R(t) = R\sqrt{2}I\cos\omega t$.

Converting voltage and current to rms phasors: $\boldsymbol{I_R} = I\underline{/0}$ and $\boldsymbol{V_R} = RI\underline{/0}$ ,
and we can conclude that the impedance of a resistor is:

$$\boxed{Z_R = \frac{V_R}{I_R} = R}$$

This very simple result tells us that the impedance of a resistor is entirely real and is the resistance itself.

### 4.3.2 Impedance of an Inductor

We have seen that for an ideal inductor for time-varying voltages and currents, as shown in figure 4.5, we get:

$$v_L(t) = L\frac{di_L}{dt}$$

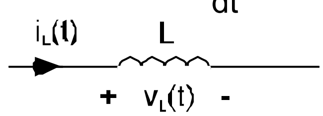


**Figure 4.5**

Once again we say: $i_L(t) = \sqrt{2}I\cos\omega t$, which leads to: $\boldsymbol{I_L} = I\underline{/0}$ (rms phasor)

Then: $\dfrac{di_L}{dt} = -\omega\sqrt{2}I\sin\omega t = \omega\sqrt{2}I\cos(\omega t + 90)$

and: $v_L(t) = L\omega\sqrt{2}I\cos(\omega t + 90)$, the corresponding phasor is: $\boldsymbol{V_L} = \omega LI\underline{/90}$ , which gives the impedance of an inductor as:

$$\boxed{Z_L = \frac{V_L}{I_L} = \frac{\omega LI\angle 90}{I\angle 0} = j\omega L}$$

The quantity $\omega L$ is referred to as the *inductive reactance* and is denoted by $X_L$.

## EXAMPLE 4.2

An inductance of 35 mH has a winding resistance of 7.2 Ω that appears in series with the inductance. It is connected across a 240 V, 60 Hz supply. Determine:
a) Impedance, b) Current drawn.

### *Solution*

**Strategy:** Impedance is made up of resistance (real part) and inductive reactance (imaginary part). To get inductive reactance we need to know the frequency in rad/s and we note that we are given the frequency in Hz. Once impedance has been determined the current is obtained from Ohm's Law.

**Assumptions:** Both resistance and inductance are linear.

**Analysis:**

a) To get inductive reactance we need to find $\omega = 2\pi \times 60 = 377$ rad/s

$\therefore$ $X_L = 377 \times 0.035 = 13.2\ \Omega$

and: $\boldsymbol{Z} = R + j X_L = 7.2 + j13.2\ \Omega$, in rectangular co-ordinates

or: $\boldsymbol{Z} = 15\underline{/61.4^o}\ \Omega$, in polar co-ordinates.

b) $\boldsymbol{I} = \frac{V}{Z} = \frac{240\angle 0}{15\angle 61.4} = 16\angle -61.4\ \text{A}$ and is the rms value since rms voltage was used.

### 4.3.3 Impedance of a Capacitor

We have seen that for an ideal capacitor for time-varying voltages and currents, as shown in figure 4.6, we get:

$$i_c(t) = C\frac{dv_c}{dt}$$

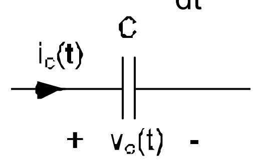


**Figure 4.6**

This time we say: $v_C(t) = \sqrt{2}V\cos\omega t$, which leads to: $\boldsymbol{V_L} = V\underline{/0}$ (rms phasor)

Then: $\frac{dv_C}{dt} = -\omega\sqrt{2}V\sin\omega t = \omega\sqrt{2}V\cos(\omega t + 90)$

and: $i_C(t) = C\omega\sqrt{2}V\cos(\omega t + 90)$, the corresponding phasor is: $\boldsymbol{I_C} = \omega CV\underline{/90}$, which gives the impedance of a capacitor as:

$$\boxed{Z_C = \frac{V_C}{I_C} = \frac{V\angle 0}{\omega CV\angle 90} = \frac{1}{j\omega C}}$$

The quantity $\frac{1}{\omega C}$ is referred to as the *capacitive reactance* and is denoted by $X_C$.

Note that the "j" in the denominator causes $X_C$ to be associated with a "-j".

## EXAMPLE 4.3

A 20 μF capacitor is connected in parallel with a 100 Ω resistor and the combination is placed across the same supply as in example 4.2. Determine:
a) Impedance, b) Current drawn.

### *Solution*

**Strategy:** Impedance is made up of the parallel combination of the resistance with the capacitive reactance. To get capacitive reactance we need to know the frequency in rad/s and we note that we are given the frequency in Hz. Once impedance has been determined the current is obtained from Ohm's Law.

**Assumptions:** Both resistance and capacitance are linear.

**Analysis:**

a) To get capacitive reactance we use $\omega$ = 377 rad/s, found in example 4.2.

$$\therefore \quad X_C = \frac{1}{377 \times 20 \times 10^{-6}} = 132.6\ \Omega$$, and this is in parallel with R.

$$\therefore \quad \frac{1}{Z} = \frac{1}{R} + \frac{1}{jX_C} = \frac{1}{100} + \frac{1}{j132.6}$$ gives: $\boldsymbol{Z}$ = 63.8 - j48.1 Ω,

or: $\boldsymbol{Z}$ = $79.8\underline{/-37^{\circ}}$ Ω, in polar co-ordinates.

b) $$\boldsymbol{I} = \frac{V}{Z} = \frac{240\angle 0}{79.8\angle -37} = 3\angle 37\ \text{A}$$ and is the rms value since rms voltage was used.

## 4.6 Summary

Sinusoidal ac voltage or current waveforms have three dimensions – Magnitude, frequency, and phase angle. The period of the wave (the length of time it takes to go through its complete cycle) is the reciprocal of the frequency in hertz. In their mathematical, function-of-time representation the peak magnitude is used. But engineers usually describe ***I***'s and ***V***'s by their rms (or effective) magnitudes, which is peak divided by √2. Frequency can be represented in terms of hertz (f) or in radians per second ($\omega$), where $\omega = 2\pi f$.

For a given frequency, an ac circuit can be represented in its phasor form. Phasors are a convenient transformation for ac sinusoids, and turn the ac function-of-time trigonometry analysis into complex algebra. With this transformation, all of the dc analysis techniques and theorems (Mesh, Nodal, Norton, Thevenin, etc.) are usable with the sinusoidal ac circuits.

Sinusoids can thus be represented in four ways: as mathematical or graphical functions of time (the time domain), and as mathematical or graphical phasors (the frequency domain).

The ratio of ***V*** to ***I*** (phasor numbers) is now complex and is named impedance ***Z***, where ***Z*** = R + jX, where R is the resistance and X is the reactance. It is important to note that the value of X changes whenever the frequency changes.