

Electromagnetic Foundations

Gauss' law $\nabla \cdot \mathbf{D} = \rho$

Conservation of magnetic flux $\nabla \cdot \mathbf{B} = 0$

Ampere's law $\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$

Faraday's law $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$

Traveling waves

Taking the curl of Faraday

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = \nabla \times \left(-\frac{\partial \mathbf{B}}{\partial t} \right) = -\frac{\partial}{\partial t} (\nabla \times \mathbf{B})$$

In a charge-free ($\rho=0$) and current-free ($\mathbf{J}=0$) region.

$$\begin{aligned} \nabla (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} &= -\nabla^2 \mathbf{E} = -\frac{\partial}{\partial t} (\nabla \times \mathbf{B}) = -\mu \frac{\partial}{\partial t} (\nabla \times \mathbf{H}) \\ \nabla^2 \mathbf{E} &= -\mu \frac{\partial}{\partial t} (\nabla \times \mathbf{H}) \end{aligned}$$

Substituting for $\nabla \times \mathbf{H}$ using Ampere's law ($\mathbf{J}=0$), we obtain the wave equation

$$\begin{aligned} \nabla^2 \mathbf{E} &= \mu \varepsilon \frac{\partial}{\partial t} \left(\frac{\partial \mathbf{E}}{\partial t} \right) \\ \nabla^2 \mathbf{E} - \mu \varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} &= 0 \end{aligned}$$

This equation is called the wave equation. A wave equation for \mathbf{H} can be found as well. Taking the curl of Ampere's law and then substituting Faraday's law [Again, in a charge-free ($\rho_v=0$) and current-free ($\mathbf{J}=0$) region.]

$$\nabla^2 \mathbf{H} - \mu \varepsilon \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0$$

Together, these equations predict that an electromagnetic wave can *travel* through space. These vector equations (one in \mathbf{E} and one in \mathbf{H}) would each give three scalar equations (one for each component of the vectors) for a total of six scalar equations. Using Cartesian coordinates, one scalar equation for E_x would be

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} = \mu \varepsilon \frac{\partial^2 E_x}{\partial t^2}$$

Waves traveling in the z direction would have the form

$$g = g \left(t \pm \frac{z}{v} \right) \quad \Rightarrow \quad \frac{\partial^2 g}{\partial z^2} = \frac{1}{v^2} \frac{\partial^2 g}{\partial t^2}$$

Two waves are possible: one traveling in the $+\mathbf{a}_z$ direction with a speed of $v = 1/\sqrt{\mu\varepsilon}$, the other traveling in the $-\mathbf{a}_z$ direction with the same speed.

Plane Waves

Assuming the electric vector to only have a component in the x-direction, E_x , which depends only on z and t.

$$\frac{\partial^2 E_x}{\partial z^2} = \mu\varepsilon \frac{\partial^2 E_x}{\partial t^2}$$

Solutions are functions where the 2nd derivative with respect to z are equal to their 2nd derivative with respect to t (aside from a constant multiplying factor). Sinusoids are an obvious choice (this particular choice travels in the $+\mathbf{a}_z$ direction).

$$\mathbf{E} = \mathbf{a}_x E \cos(\omega t - \beta z + \varphi^+)$$

Substituting this possible solution,

$$\begin{aligned} \beta^2 \mathbf{a}_x E \cos(\omega t - \beta z + \varphi^+) &= \varepsilon\mu \omega^2 \mathbf{a}_x E \cos(\omega t - \beta z + \varphi^+) \\ \beta^2 &= \varepsilon\mu \omega^2 \quad \Rightarrow \quad \beta = \omega\sqrt{\varepsilon\mu} \end{aligned}$$

Just as for sinusoidal steady state analysis in electrical circuits, harmonic analysis is provides a convenient notation.

$$\mathbf{E} = \mathbf{a}_x E \cos(\omega t - \beta z + \varphi^+) = \text{Re} \left[\mathbf{a}_x E e^{j\varphi^+} e^{-j\beta z} e^{j\omega t} \right] = \text{Re} \left[\tilde{\mathbf{E}} e^{j\omega t} \right]$$

That is, for fields that vary harmonically with time, each Fourier component can be described as

$$\begin{aligned} \mathbf{E}(\mathbf{r}) \cos(\omega t + \varphi) &= \text{Re} \left\{ \mathbf{E}(\mathbf{r}) e^{j(\omega t + \varphi)} \right\} = \text{Re} \left\{ \mathbf{E}(\mathbf{r}) e^{j\varphi} e^{j\omega t} \right\} \\ \mathbf{E} &= \text{Re} \left\{ \tilde{\mathbf{E}} e^{j\omega t} \right\} \\ \mathbf{E} &\leftrightarrow \tilde{\mathbf{E}} \end{aligned}$$

Just as in phasor analysis, the derivative with respect to time corresponds to a $j\omega$ multiplier in the frequency domain.

$$\begin{aligned} \frac{\partial}{\partial t} \mathbf{E} &= \frac{\partial}{\partial t} \text{Re} \left\{ \tilde{\mathbf{E}} e^{j\omega t} \right\} = \text{Re} \left\{ j\omega \tilde{\mathbf{E}} e^{j\omega t} \right\} \\ \frac{\partial}{\partial t} \mathbf{E} &\leftrightarrow j\omega \tilde{\mathbf{E}} \end{aligned}$$

In the frequency domain, Maxwell's equations become,

$$\begin{aligned}\nabla \times \tilde{\mathbf{E}} &= -j\omega\tilde{\mathbf{B}} & \nabla \cdot \tilde{\mathbf{D}} &= \tilde{\rho} \\ \nabla \times \tilde{\mathbf{H}} &= \tilde{\mathbf{J}} + j\omega\tilde{\mathbf{D}} & \nabla \cdot \tilde{\mathbf{B}} &= 0\end{aligned}$$

Example 1

Suppose, as above, an electric vector, pointing in the \mathbf{a}_x direction travels in the \mathbf{a}_z direction. How can the magnetic vector be found? Answer: Faraday's Law.

$$\begin{aligned}\nabla \times \tilde{\mathbf{E}} &= -j\omega\mu\tilde{\mathbf{H}} \\ \tilde{\mathbf{H}} &= \frac{j}{\omega\mu} \nabla \times \tilde{\mathbf{E}} = \frac{j}{\omega\mu} \left[\left(\frac{\partial 0}{\partial y} - \frac{\partial 0}{\partial z} \right) \mathbf{a}_x + \left(\frac{\partial \tilde{E}e^{-j\beta z}}{\partial z} - \frac{\partial 0}{\partial x} \right) \mathbf{a}_y + \left(\frac{\partial 0}{\partial x} - \frac{\partial \tilde{E}e^{-j\beta z}}{\partial y} \right) \mathbf{a}_z \right] \\ \tilde{\mathbf{H}} &= \frac{j}{\omega\mu} (-j\beta\tilde{E}e^{-j\beta z} \mathbf{a}_y) = \frac{\beta}{\omega\mu} \tilde{E}e^{-j\beta z} \mathbf{a}_y = \frac{\omega\sqrt{\varepsilon\mu}}{\omega\mu} \tilde{E}e^{-j\beta z} \mathbf{a}_y \\ \tilde{\mathbf{H}} &= \sqrt{\frac{\varepsilon}{\mu}} \tilde{E}e^{-j\beta z} \mathbf{a}_y\end{aligned}$$

Notice that $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{H}}$ are related by the *field analog* of Ohm's law.

$$\tilde{\mathbf{E}} = \sqrt{\frac{\mu}{\varepsilon}} \tilde{\mathbf{H}} = \eta \tilde{\mathbf{H}}, \text{ where } \eta = \sqrt{\frac{\mu}{\varepsilon}} \text{ is the intrinsic, or wave, impedance (in } \Omega\text{).}$$

The wave impedance for an electromagnetic wave in free space (nearly the same for a wave traveling in air, since, in air, $\varepsilon \cong \varepsilon_0$ and $\mu \cong \mu_0$) is

$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \cong 377\Omega$$

The relative directions of $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{H}}$ can be determined from the fact that $\mathbf{E} \times \mathbf{H}$ is in the direction of travel. In fact $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ is known as Poynting's vector and is the instantaneous power flux density, with units of W/m^2 , transmitted by an EM wave.

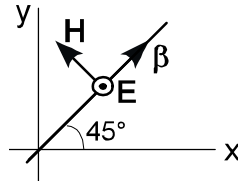
For periodic fields, the time-averaged power flux density,

$$\langle \mathbf{S} \rangle = \frac{1}{2} \text{Re}(\tilde{\mathbf{E}} \times \tilde{\mathbf{H}}^*)$$

Example 2

An electric field, of amplitude 37.7 V/m points in the \mathbf{a}_z direction and travels along $\phi = 45^\circ$ in the x-y plane. The wavelength is 700 nm and the medium is air.

- Give the frequency domain description of \mathbf{E} and \mathbf{H} . Take the phase of $\tilde{\mathbf{E}}$ to be zero degrees.
- What is the frequency of the waves? Give their time-domain descriptions.



Solution

- The wave number, or phase constant, β , is found from the wavelength.

$$\beta = \frac{2\pi}{\lambda} = \frac{2\pi}{700 \text{ nm}} \cong 9(10^6) \text{ m}^{-1}$$

$$\tilde{\mathbf{E}} = \mathbf{a}_z 37.7 e^{-j9(10^6)(x\cos 45^\circ + y\sin 45^\circ)} \text{ V/m}$$

The magnitude of the magnetic vector can be determined from the field analog of Ohm's law and the direction of the vector can be determined from the fact that $\mathbf{E} \times \mathbf{H}$ is in the direction of travel (as shown in the diagram above). The magnetic vector travels with the electric vector (after all, we have a traveling EM wave of which the two vectors are part).

$$\tilde{\mathbf{H}} = (\mathbf{a}_y \sin 45^\circ - \mathbf{a}_x \cos 45^\circ) \frac{37.7 \text{ V/m}}{377 \Omega} e^{-j9(10^6)(x\cos 45^\circ + y\sin 45^\circ)}$$

$$\tilde{\mathbf{H}} = 0.1(\mathbf{a}_y \sin 45^\circ - \mathbf{a}_x \cos 45^\circ) e^{-j9(10^6)(x\cos 45^\circ + y\sin 45^\circ)} \text{ A/m}$$

- The speed of propagation can be determined from the physical properties of the medium, in this case air. Knowing the speed of propagation and the wavelength permits the frequency to be determined.

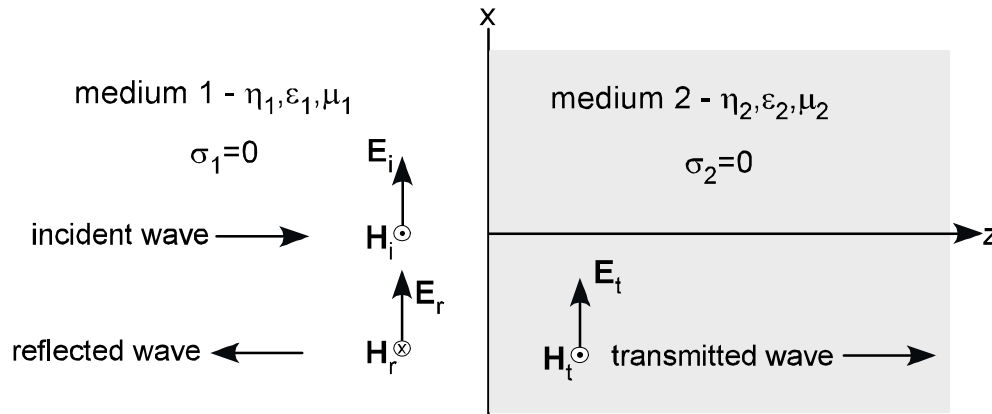
$$v_p = \frac{1}{\sqrt{\epsilon\mu}} \cong \frac{1}{\sqrt{\epsilon_0\mu_0}}, \text{ for air}$$

$$v_p = \frac{1}{\sqrt{8.854(10^{-12}) \text{ F/m } 4\pi(10^{-7}) \text{ H/m}}} \cong 3(10^8) \text{ m/s}$$

$$f = \frac{3(10^8) \text{ m/s}}{700(10^{-9}) \text{ m}} = 4.29(10^{14}) \text{ Hz}$$

Reflection and Transmission

How are EM waves reflected and transmitted when a plane dielectric boundary is encountered? Consider the situation below:



Define a transmission coefficient, T , and reflection coefficient, Γ .

$$T = \frac{E_t}{E_i} \quad \Gamma = \frac{E_r}{E_i}$$

Let $\tilde{\mathbf{E}}_i = \mathbf{a}_x E_i e^{-j\beta_1 z}$

From EM theory, the boundary conditions that must be met are that the tangential components of the electric field and the magnetic field (assuming no surface currents) are equal on either side of the boundary.

Boundary conditions: $E_{1tan}|_{z=0} = E_{2tan}|_{z=0}$ $H_{1tan}|_{z=0} = H_{2tan}|_{z=0}$

The first equation is obtained from the boundary conditions on the electric vectors.

$$(E_i + E_r)\mathbf{a}_x = E_t \mathbf{a}_x \quad \Rightarrow \quad E_i + \Gamma E_i = T E_i$$

The second is obtained from the boundary conditions on the magnetic vectors.

$$(H_i - H_r)\mathbf{a}_y = H_t \mathbf{a}_y \quad \Rightarrow \quad \frac{E_i}{\eta_1} - \frac{\Gamma E_i}{\eta_1} = \frac{T E_i}{\eta_2}$$

So the result is two equations in Γ and T .

$$1 + \Gamma = T \quad \text{and} \quad \frac{1}{\eta_1} - \frac{\Gamma}{\eta_1} = \frac{T}{\eta_2}$$

Solving for Γ and T .

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \quad T = \frac{2\eta_2}{\eta_2 + \eta_1}$$

Using these coefficients and the intrinsic impedances of the materials, one can find the incident, reflected, and transmitted electric and magnetic vectors in terms of the incident electric vector.

$$\begin{aligned}\tilde{\mathbf{E}}_i &= \mathbf{a}_x E_i e^{-j\beta_1 z} & \tilde{\mathbf{E}}_r &= \mathbf{a}_x \Gamma E_i e^{j\beta_1 z} & \tilde{\mathbf{E}}_t &= \mathbf{a}_x T E_i e^{-j\beta_2 z} \\ \tilde{\mathbf{H}}_i &= \mathbf{a}_y \frac{E_i}{\eta_1} e^{-j\beta_1 z} & \tilde{\mathbf{H}}_r &= -\mathbf{a}_y \frac{\Gamma E_i}{\eta_1} e^{j\beta_1 z} & \tilde{\mathbf{H}}_t &= \mathbf{a}_y \frac{T E_i}{\eta_2} e^{-j\beta_2 z}\end{aligned}$$

From the field vectors, the power densities can be determined.

$$\begin{aligned}\langle \mathbf{S}_i \rangle &= \frac{1}{2} \text{Re}(\tilde{\mathbf{E}}_i \times \tilde{\mathbf{H}}_i^*) = \frac{1}{2} \text{Re} \left[\mathbf{a}_x E_i e^{-j\beta_1 z} \times \left(\mathbf{a}_y \frac{E_i}{\eta_1} e^{-j\beta_1 z} \right)^* \right] \\ \langle \mathbf{S}_i \rangle &= \frac{1}{2} \text{Re} \left[\mathbf{a}_x E_i e^{-j\beta_1 z} \times \mathbf{a}_y \frac{E_i}{\eta_1} e^{j\beta_1 z} \right] = \mathbf{a}_z \frac{E_i^2}{2\eta_1}\end{aligned}$$

Similarly,

$$\langle \mathbf{S}_r \rangle = -\mathbf{a}_z \frac{\Gamma^2 E_i^2}{2\eta_1} \quad \text{and} \quad \langle \mathbf{S}_t \rangle = \mathbf{a}_z \frac{T^2 E_i^2}{2\eta_2}$$

Note, if the materials are conductive, the reflection and transmission coefficients, as well as the intrinsic impedances are complex.

Standing Waves

When reflection is present, the incident and reflected waves will interfere to produce standing waves in region 1.

$$\begin{aligned}\tilde{\mathbf{E}}_1 &= \tilde{\mathbf{E}}_i + \tilde{\mathbf{E}}_r = \mathbf{a}_x E_i e^{-j\beta_1 z} + \mathbf{a}_x \Gamma E_i e^{j\beta_1 z} = \mathbf{a}_x (1-\Gamma) e^{-j\beta_1 z} + \mathbf{a}_x \Gamma E_i (e^{j\beta_1 z} + e^{-j\beta_1 z}) \\ \tilde{\mathbf{E}}_1 &= \mathbf{a}_x (1-\Gamma) e^{-j\beta_1 z} + \mathbf{a}_x 2\Gamma E_i \cos\beta_1 z \quad \text{where the 2nd term in the standing wave}\end{aligned}$$

In the time domain, we have

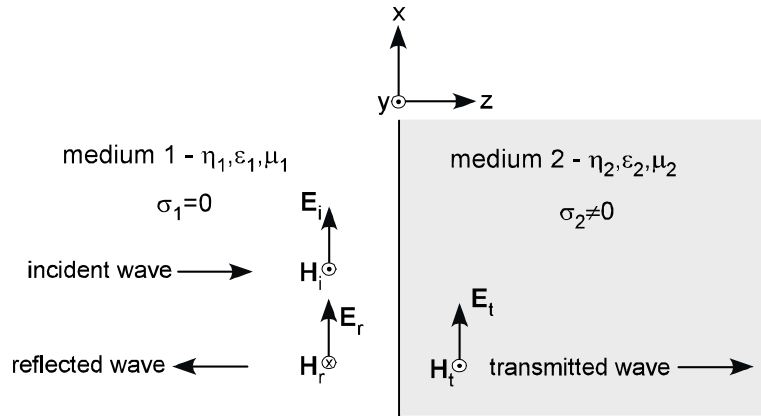
$$\mathbf{E}_1 = \text{Re}(\tilde{\mathbf{E}}_1 e^{j\omega t}) = \mathbf{a}_x (1-\Gamma) \cos(\omega t - \beta_1 z) + \mathbf{a}_x 2\Gamma E_i \cos\beta_1 z \cos\beta_1 z$$

The standing wave ratio (SWR) is commonly used to characterize reflections from a surface. The SWR is defined as the ratio between the maximum amplitude electric field in region 1 to the minimum electric field.

$$\text{SWR} = \frac{E_{\max}}{E_{\min}} = \frac{1 - \Gamma + 2\Gamma}{1 - \Gamma} = \frac{1 + \Gamma}{1 - \Gamma} \quad \Rightarrow \quad \Gamma = \frac{\text{SWR} - 1}{\text{SWR} + 1}$$

Where, Γ is the magnitude of $\tilde{\Gamma}$.

Reflection and Transmission ($\sigma_2 \neq 0$)



The conductivity in region 2 leads to complex values for the reflection and transmission coefficients as well as for the intrinsic impedance in region 2.

$$\tilde{\eta}_2 = \sqrt{\frac{\mu_2}{\tilde{\epsilon}_2}} = \sqrt{\frac{\mu_2}{\epsilon_2 - j\frac{\sigma_2}{\omega}}} = \eta_2 e^{j\varphi_\eta} \quad \tilde{\Gamma} = \frac{\tilde{\eta}_2 - \eta_1}{\tilde{\eta}_2 + \eta_1} = \Gamma e^{j\varphi_\Gamma} \quad \tilde{T} = \frac{2\tilde{\eta}_2}{\tilde{\eta}_2 + \eta_1} = T e^{j\varphi_T}$$

This results in a phase shift between the incident wave and waves that are reflected and transmitted. In addition, the fact that the intrinsic impedance in region 2 is complex leads to their being phase shifts between the electric vector and the magnetic vector in region 2.

$$\begin{aligned} \tilde{\mathbf{E}}_i &= \mathbf{a}_x E_i e^{-j\beta_1 z} & \tilde{\mathbf{E}}_r &= \mathbf{a}_x \Gamma E_i e^{j(\beta_1 z + \varphi_\Gamma)} & \tilde{\mathbf{E}}_t &= \mathbf{a}_x T E_i e^{-j(\beta_2 z - \varphi_T)} \\ \tilde{\mathbf{H}}_i &= \mathbf{a}_y \frac{E_i}{\eta_1} e^{-j\beta_1 z} & \tilde{\mathbf{H}}_r &= -\mathbf{a}_y \frac{\Gamma E_i}{\eta_1} e^{j(\beta_1 z + \varphi_\Gamma)} & \tilde{\mathbf{H}}_t &= \mathbf{a}_y \frac{T E_i}{\eta_2} e^{-j(\beta_2 z + \varphi_\eta - \varphi_T)} \end{aligned}$$

The incident power is unchanged from the previous case, $\langle \mathbf{S}_i \rangle = \mathbf{a}_z \frac{E_i^2}{2\eta_1}$.

The reflected power density is

$$\begin{aligned} \langle \mathbf{S}_r \rangle &= \frac{1}{2} \text{Re}(\tilde{\mathbf{E}}_r \times \tilde{\mathbf{H}}_r^*) = \frac{1}{2} \text{Re} \left[\mathbf{a}_x \Gamma E_i e^{j(\beta_1 z + \varphi_\Gamma)} \times \left(-\mathbf{a}_y \frac{\Gamma E_i}{\eta_1} e^{j(\beta_1 z + \varphi_\Gamma)} \right)^* \right] \\ \langle \mathbf{S}_r \rangle &= \frac{1}{2} \text{Re} \left[\mathbf{a}_x \Gamma E_i e^{j(\beta_1 z + \varphi_\Gamma)} \times -\mathbf{a}_y \frac{\Gamma E_i}{\eta_1} e^{-j(\beta_1 z + \varphi_\Gamma)} \right] = -\mathbf{a}_z \frac{\Gamma^2 E_i^2}{2\eta_1} \end{aligned}$$

The transmitted power density is

$$\langle \mathbf{S}_t \rangle = \frac{1}{2} \text{Re}(\tilde{\mathbf{E}}_t \times \tilde{\mathbf{H}}_t^*) = \frac{1}{2} \text{Re} \left[\mathbf{a}_x \Gamma E_i e^{-j(\beta_1 z - \varphi_\Gamma)} \times \left(\mathbf{a}_y \frac{\Gamma E_i}{\eta_2} e^{-j(\beta_1 z + \varphi_\Gamma - \varphi_\eta)} \right)^* \right]$$

$$\langle \mathbf{S}_t \rangle = \frac{1}{2} \text{Re} \left[\mathbf{a}_x \Gamma E_i e^{-j(\beta_1 z - \varphi_\Gamma)} \times \mathbf{a}_y \frac{\Gamma E_i}{\eta_2} e^{j(\beta_1 z + \varphi_\Gamma - \varphi_\eta)} \right] = \mathbf{a}_z \frac{\Gamma^2 E_i^2}{2\eta_1} \cos \varphi_\eta$$

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When reflection is present, the incident and reflected waves will interfere to produce standing waves in region 1.

$$\tilde{\mathbf{E}}_1 = \tilde{\mathbf{E}}_i + \tilde{\mathbf{E}}_r = \mathbf{a}_x E_i e^{-j\beta_1 z} + \mathbf{a}_x \Gamma E_i e^{j(\beta_1 z + \varphi_\Gamma)} = \mathbf{a}_x (1-\Gamma) e^{-j\beta_1 z} + \mathbf{a}_x \Gamma E_i (e^{j(\beta_1 z + \varphi_\Gamma)} + e^{-j\beta_1 z})$$

$$\tilde{\mathbf{E}}_1 = \mathbf{a}_x (1-\Gamma) e^{-j\beta_1 z} + \mathbf{a}_x \Gamma E_i e^{j\frac{\varphi_\Gamma}{2}} \left(e^{j\left(\beta_1 z + \frac{\varphi_\Gamma}{2}\right)} + e^{-j\left(\beta_1 z + \frac{\varphi_\Gamma}{2}\right)} \right)$$

$$\tilde{\mathbf{E}}_1 = \mathbf{a}_x (1-\Gamma) e^{-j\beta_1 z} + \mathbf{a}_x 2\Gamma E_i e^{j\frac{\varphi_\Gamma}{2}} \cos\left(\beta_1 z + \frac{\varphi_\Gamma}{2}\right), \quad \text{where the 2}^{\text{nd}} \text{ term in the standing wave}$$

In the time domain, we have

$$\mathbf{E}_1 = \text{Re}(\tilde{\mathbf{E}}_1 e^{j\omega t}) = \mathbf{a}_x (1-\Gamma) \cos(\omega t - \beta_1 z) + \mathbf{a}_x 2\Gamma E_i \cos\left(\beta_1 z + \frac{\varphi_\Gamma}{2}\right) \cos\left(\omega t + \frac{\varphi_\Gamma}{2}\right)$$

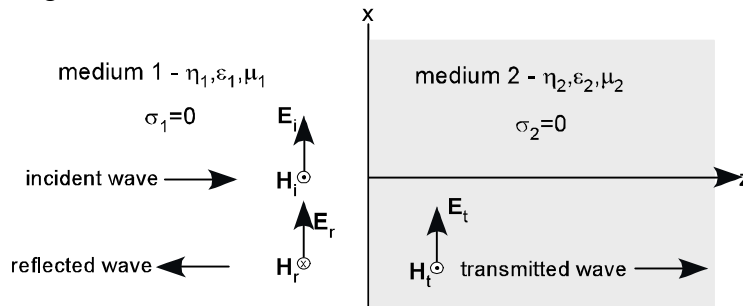
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$$\text{SWR} = \frac{E_{\max}}{E_{\min}} = \frac{1 - \Gamma + 2\Gamma}{1 - \Gamma} = \frac{1 + \Gamma}{1 - \Gamma} \Rightarrow \Gamma = \frac{\text{SWR} - 1}{\text{SWR} + 1}$$

Where, Γ is the magnitude of $\tilde{\Gamma}$.

Example 3

- Find the transmission and reflection coefficients for the situation shown below if medium 1 is air and medium 2 is PCB material ($\sigma=0$, $\epsilon_r=4.4$, $\mu_r=1$).
- Find the portion of the incident energy that is transported into the FR-4 material.
- Find the standing wave ratio in medium 1.



Solution

$$i) \quad \eta_1 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 377 \, \Omega \qquad \eta_2 = \sqrt{\frac{\mu_0}{\epsilon_2}} = \sqrt{\frac{\mu_0}{4.4 \epsilon_0}} = \frac{377 \, \Omega}{\sqrt{4.4}}$$

Knowing the intrinsic impedances, Γ and T are readily calculated.

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{\frac{377 \, \Omega}{\sqrt{4.4}} - 377 \, \Omega}{\frac{377 \, \Omega}{\sqrt{4.4}} + 377 \, \Omega} = \frac{1 - \sqrt{4.4}}{1 + \sqrt{4.4}}$$

$$T = \frac{2\eta_2}{\eta_2 + \eta_1} = \frac{2}{1 + \sqrt{4.4}}$$

$$ii) \quad \frac{\langle \mathbf{S}_t \rangle}{\langle \mathbf{S}_i \rangle} = \frac{\mathbf{a}_z \frac{T^2 E_i^2}{2\eta_2}}{\mathbf{a}_z \frac{E_i^2}{2\eta_1}} = \frac{T^2 \eta_1}{\eta_2} = 0.874$$

$$\text{Note that } \frac{\langle \mathbf{S}_t \rangle}{\langle \mathbf{S}_i \rangle} + \frac{\langle \mathbf{S}_r \rangle}{\langle \mathbf{S}_i \rangle} = 1$$

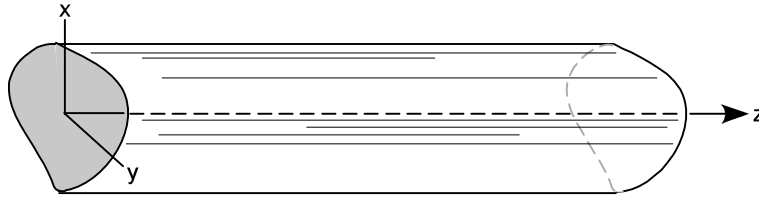
This is just the conservation of energy, $\langle \mathbf{S}_i \rangle = \langle \mathbf{S}_r \rangle + \langle \mathbf{S}_t \rangle$

$$iii) \quad \text{SWR} = \frac{1 + \Gamma}{1 - \Gamma} \cong 2.06$$

Notice that the standing wave ratio must be between 1 and ∞ .

Transmission Line Fundamentals

Transmission lines are a special case of solutions to Maxwell's equations when the properties of the physical system do not change in one direction (typically taken to be the z-direction).



Employing frequency domain analysis is convenient when working with transmission lines. In frequency domain analysis, one assumes that all fields vary sinusoidally in time. Suppose a source $\mathbf{a}_x E_s \cos \omega t$ produces, in response, an electric vector which travels in the \mathbf{a}_z direction.

$$\mathbf{E} = \mathbf{a}_x E_o \cos(\omega t - \beta z)$$

If the system is linear, the complex source $\mathbf{a}_x E_s (\cos \omega t + j \sin \omega t)$ would produce a complex response.

$$\mathbf{E} = \mathbf{a}_x E_o [\cos(\omega t - \beta z) + j \sin(\omega t - \beta z)]$$

Using Euler's formula ($e^{j\alpha} = \cos \alpha + j \sin \alpha$), this can be written as

$$\mathbf{E} = \mathbf{a}_x E_o e^{j(\omega t - \beta z)} = \mathbf{a}_x E_o e^{-j\beta z} e^{j\omega t} = \tilde{\mathbf{E}} e^{j\omega t}$$

Example 4

An example will illustrate the form that Maxwell's equations have in the frequency domain. Using Faraday's law with $\mathbf{E} = \tilde{\mathbf{E}} e^{j\omega t}$ and $\mathbf{B} = \tilde{\mathbf{B}} e^{j\omega t}$.

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \Rightarrow \nabla \times \tilde{\mathbf{E}} e^{j\omega t} = -\frac{\partial (\tilde{\mathbf{B}} e^{j\omega t})}{\partial t}$$

Note that $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{B}}$ have no time dependence and that $e^{j\omega t}$ has no spatial dependence. This allows the curl to pass through $e^{j\omega t}$ and $\tilde{\mathbf{B}}$ to pass through the time derivative.

$$e^{j\omega t} (\nabla \times \tilde{\mathbf{E}}) = -\tilde{\mathbf{B}} \frac{\partial e^{j\omega t}}{\partial t} = -j\omega \tilde{\mathbf{B}} e^{j\omega t} \Rightarrow \nabla \times \tilde{\mathbf{E}} = -j\omega \tilde{\mathbf{B}}$$

In the frequency-domain version of Faraday shown above, $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{B}}$ are the frequency domain field vectors. They are not the time domain vectors. The time domain vectors can be recovered by putting the time dependence back and taking the real part.

Taking the electric vector used in this example to demonstrate,

$$\mathbf{E} = \text{Re}(\tilde{\mathbf{E}} e^{j\omega t}) = \text{Re}(\mathbf{a}_x E_0 e^{-j\beta z} e^{j\omega t}) = \text{Re}(\mathbf{a}_x E_0 e^{j(\omega t - \beta z)})$$

$$\mathbf{E} = \text{Re}\{\mathbf{a}_x E_0 [\cos(\omega t - \beta z) + j \sin(\omega t - \beta z)]\} = \mathbf{a}_x E_0 \cos(\omega t - \beta z)$$

This result can be generalized to express Maxwell's equations in the frequency domain (current and charge-free)

$$\nabla \times \tilde{\mathbf{E}} = -j\omega \tilde{\mathbf{B}}$$

$$\nabla \times \tilde{\mathbf{H}} = j\omega \tilde{\mathbf{D}}$$

$$\nabla \cdot \tilde{\mathbf{D}} = 0$$

$$\nabla \cdot \tilde{\mathbf{B}} = 0$$

Combining the equations (as was done above in the time-domain), one obtains the Helmholtz wave equation

$$(\nabla^2 + \omega^2 \mu \epsilon) \begin{Bmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{H}} \end{Bmatrix} = 0$$

Plane waves are solutions.

$$\begin{Bmatrix} \tilde{\mathbf{E}}(x,y,z,\omega) \\ \tilde{\mathbf{H}}(x,y,z,\omega) \end{Bmatrix} = \begin{Bmatrix} \tilde{\mathbf{E}}(x,y,\omega) e^{\pm j\beta z} \\ \tilde{\mathbf{H}}(x,y,\omega) e^{\pm j\beta z} \end{Bmatrix}$$

It is important to note that the Laplacian operator can always be split into two pieces—one piece which depends only on a variable which varies in the direction of travel (in this case, z) and the other piece which depends only on variables which vary in directions transverse to that of travel (in this case, x and y).

$$(\nabla_t^2 + \omega^2 \mu \epsilon - \beta^2) \begin{Bmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{H}} \end{Bmatrix} = 0 \quad \text{where } \nabla_t^2 = \nabla^2 - \frac{\partial^2}{\partial z^2}$$

Special types of solutions exist in transmission lines (coaxial lines, microstrip, twin-lead) for which the \mathbf{a}_z component of \mathbf{E} and \mathbf{H} are zero. These solutions are transverse electromagnetic waves.

The equation above can be split into two pieces, both of which must be satisfied.

$$\nabla_t^2 \begin{Bmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{H}} \end{Bmatrix} = 0 \quad \text{and} \quad (\omega^2 \mu \epsilon - \beta^2) \begin{Bmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{H}} \end{Bmatrix} = 0$$

In the first, since ∇_t^2 does not involve the variable along the direction of travel,

$$\nabla_t^2 \begin{Bmatrix} \tilde{\mathbf{E}} \\ \tilde{\mathbf{H}} \end{Bmatrix} = \nabla_t^2 \begin{Bmatrix} \tilde{\mathbf{E}}_t + \tilde{\mathbf{E}}_z \\ \tilde{\mathbf{H}}_t + \tilde{\mathbf{H}}_z \end{Bmatrix} = \nabla_t^2 \begin{Bmatrix} \tilde{\mathbf{E}}_t \\ \tilde{\mathbf{H}}_t \end{Bmatrix} = 0 \quad \text{since } \nabla_t^2 \tilde{\mathbf{E}}_z = \nabla_t^2 \tilde{\mathbf{H}}_z = 0$$

In the second, since non-zero solutions for $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{B}}$ are sought, we must have $\omega^2 \mu \epsilon - \beta^2 = 0$ for traveling waves along a transmission line.

Transverse Electromagnetic (TEM) waves

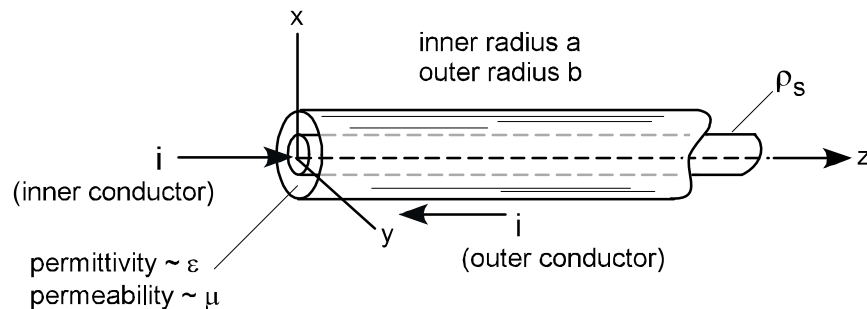
The fields are of the form

$$\begin{Bmatrix} \tilde{\mathbf{E}}_t(x,y,z,\omega) \\ \tilde{\mathbf{H}}_t(x,y,z,\omega) \end{Bmatrix} = \begin{Bmatrix} \tilde{\mathbf{E}}_t(x,y,\omega)e^{\pm j\beta z} \\ \tilde{\mathbf{H}}_t(x,y,\omega)e^{\pm j\beta z} \end{Bmatrix}$$

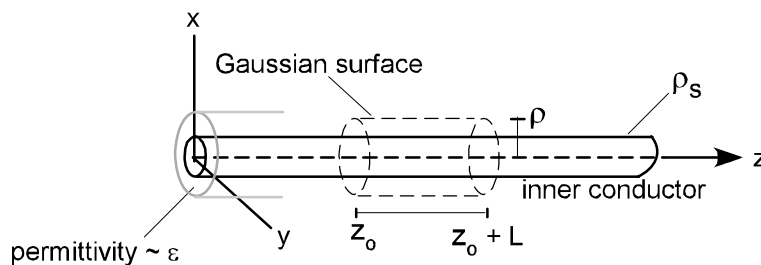
Here the direction of travel is $\pm \mathbf{a}_z$. Waves having a z-dependence of $e^{-j\beta z}$ travel in the $+\mathbf{a}_z$ direction and waves with $e^{j\beta z}$ dependence travel in the $-\mathbf{a}_z$ direction.

Since the transverse fields satisfy $\nabla_t^2 \tilde{\mathbf{E}}_t = 0$ and $\nabla_t^2 \tilde{\mathbf{H}}_t = 0$, the form of the waves in transverse directions are static solutions.

Example 5



To find the *electric field*, first note that, by symmetry, the field must point in the \mathbf{a}_ρ direction (using cylindrical coordinates) and its magnitude can only depend on ρ . This allows the integral form of Gauss' law to be readily used.



Gauss' law, in integral form, reads

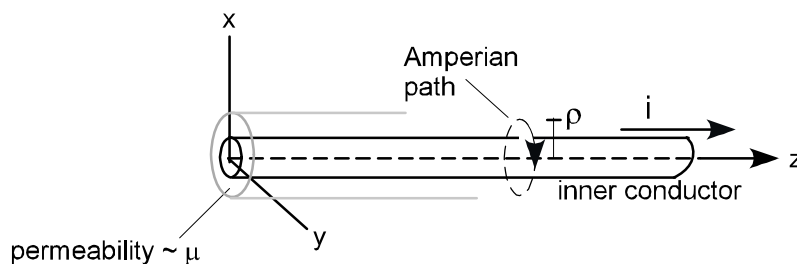
$$\iint_{\text{gaussian surface}} \epsilon \mathbf{E} \cdot d\mathbf{s} = Q_{\text{inside}}$$

Applying Gauss' law for the Gaussian surface shown, for $a < \rho < b$,

$$\int_{z=z_0}^{z_0+L} \int_{\phi=0}^{2\pi} \epsilon \mathbf{E} \mathbf{a}_\rho \cdot \rho d\phi dz \mathbf{a}_\rho = \int_{z=z_0}^{z_0+L} \int_{\phi=0}^{2\pi} \rho_s a d\phi dz$$

$$2\pi \rho L \epsilon E = 2\pi a L \rho_s \Rightarrow \mathbf{E} = \mathbf{a}_\rho \frac{a \rho_s}{\epsilon \rho}$$

To find the *magnetic field*, first note that, by symmetry, the field must point in the \mathbf{a}_ϕ direction (still using cylindrical) and its magnitude can only depend on ρ . This allows the integral form of Ampere's law to be readily used.



Ampere's law, in integral form, reads

$$\oint_{\text{Amperian path}} \mathbf{H} \cdot d\mathbf{l} = I_{\text{inside}}$$

Applying Ampere's law for the Amperian path shown, for $a < \rho < b$,

$$\int_{\phi=0}^{2\pi} \mathbf{H} \mathbf{a}_\rho \cdot \rho d\phi \mathbf{a}_\rho = i \Rightarrow \mathbf{H} = \frac{i}{2\pi\rho} \mathbf{a}_\phi$$

The transverse fields, found using statics, can be then used to find the traveling wave. In the frequency domain these fields are

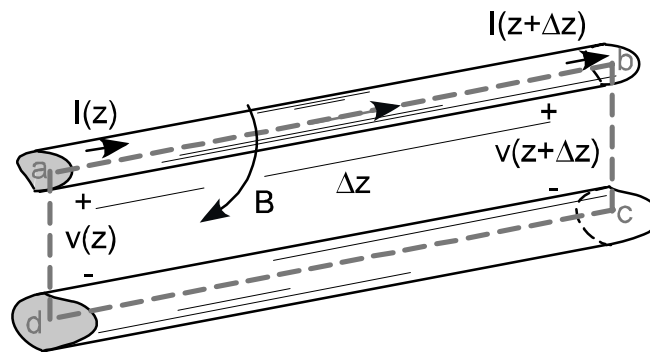
$$\tilde{\mathbf{E}}_t = \mathbf{a}_\rho \frac{a \rho_s}{\epsilon \rho} \Rightarrow \tilde{\mathbf{E}} = \mathbf{a}_\rho \frac{a \rho_s}{\epsilon \rho} e^{\pm j\beta z}$$

$$\tilde{\mathbf{H}}_t = \mathbf{a}_\phi \frac{i}{2\pi\rho} \Rightarrow \tilde{\mathbf{H}} = \mathbf{a}_\phi \frac{i}{2\pi\rho} e^{\pm j\beta z}$$

With transmission lines, one is often not concerned with the details of the fields but rather is often more interested in the terminal current voltage equations.

In his work with transmission lines, Oliver Heaviside developed the telegrapher equations which involve traveling voltage and current waves rather than traveling electric and magnetic fields.

Voltage & Current Waves: Telegrapher's equations



Using Faraday's law, $\oint_{\text{path}} \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \iint_{\text{area bounded by path}} \mathbf{B} \cdot d\mathbf{s}$, where $d\mathbf{l}$ and $d\mathbf{s}$ are related via

the right hand rule. Using frequency-domain field vectors, $\oint_{\text{path}} \tilde{\mathbf{E}} \cdot d\mathbf{l} = -j\omega \iint_{\text{area bounded by path}} \tilde{\mathbf{B}} \cdot d\mathbf{s}$.

Carrying out the path integral, Faraday's law shows that

$$\tilde{V}(z+\Delta z) - \tilde{V}(z) = -j\omega \iint_{\text{area}} \tilde{\mathbf{B}} \cdot d\mathbf{s} = -j\omega \tilde{I} \frac{\iint_{\text{area}} \tilde{\mathbf{B}} \cdot d\mathbf{s}}{\tilde{I}}$$

Where the quantity, $\frac{\iint_{\text{area}} \tilde{\mathbf{B}} \cdot d\mathbf{s}}{\tilde{I}}$, being a ratio of flux to current is clearly the inductance of the loop.

$$\tilde{V}(z+\Delta z) - \tilde{V}(z) = -j\omega \tilde{I} L$$

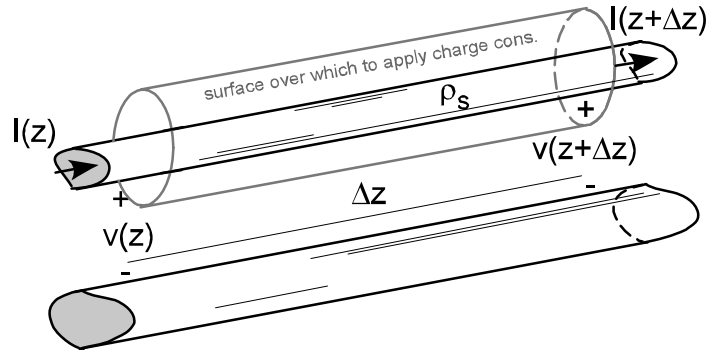
By dividing the the interval Δz , and taking the limit as $\Delta z \rightarrow 0$, one obtains,

$$\lim_{\Delta z \rightarrow 0} \left(\frac{\tilde{V}(z+\Delta z) - \tilde{V}(z)}{\Delta z} \right) = -j\omega \tilde{I} \lim_{\Delta z \rightarrow 0} \frac{L}{\Delta z} = -j\omega \tilde{I} \ell$$

Where L is the inductance per unit length of the transmission line. The result is the first of the Telegrapher equations (this equation is for a line without loss).

$$\frac{d\tilde{V}}{dz} = -j\omega\tilde{L} \mathcal{I}$$

The second telegrapher equation is revealed through charge conservation.



From charge conservation, $I(z) - I(z+\Delta z) = \frac{dQ}{dt} = \frac{d}{dt} \iint_{\text{surface}} \rho_s \, ds = \frac{d}{dt} \iint_{\text{surface}} \mathbf{D} \cdot d\mathbf{s}$.

Using frequency-domain field vectors,

$$\tilde{I}(z) - \tilde{I}(z+\Delta z) = j\omega \iint_{\text{surface}} \tilde{\mathbf{D}} \cdot d\mathbf{s} = j\omega\tilde{V} \frac{\iint_{\text{surface}} \tilde{\mathbf{D}} \cdot d\mathbf{s}}{\tilde{V}}$$

Where the quantity $\frac{\iint_{\text{surface}} \tilde{\mathbf{D}} \cdot d\mathbf{s}}{\tilde{V}}$ is clearly the capacitance of the line section, C .

$$\tilde{I}(z) - \tilde{I}(z+\Delta z) = j\omega\tilde{V} C \quad \Rightarrow \quad \tilde{I}(z+\Delta z) - \tilde{I}(z) = -j\omega\tilde{V} C$$

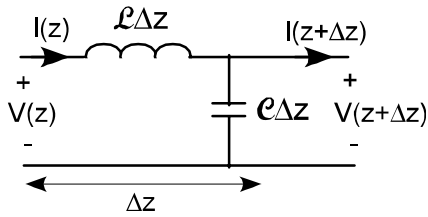
By dividing by Δz and taking the limit as $\Delta z \rightarrow 0$, one arrives at the second Telegrapher equation (again for the case of the lossless transmission line).

$$\lim_{\Delta z \rightarrow 0} \frac{\tilde{I}(z+\Delta z) - \tilde{I}(z)}{\Delta z} = -j\omega\tilde{V} \lim_{\Delta z \rightarrow 0} \frac{C}{\Delta z}$$

$$\frac{d\tilde{I}}{dz} = -j\omega\tilde{V} c$$

Lossless Transmission Lines

circuit model



governing equations

$$\frac{d\tilde{V}}{dz} = -j\omega\tilde{I} \mathcal{L}$$

$$\frac{d\tilde{I}}{dz} = -j\omega\tilde{V} \mathcal{C}$$

Taking a derivative with respect to z of the first Telegrapher equation and substituting the second equation, one obtains:

$$\begin{aligned} \frac{d^2\tilde{V}}{dz^2} = -j\omega\mathcal{L} \frac{d\tilde{I}}{dz} &\Rightarrow \frac{d^2\tilde{V}}{dz^2} = -j\omega\mathcal{L} (-j\omega\mathcal{C}) \tilde{V} \\ &\frac{d^2\tilde{V}}{dz^2} + \omega^2\mathcal{L}\mathcal{C} \tilde{V} = 0 \end{aligned}$$

Similarly,

$$\begin{aligned} \frac{d^2\tilde{I}}{dz^2} = -j\omega\mathcal{C} \frac{d\tilde{V}}{dz} &\Rightarrow \frac{d^2\tilde{I}}{dz^2} = -j\omega\mathcal{C} (-j\omega\mathcal{L}) \tilde{I} \\ &\frac{d^2\tilde{I}}{dz^2} + \omega^2\mathcal{L}\mathcal{C} \tilde{I} = 0 \end{aligned}$$

Solutions are traveling waves

$$\begin{aligned} \tilde{V} &= \tilde{V}^+ e^{-j\beta z} + \tilde{V}^- e^{j\beta z} \quad \text{where} \quad \tilde{V}^+ = V^+ e^{j\phi_v^+} \quad \text{and} \quad \tilde{V}^- = V^- e^{j\phi_v^-} \\ \tilde{I} &= \tilde{I}^+ e^{-j\beta z} + \tilde{I}^- e^{j\beta z} \quad \text{where} \quad \tilde{I}^+ = I^+ e^{j\phi_i^+} \quad \text{and} \quad \tilde{I}^- = I^- e^{j\phi_i^-} \end{aligned}$$

where $\beta = \omega\sqrt{\mathcal{L}\mathcal{C}}$ is the phase constant.

Consider a voltage wave traveling in the $+z$ direction and determine the relation between voltage and current. The ratio between the voltage and current waves is the transmission line's *characteristic impedance*, Z_c .

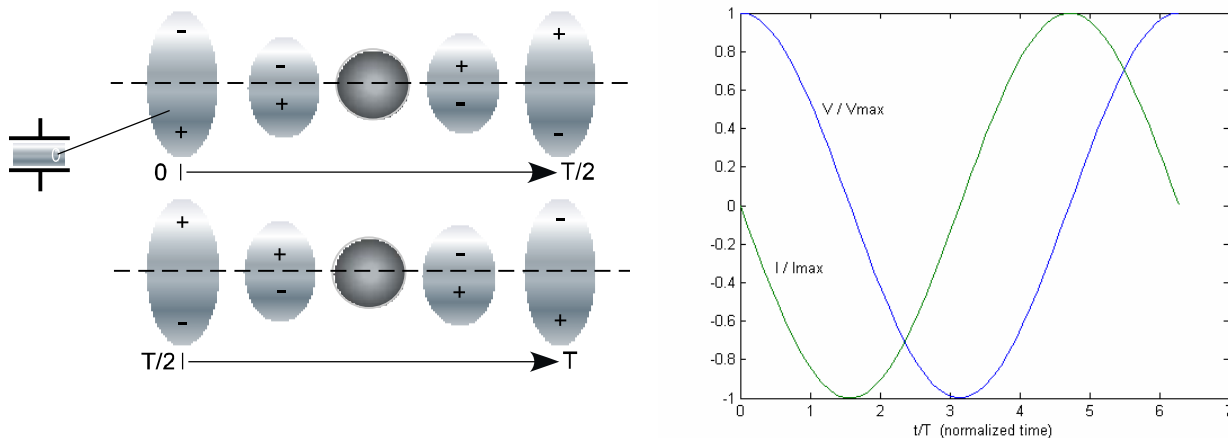
$$\begin{aligned} \text{using } \tilde{V} &= \tilde{V}^+ e^{-j\beta z} \text{ in the Telegrapher eq. } \frac{d\tilde{V}}{dz} = -j\omega\tilde{I} \mathcal{L}, \\ -j\beta\tilde{V}^+ e^{-j\beta z} &= -j\omega\mathcal{L}\tilde{I} \quad \Rightarrow \quad \tilde{I} = \frac{-j\beta}{-j\omega\mathcal{L}} \tilde{V}^+ e^{-j\beta z} = \frac{-j\omega\sqrt{\mathcal{L}\mathcal{C}}}{-j\omega\mathcal{L}} \tilde{V}^+ e^{-j\beta z} \\ \tilde{I} &= \frac{\tilde{V}^+}{\sqrt{\mathcal{L}/\mathcal{C}}} e^{-j\beta z} \quad \Rightarrow \quad Z_c = \sqrt{\frac{\mathcal{L}}{\mathcal{C}}} \end{aligned}$$

Lossy Transmission Lines

Transmission lines serve to guide electromagnetic waves. As the waves travel along the transmission lines, there are three main sources of loss: 1) Joule heating or copper losses, 2) dielectric losses, and 3) radiation.

Joule Heating – as the EM wave travels along the conductors of the transmission line, the electromagnetic wave enters the conductors—typically a metal—where it decays exponentially. Even though the wave decays once it enters the TL conductors, the EM wave does enter them to a degree. Inside the conductor, the EM field accelerates free electrons, which then collide with metal atoms to make them vibrate. The result is that the EM wave heats the metal. This process is called Joule heating and is usually the dominant loss mechanism in transmission lines.

Dielectric Losses – a dielectric is formed by fixed positive nuclei to which are bound negatively charged electrons. In an ideal dielectric, an applied electric field results in charge separation as shown below. Notice that, for an ideal dielectric, the force and



motion are in the same direction for $\frac{1}{4}$ of a period (or equivalently the voltage and current have the same sign for $\frac{1}{4}$ of a period) and then are in opposite directions for the next $\frac{1}{4}$ period.

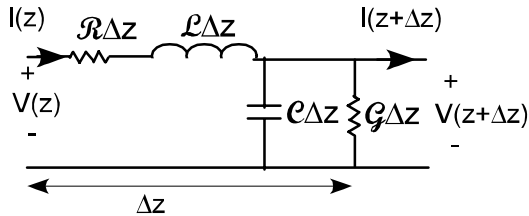
The result is that work is done on the dielectric (energy is stored) for $\frac{1}{4}$ of a period which is then given back in the next $\frac{1}{4}$ period. This is the physical basis of capacitance.

For a non-ideal dielectric, the charges have non-zero mass and their motion lags that shown. The result is that the current is no longer 90° out of phase with the voltage which results in energy loss from the EM wave.

Radiation Losses – anytime charge is accelerated, an electromagnetic wave is emitted. This loss mechanism tends to become important at higher frequencies or whenever the physical structure becomes a significant portion of a wavelength—especially when the current path is separated from its return path.

The transmission line model which includes loss is
circuit model

governing equations



$$\frac{d\tilde{V}}{dz} = -(\mathcal{R} + j\omega\mathcal{L})\tilde{I}$$

$$\frac{d\tilde{I}}{dz} = -(\mathcal{G} + j\omega\mathcal{C})\tilde{V}$$

As with the lossless lines, traveling waves result:

$$\frac{d^2\tilde{V}}{dz^2} = -(\mathcal{R} + j\omega\mathcal{L})\frac{d\tilde{I}}{dz} \Rightarrow \frac{d^2\tilde{V}}{dz^2} = -(\mathcal{R} + j\omega\mathcal{L})[-(\mathcal{G} + j\omega\mathcal{C})]\tilde{V} = \tilde{\mathcal{Z}}\tilde{\mathcal{Y}}\tilde{V} = \tilde{\gamma}^2\tilde{V}$$

$$\frac{d^2\tilde{V}}{dz^2} - \tilde{\gamma}^2\tilde{V} = 0$$

Similarly for the current wave

$$\frac{d^2\tilde{I}}{dz^2} - \tilde{\gamma}^2\tilde{I} = 0$$

where

$\tilde{\mathcal{Z}} = (\mathcal{R} + j\omega\mathcal{L})$ is the series impedance,
 $\tilde{\mathcal{Y}} = \mathcal{G} + j\omega\mathcal{C}$ is the parallel admittance,
 and $\tilde{\gamma} = \sqrt{\tilde{\mathcal{Z}}\tilde{\mathcal{Y}}}$ is the propagation constant.

The propagation constant, $\tilde{\gamma}$, is complex.

$\tilde{\gamma} = \alpha + j\beta$, where α is the attenuation constant and β is the phase constant.

Traveling wave solutions

$$\tilde{V} = \tilde{V}^+e^{-\tilde{\gamma}z} + \tilde{V}^-e^{\tilde{\gamma}z} = \tilde{V}^+e^{-\alpha z}e^{-j\beta z} + \tilde{V}^-e^{\alpha z}e^{j\beta z}$$

$$\tilde{I} = \tilde{I}^+e^{-\tilde{\gamma}z} + \tilde{I}^-e^{\tilde{\gamma}z} = \tilde{I}^+e^{-\alpha z}e^{-j\beta z} + \tilde{I}^-e^{\alpha z}e^{j\beta z}$$

In the time domain

$$v(t) = V^+e^{-\alpha z}\cos(\omega t - \beta z + \phi_v^+) + V^-e^{\alpha z}\cos(\omega t + \beta z + \phi_v^-)$$

$$i(t) = I^+e^{-\alpha z}\cos(\omega t - \beta z + \phi_i^+) + I^-e^{\alpha z}\cos(\omega t + \beta z + \phi_i^-)$$

Low Loss Transmission Lines

When transmission lines are lossy, nearly always they are low-loss ($\mathcal{R} \ll \omega\mathcal{L}$ and $\mathcal{G} \ll \omega\mathcal{C}$). The general case is seldom and will not be treated here. For the more common low loss case, the expression for the propagation constant, $\tilde{\gamma}$, can be simplified.

$$\tilde{\gamma} = \sqrt{\tilde{\mathcal{Z}}\tilde{\mathcal{Y}}} = \sqrt{(\mathcal{R} + j\omega\mathcal{L})(\mathcal{G} + j\omega\mathcal{C})} = j\omega\sqrt{\mathcal{L}\mathcal{C}} \sqrt{\left(1 - j\frac{\mathcal{R}}{\omega\mathcal{L}}\right)\left(1 - j\frac{\mathcal{G}}{\omega\mathcal{C}}\right)}$$

$$\tilde{\gamma} \cong j\omega\sqrt{\mathcal{L}\mathcal{C}} \left(1 - j\frac{\mathcal{R}}{2\omega\mathcal{L}}\right)\left(1 - j\frac{\mathcal{G}}{2\omega\mathcal{C}}\right) \text{ for } \mathcal{R} \ll \omega\mathcal{L} \text{ and } \mathcal{G} \ll \omega\mathcal{C}$$

This can be further simplified.

$$\tilde{\gamma} \cong j\omega\sqrt{\mathcal{L}\mathcal{C}} \left(1 - \frac{\mathcal{G}\mathcal{R}}{4\omega^2\mathcal{L}\mathcal{C}} - j\frac{\mathcal{R}}{2\omega\mathcal{L}} - j\frac{\mathcal{G}}{2\omega\mathcal{C}}\right)$$

The second-order term can be neglected.

$$\tilde{\gamma} \cong j\omega\sqrt{\mathcal{L}\mathcal{C}} \left(1 - j\frac{\mathcal{R}}{2\omega\mathcal{L}} - j\frac{\mathcal{G}}{2\omega\mathcal{C}}\right) = \frac{\mathcal{R}}{2}\sqrt{\frac{\mathcal{C}}{\mathcal{L}}} + \frac{\mathcal{G}}{2}\sqrt{\frac{\mathcal{L}}{\mathcal{C}}} + j\omega\sqrt{\mathcal{L}\mathcal{C}}$$

$$\tilde{\gamma} = \alpha + j\beta \cong \frac{\mathcal{R}}{2}\sqrt{\frac{\mathcal{C}}{\mathcal{L}}} + \frac{\mathcal{G}}{2}\sqrt{\frac{\mathcal{L}}{\mathcal{C}}} + j\omega\sqrt{\mathcal{L}\mathcal{C}}$$

Most often copper loss dominates the attenuation constant. In this case

$\tilde{\gamma} = \alpha + j\beta \cong \frac{\mathcal{R}}{2}\sqrt{\frac{\mathcal{C}}{\mathcal{L}}} + j\omega\sqrt{\mathcal{L}\mathcal{C}}$	(low-loss where α is determined by copper loss)
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What is the nature of this propagation constant? First, the phase constant is the same as that found for the lossless case. This gives the important result that the phase velocity of the wave is the same as for the lossless case.

$$v_p = \frac{\omega}{\beta} = \sqrt{\frac{1}{\mathcal{L}\mathcal{C}}}$$

Second, α is usually small, which might lead one to question whether it is really necessary or whether one might assume that $\alpha = 0$. Since the attenuation factor is the product of αz , the answer to this question depends on 1) how long the transmission line is and 2) what magnitude changes are considered important in a particular application.

Example 6

Consider a transmission line with

$$\mathcal{L} = 5 \mu\text{H/m}, \mathcal{C} = 0.02 \mu\text{F/m}, \mathcal{R} = 0.01 \Omega/\text{m}, \mathcal{G} = 0 \text{ and } f = 1 \text{ GHz}$$

- i) How long would the transmission line need to be for a voltage wave to suffer 5% attenuation?
- ii) What is the speed of propagation?
- iii) What is the wavelength?
- iv) What is the characteristic impedance?

Solutions

Using these values, the propagation constant can be determined.

$$\tilde{\gamma} = \alpha + j\beta \cong \frac{\mathcal{R}}{2} \sqrt{\frac{\mathcal{C}}{\mathcal{L}}} + j\omega\sqrt{\mathcal{L}\mathcal{C}} = (0.000316 + j1986.9) \text{ m}^{-1}$$

- i) Let the length of line = d.

$$e^{-0.000316d} = 0.95 \quad \Rightarrow \quad d = 162.3 \text{ m}$$

In this case, even though α is quite small, if $d = 162.3 \text{ m}$, there will be 5% attenuation. α , even if small, cannot be ignored for long lines.

ii) $v_p = \frac{\omega}{\beta} = \frac{2\pi f}{\beta} = \sqrt{\frac{1}{\mathcal{L}\mathcal{C}}} = 3.16(10^6) \text{ m/s}$

iii) $\lambda = \frac{2\pi}{\beta} = \frac{v_p}{f} = 3.16 \text{ mm}$

- iv) Strictly speaking the characteristic equation for a lossy line is complex.

$$\tilde{Z}_c = \sqrt{\frac{\tilde{Z}}{\tilde{Y}}} = \sqrt{\frac{\mathcal{R} + j\omega\mathcal{L}}{\mathcal{G} + j\omega\mathcal{C}}} = [15.8 - j2.5(10^{-6})] \Omega = 15.8 \angle -9.1(10^{-6})^\circ \Omega \cong 15.8 \Omega$$

This example makes it clear that, even though \tilde{Z}_c is actually complex, it is so nearly real that it hardly matters in nearly all practical cases. Therefore, to determine the characteristic impedance, one would use

$$Z_c = \sqrt{\frac{\mathcal{L}}{\mathcal{C}}} = 15.8 \Omega$$

Example 7

Consider the following RG-58 coaxial cable operating at 100 MHz. The specifications provided by the manufacturer are

$$\epsilon = 30 \text{ pF/ft} \quad v_p = 2(10)^8 \text{ m/s} \quad Z_c = 50 \Omega \quad \alpha = 4.5 \text{ dB/100ft}$$

- i) Convert ϵ to pF/m
- ii) From ϵ and Z_c , determine \mathcal{L} (could also use ϵ and v_p **or** v_p and Z_c)
- iii) Convert α to nepers/m, then find \mathcal{R} .

$$\text{i) } \epsilon = 30 \frac{\text{pF}}{\text{ft}} \left(\frac{1 \text{ ft}}{12 \text{ in}} \right) \left(\frac{1 \text{ in}}{0.0254 \text{ m}} \right) = 98.4 \text{ pF/m}$$

$$\text{ii) } Z_c = \sqrt{\frac{\mathcal{L}}{\epsilon}} \Rightarrow \mathcal{L} = Z_c^2 \epsilon = (50 \Omega)^2 98.4(10^{-12}) \text{ F/m} = 0.246 \mu\text{H/m}$$

$$\text{iii) } \alpha = 0.045 \frac{\text{dB}}{\text{ft}} \left(\frac{1 \text{ neper}}{8.686 \text{ dB}} \right) \left(\frac{1 \text{ ft}}{12 \text{ in}} \right) \left(\frac{1 \text{ in}}{0.0254 \text{ m}} \right) = 0.017 \frac{\text{neper}}{\text{m}} = 0.017 \text{ m}^{-1}$$

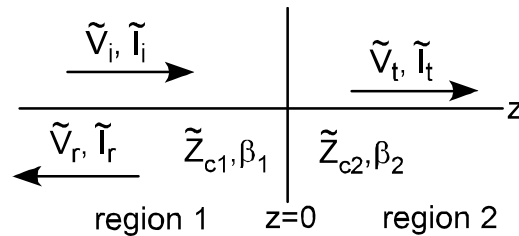
$$\text{for } G = 0, \alpha = \frac{\mathcal{R}}{2} \sqrt{\frac{\epsilon}{\mathcal{L}}} \Rightarrow \mathcal{R} = 2\alpha \sqrt{\frac{\mathcal{L}}{\epsilon}} = 2\alpha Z_c = 1.7 \Omega/\text{m}$$

$$\text{Check } Z_c = \sqrt{\frac{\mathcal{L}}{\epsilon}} = \sqrt{\frac{2.46(10^{-7}) \text{ H/m}}{9.84(10^{-11}) \text{ F/m}}} = 50 \Omega$$

$$\text{Check } v_p = \sqrt{\frac{1}{\mathcal{L}\epsilon}} = \sqrt{\frac{1}{[2.46(10^{-7}) \text{ H/m}] [9.84(10^{-11}) \text{ F/m}]}} = 2.0(10^8) \text{ m/s}$$

Time Domain Reflectometry

Just as traveling electromagnetic waves are both reflected and transmitted at discontinuities, so are traveling voltage and current waves. With EM waves, the boundary conditions were continuity of the tangential components of the electric and magnetic vectors. With voltage and current waves, Kirchoff's Voltage Law (KVL) and Kirchoff's Current Law (KCL) furnish the necessary boundary conditions.



$$\tilde{\Gamma} = \left. \frac{\tilde{V}_r}{\tilde{V}_i} \right|_{z=0} \quad \tilde{T} = \left. \frac{\tilde{V}_t}{\tilde{V}_i} \right|_{z=0}$$

$$\tilde{V}_i + \tilde{V}_r = \tilde{V}_t \quad \Rightarrow \quad \tilde{V}_i(1 + \tilde{\Gamma}) = \tilde{V}_i\tilde{T}$$

$$\tilde{I}_i - \tilde{I}_r = \tilde{I}_t \quad \Rightarrow \quad \frac{\tilde{V}_i}{\tilde{Z}_{c1}}(1 - \tilde{\Gamma}) = \frac{\tilde{V}_i}{\tilde{Z}_{c2}}\tilde{T} \Rightarrow$$

Solving for $\tilde{\Gamma}$ and \tilde{T}

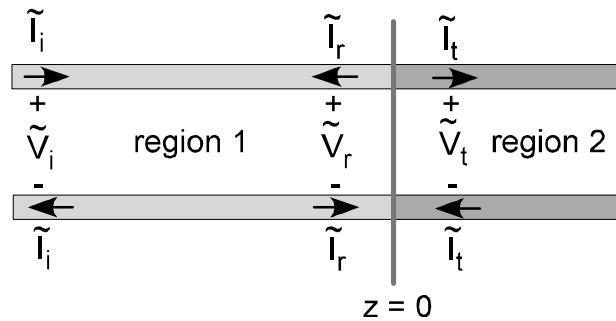
$$\tilde{\Gamma} = \frac{\tilde{Z}_{c2} - \tilde{Z}_{c1}}{\tilde{Z}_{c2} + \tilde{Z}_{c1}}$$

$$\tilde{T} = \frac{2\tilde{Z}_{c2}}{\tilde{Z}_{c2} + \tilde{Z}_{c1}}$$

The above boundary conditions may need some explanation. The boundary conditions state that, at $z=0$, the total voltage in region 1 ($\tilde{V}_i + \tilde{V}_r$) must equal the total voltage in region 2 (\tilde{V}_t). Likewise the total current in the \mathbf{a}_z direction in region 1 at $z=0$ ($\tilde{I}_i - \tilde{I}_r$) must equal the current in the \mathbf{a}_z direction at $z=0$ in region 2 (\tilde{I}_t).

Sometimes it seems unclear why the total current in region 1 is $\tilde{I}_i - \tilde{I}_r$ and **not** $\tilde{I}_i + \tilde{I}_r$.

The more detailed diagram below may help.



From this diagram, it is clear that the total voltage in region 1 is $\tilde{V}_i + \tilde{V}_r$ and that the total current in region 1 is $\tilde{I}_i - \tilde{I}_r$.

Lumped-element loads

The reflection coefficient would be unchanged if, instead of a second medium with characteristic impedance \tilde{Z}_{c2} , there were a lumped element with this same impedance,

$$\tilde{Z}_L = \tilde{Z}_{c2}.$$

$$\tilde{\Gamma} = \frac{\tilde{Z}_L - \tilde{Z}_{c1}}{\tilde{Z}_L + \tilde{Z}_{c1}}$$

$$\tilde{T} = \frac{2\tilde{Z}_L}{\tilde{Z}_L + \tilde{Z}_{c1}}$$

Consider three important special cases.

- 1) Matched load ($\tilde{Z}_L = \tilde{Z}_{c1}$)

$$\tilde{\Gamma} = 0, \text{ no reflections.}$$

- 2) Short circuit ($\tilde{Z}_L = 0$)

$$\tilde{\Gamma} = -1, \text{ reflected voltage cancels incident voltage } (\tilde{V}_1|_{z=0} = (\tilde{V}_i + \tilde{V}_r)|_{z=0} = 0)$$

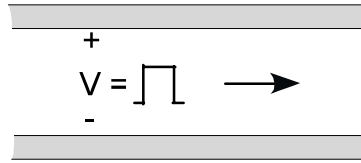
- 3) Open circuit ($\tilde{Z}_L = \infty$)

$$\tilde{\Gamma} = 1, \text{ reflected voltage is equal to the incident voltage. The total voltage at } z=0 \text{ is twice the incident voltage}$$

In general, for a passive load, the reflection coefficient $\tilde{\Gamma}$ exists on or within the unit circle in the complex plane.

Traveling Waves

Consider a voltage pulse traveling down a transmission line.

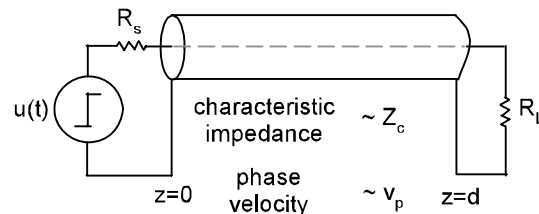


In the frequency domain, this voltage pulse consists of a range of frequencies. If losses are not a function of frequency (that is, if α is not a function of frequency) and if the phase velocity, v_p , is not a function of frequency, the relationships between the pulse's frequency components will remain unchanged as it travels and the pulse will retain its shape. Losses result in the pulse decreasing in magnitude, but, if v_p is independent of frequency, the pulse's shape will be unchanged as it travels.

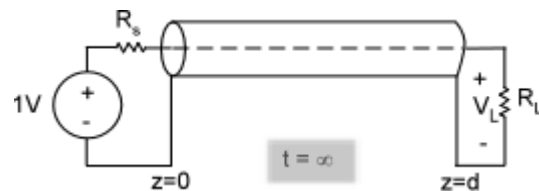
Going further, if the transmission line is lossless (if the losses can be neglected), and if v_p is not a function of frequency, the pulse will remain unchanged, both in amplitude and in shape as it travels along the transmission line. This is the important special case that will be considered in regards to Time Domain Reflectometry (TDR).

Transient Waves on Lossless lines

Consider the situation below where a transmission line is fed from a voltage source with a source resistance, R_s .



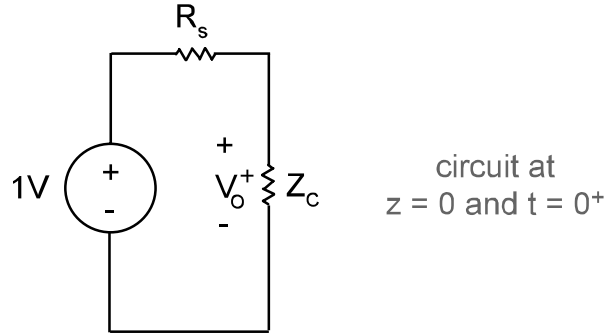
How will this system's transient response evolve? There is no doubt regarding the eventual response. As time elapses, the response will be that of a simple two resistor circuit—the transmission line having no effect as it is just two conductors for DC.



$$\text{Where } V_L = \frac{R_L}{R_s + R_L} 1V.$$

How does the system evolve, beginning at $t = 0^+$ to $t \rightarrow \infty$?

At $t = 0^+$, the only location involved is at $z = 0$. At $t = 0^+$, the system, at $z = 0$, can be modeled by two resistors in series— R_s and Z_c .



Where $V_o^+ = \frac{Z_c}{R_s + Z_c} 1V$. The initial traveling step voltage can be described as

$$V_o^+(z,t) = \frac{Z_c}{R_s + Z_c} u\left(t - \frac{z}{v_p}\right)$$

This initial step voltage will travel down the transmission line until it encounters the discontinuity at the load with a reflection coefficient, $\tilde{\Gamma}_L$, at which point a reflected wave, $V_o^-(z,t)$, traveling in the $-\mathbf{a}_z$ direction, will be created.

$$\tilde{\Gamma}_L = \frac{R_L - Z_c}{R_L + Z_c}$$

$$V_o^-(z,t) = \frac{Z_c}{R_s + Z_c} \tilde{\Gamma}_L u\left[t - \frac{d}{v_p} - \frac{(d-z)}{v_p}\right] = \frac{Z_c}{R_s + Z_c} \tilde{\Gamma}_L u\left[t - \frac{2d}{v_p} + \frac{z}{v_p}\right]$$

This reflected wave will then travel in the $-\mathbf{a}_z$ direction until it encounters the discontinuity at the source and will, itself be reflected with a reflection coefficient, $\tilde{\Gamma}_s$.

$$\tilde{\Gamma}_s = \frac{R_s - Z_c}{R_s + Z_c}$$

$$V_1^+(z,t) = \frac{Z_c}{R_s + Z_c} \tilde{\Gamma}_L \tilde{\Gamma}_s u\left[t - \frac{2d}{v_p} - \frac{z}{v_p}\right]$$

The wave reflected at the source will then travel to the load where a wave will be reflected to travel to the source, etc.

For the n^{th} wave traveling in the $+\mathbf{a}_z$ direction

$$V_n^+(z,t) = \frac{Z_c}{R_s + Z_c} (\tilde{\Gamma}_L \tilde{\Gamma}_s)^n u \left[t - \frac{2nd}{v_p} - \frac{z}{v_p} \right]$$

For the n^{th} wave traveling in the $-\mathbf{a}_z$ direction

$$V_n^-(z,t) = \frac{Z_c}{R_s + Z_c} \tilde{\Gamma}_L (\tilde{\Gamma}_L \tilde{\Gamma}_s)^n u \left[t - \frac{2(n+1)d}{v_p} + \frac{z}{v_p} \right]$$

The voltage, $v(z,t)$, is the sum of these waves

$$V(z,t) = \sum_{n=0}^N \frac{Z_c}{R_s + Z_c} (\tilde{\Gamma}_L \tilde{\Gamma}_s)^n u \left[t - \frac{2nd}{v_p} - \frac{z}{v_p} \right] + \sum_{n=0}^N \frac{Z_c}{R_s + Z_c} \tilde{\Gamma}_L (\tilde{\Gamma}_L \tilde{\Gamma}_s)^n u \left[t - \frac{2(n+1)d}{v_p} + \frac{z}{v_p} \right]$$

What form does this expression have for $t \rightarrow \infty$?

$$V(\infty) = \sum_{n=0}^{\infty} \frac{Z_c}{R_s + Z_c} (\tilde{\Gamma}_L \tilde{\Gamma}_s)^n u \left[\infty - \frac{2nd}{v_p} - \frac{z}{v_p} \right] + \sum_{n=0}^{\infty} \frac{Z_c}{R_s + Z_c} \tilde{\Gamma}_L (\tilde{\Gamma}_L \tilde{\Gamma}_s)^n u \left[\infty - \frac{2(n+1)d}{v_p} + \frac{z}{v_p} \right]$$

$$V(\infty) = \sum_{n=0}^{\infty} \frac{Z_c}{R_s + Z_c} (\tilde{\Gamma}_L \tilde{\Gamma}_s)^n (1 + \tilde{\Gamma}_L) = \frac{Z_c}{R_s + Z_c} (1 + \tilde{\Gamma}_L) \sum_{n=0}^{\infty} (\tilde{\Gamma}_L \tilde{\Gamma}_s)^n$$

$$V(\infty) = \frac{Z_c}{R_s + Z_c} \frac{(1 + \tilde{\Gamma}_L)}{1 - \tilde{\Gamma}_L \tilde{\Gamma}_s}$$

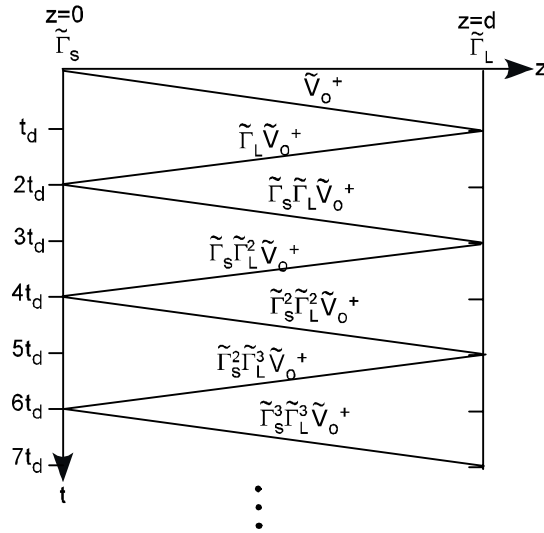
Where the formula for the geometric series, $\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$, for $a < 1$ was utilized.

A bit of algebra shows that this simplifies (as it must) to

$$V(\infty) = \frac{R_L}{R_s + R_L}$$

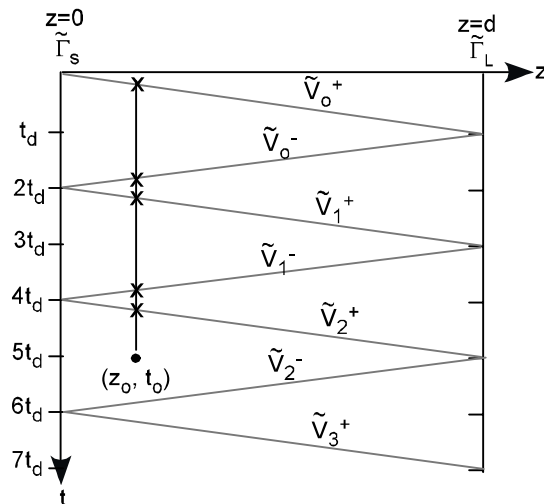
Bounce Diagrams

This process is often analyzed using bounce diagrams. A bounce diagram is a plot in the z - t plane of the positive and negatively traveling waves.



A signal travels the length of the transmission line ($z=d$) in one delay time, t_d . As can be seen in the above diagram, the slope of the transmission line is inversely proportional to the phase velocity. The value of the voltage at particular values of z and t is found by summing the voltages above this point in the bounce diagram.

Example 8

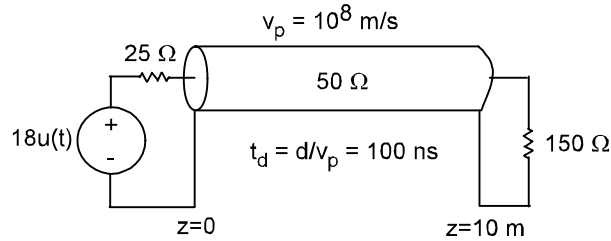


The voltage at a particular position and time (z_o and t_o in this example) is found by adding all the voltages above the point in the bounce diagram plot.

$$\tilde{V}(z_o, t_o) = \tilde{V}_o^+ + \tilde{V}_o^- + \tilde{V}_1^+ + \tilde{V}_1^- + \tilde{V}_2^+$$

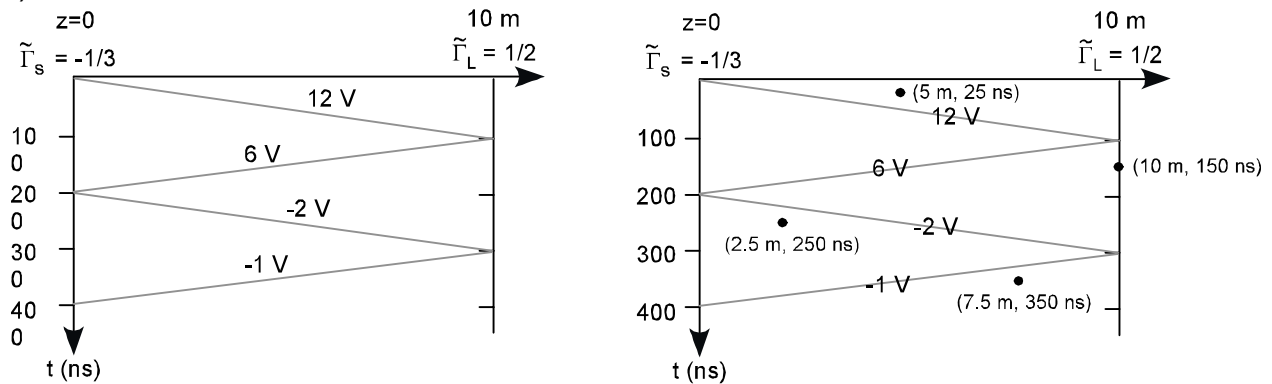
Example 9

Given the system below



- Complete a bound diagram to $t = 4t_d$.
- Find $v(5 \text{ m}, 25 \text{ ns})$, $v(10 \text{ m}, 150 \text{ ns})$, $v(7.5 \text{ m}, 350 \text{ ns})$, and $v(2.5 \text{ m}, 250 \text{ ns})$
- Determine $v_L(t \rightarrow \infty)$
- Sketch $v(5 \text{ m}, t)$ and $v(10 \text{ m}, t)$ for $0 < t < 4t_d$.

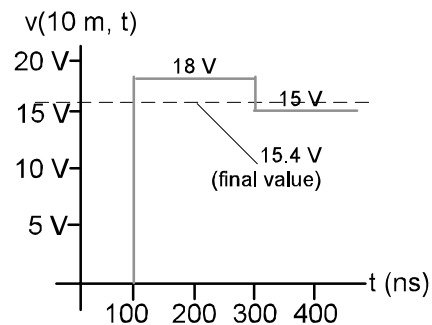
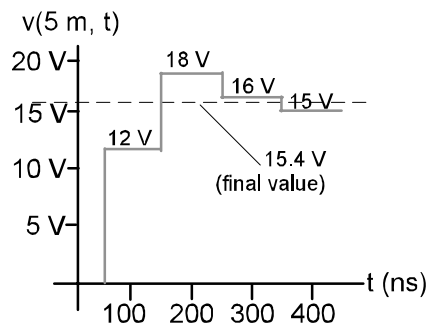
i)



- $v(5 \text{ m}, 25 \text{ ns}) = 0 \text{ V}$
 $v(10 \text{ m}, 150 \text{ ns}) = 12 \text{ V} + 6 \text{ V} = 18 \text{ V}$
 $v(7.5 \text{ m}, 350 \text{ ns}) = 12 \text{ V} + 6 \text{ V} - 2 \text{ V} - 1 \text{ V} = 15 \text{ V}$
 $v(2.5 \text{ m}, 250 \text{ ns}) = 12 \text{ V} + 6 \text{ V} - 2 \text{ V} = 16 \text{ V}$

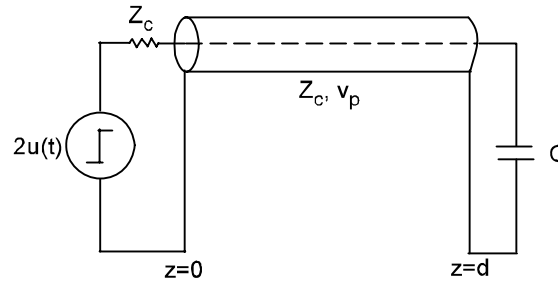
- $$v_L(t \rightarrow \infty) = 18 \text{ V} \left(\frac{150 \Omega}{25 \Omega + 150 \Omega} \right) \cong 15.4 \text{ V}$$

iv)



Reactive Loads

When transmission lines are terminated in reactive elements, the reflection varies with time.



For this case, the transmission line terminated in a capacitance, the reflection coefficient at the load will vary from -1 (when the incident wave first encounters the capacitance) to 1 (after the capacitor is fully charged).

Physical Explanation

When the positively traveling voltage wave impinges upon the capacitance, the capacitance will be uncharged. The voltage across a capacitance cannot change instantaneously. This fact can be seen in the element relation for the capacitance.

$$i = C \frac{dv}{dt} \Rightarrow dv = \frac{1}{C} i dt = \frac{dq}{C}$$

Clearly, charge cannot accumulate instantaneously which then prevents an instantaneous change in capacitance voltage.

This is important here because, when the positively traveling wave first impinges on the capacitive load, the capacitor will remain uncharged and the voltage across it will remain zero. At that instant, the capacitor will act as a short circuit, so will require the sum of the incident wave and reflected wave—at that instant—to be zero. For $t=t_d$, $\tilde{\Gamma}_L = -1$.

As time elapses and the capacitance becomes fully charged, its current will become zero and the capacitance will act as an open circuit. For large t , $\tilde{\Gamma}_L = 1$.

For $t > t_d$, the reflection coefficient changes from -1 to +1. The rate of change is that of a first-order circuit with a time constant $\tau = Z_c C$.

Analysis

$$V_o^+(z,t) = u\left(t - \frac{z}{v_p}\right) \Rightarrow V_o^+(d,t) = u\left(t - \frac{d}{v_p}\right)$$

In the s-domain $V_o^+(d,s) = \frac{e^{-\frac{d}{v_p}s}}{s}$

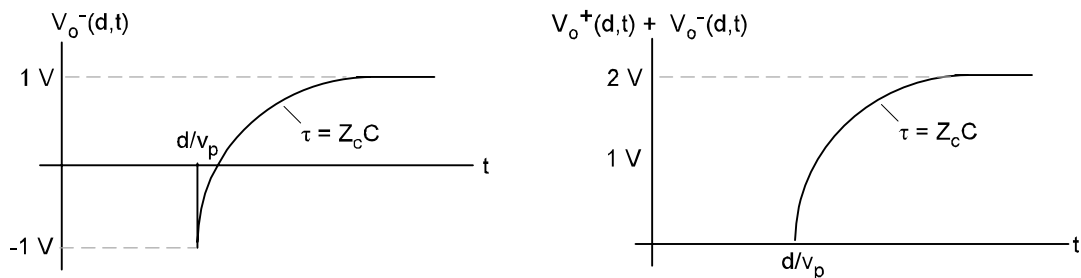
$$V_o^-(d,s) = \Gamma_L \frac{e^{-\frac{d}{v_p}s}}{s} = \frac{1}{\frac{1}{sC} + Z_c} - Z_c \frac{e^{-\frac{d}{v_p}s}}{s}, \text{ which, using partial fraction expansion, becomes}$$

$$V_o^-(d,s) = \frac{e^{-\frac{d}{v_p}s}}{s} - 2 \frac{e^{-\frac{d}{v_p}s}}{s + \frac{1}{Z_c C}}$$

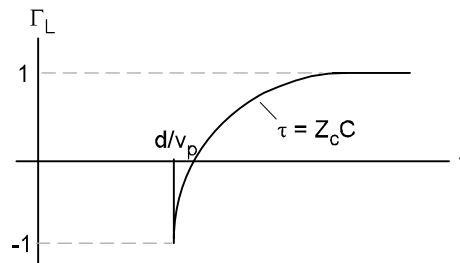
The inverse transform for this s-domain expression is readily found

$$V_o^-(d,t) = \left[1 - 2 e^{-\frac{(t - d/v_p)}{Z_c C}} \right] u\left(t - \frac{d}{v_p}\right)$$

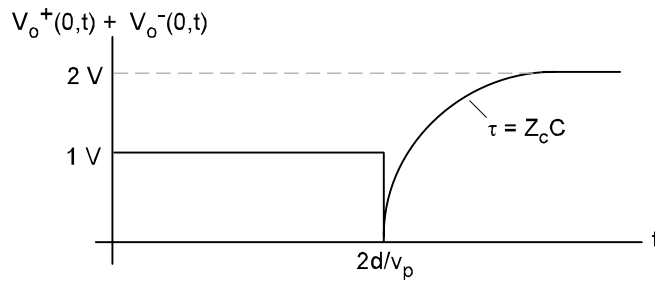
The reflected voltage and the total voltage at the load can be sketched.



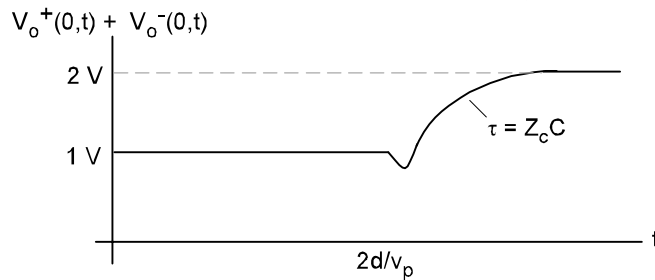
The reflection coefficient evolves with a time constant of $Z_c C$,



Consider, for the same system, the voltage at the source. Note that the source is matched to the transmission line so that $\tilde{\Gamma}_s = 0$.

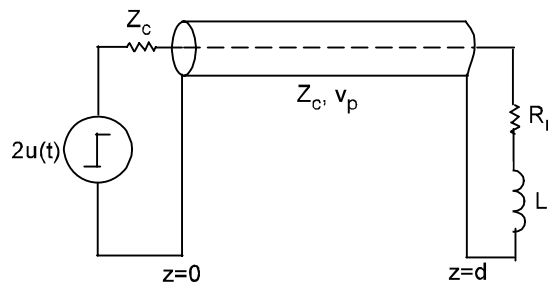


Note: Due to losses, curves measured in the laboratory will not have as sharp as the ideal model predicts. An actual curve may look something like



Example 10

As a second example of reactive loads consider a RL load—an inductance in series with a resistance as shown below.



As with the previous example, the source is matched to the line so that there will be no \tilde{V}_1^+ wave (or \tilde{V}_1^- , \tilde{V}_2^+ , \tilde{V}_2^- , etc.).

Analysis

$$V_o^+(d,t) = u\left(t - \frac{d}{v_p}\right) \quad \Rightarrow \quad V_o^+(d,s) = \frac{e^{-\frac{d}{v_p}s}}{s}$$

$$V_o^-(d,s) = \Gamma_L \frac{e^{-\frac{d}{v_p}s}}{s} = \frac{R_L + sL - Z_c}{R_L + sL + Z_c} \frac{e^{-\frac{d}{v_p}s}}{s}, \text{ which, using partial fraction expansion, becomes}$$

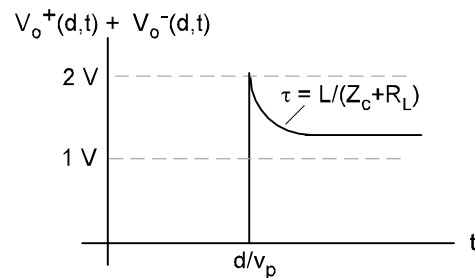
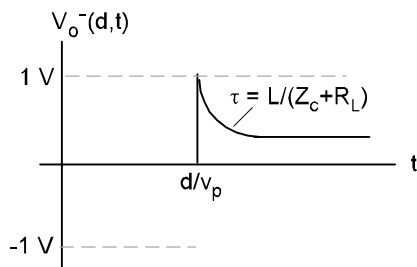
$$V_o^-(d,s) = \frac{R_L - Z_c}{R_L + Z_c} \frac{e^{-\frac{d}{v_p}s}}{s} + \frac{2Z_c}{R_L + Z_c} \frac{e^{-\frac{d}{v_p}s}}{s + \frac{R_L + Z_c}{L}}$$

$$V_o^-(d,t) = \frac{R_L}{R_L + Z_c} u\left(t - \frac{d}{v_p}\right) + \frac{Z_c}{R_L + Z_c} \left(2e^{-\left(\frac{R_L + Z_c}{L}\right)\left(t - \frac{d}{v_p}\right)} - 1 \right) u\left(t - \frac{d}{v_p}\right)$$

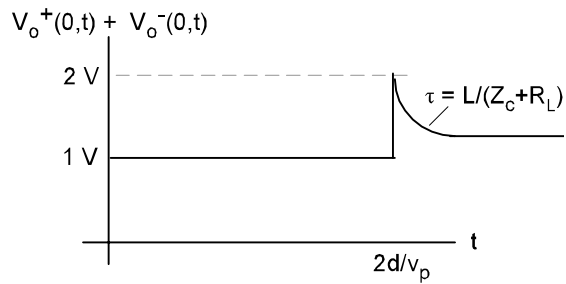
check: $V_o^-(d, t_d) = 1$ (for $t = t_d^+$, the inductance acts as an open circuit)

check: $V_o^-(t \rightarrow \infty) = \frac{R_L - Z_c}{R_L + Z_c}$ (for $t \rightarrow \infty$, the inductance becomes a short circuit)

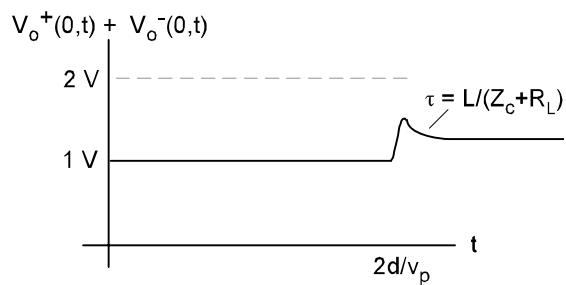
At the load, $z=d$, for the reflected wave (left graph) and for the total voltage (right graph).



Total voltage at the input to the transmission line, $z=0$.

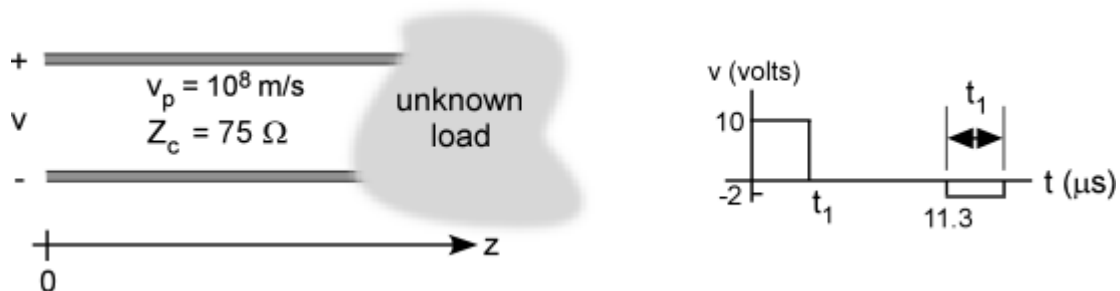


Note: Due to increased losses at high frequencies, curves measured in the laboratory will not have as sharp as the ideal model predicts. Actual curves may look like



Example 11

A 10 volt pulse is introduced into a 75Ω transmission line having $v_p = 10^8$ m/s. $11.3 \mu\text{s}$ later, a -2 volt pulse is observed at the TDR unit. What information is available about the discontinuity and its location?



Solution

From $v(t)$ at the input to the transmission line, the reflection coefficient can be determined. From this measured Γ , the load impedance can be calculated.

$$\Gamma = \frac{V^-}{V^+} = -0.2 = \frac{Z_L - Z_c}{Z_L + Z_c} \Rightarrow Z_L = 50 \Omega$$

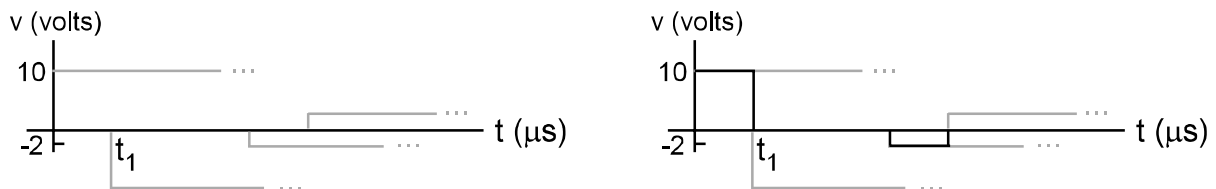
From the time of the reflected pulse, the distance to the load ($z = d$) can be determined. Note that the time is the sum of the time required for the incident pulse to reach the load and the time required for the reflected wave to reach the input.

$$t_r = \frac{2d}{v_p} \Rightarrow d = \frac{t_r v_p}{2}$$

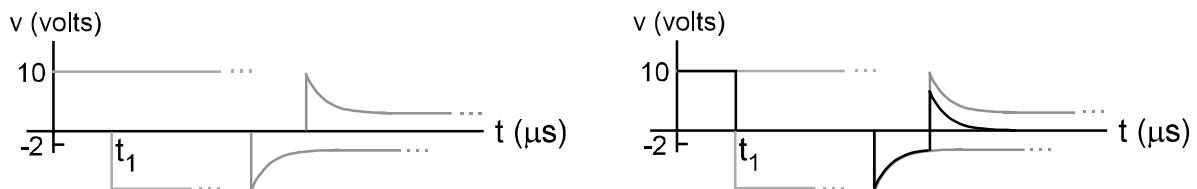
$$d = 56.5 \text{ m}$$

Further discussion

Note that a pulse can be considered as the sum of two step functions and that, likewise, the reflection can be considered as the sum of two step functions.

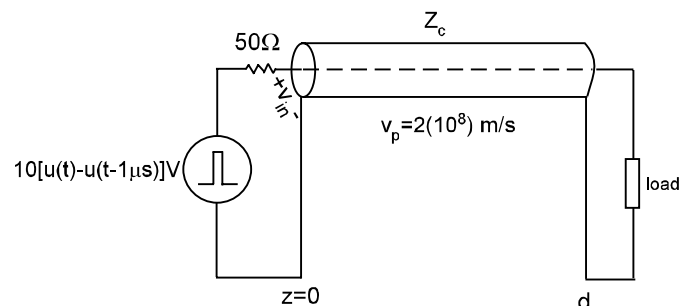


This viewpoint is especially valuable when considering reflections from reactive loads. For example, consider the results when a pulse is applied to the above 75Ω line with a load consisting of a capacitance in parallel with a 50Ω resistance. In this case, the Γ will initially be -1 and will decay to -0.2 as the capacitance charges.

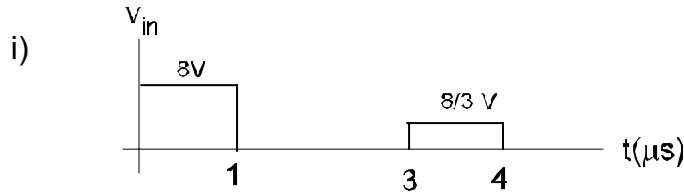


Example 12

Consider the system below.



For the input voltages shown below, determine Z_c , d , and the load.



- From v^+ , $Z_c = 200 \Omega$.
- From the time of reflected waves ($t_r = 3 \mu\text{s}$), $d = 300$ m.
- From the shape of the reflected waves, the load is a resistance combination. From its amplitude, $R = 2200 \Omega$.



- From v^+ , $Z_c = 50 \Omega$.
- From the time of reflected waves ($t_r = 4 \mu\text{s}$), $d = 400$ m.
- From the shape of the reflected waves, the load is a parallel RC combination. Since $\Gamma \rightarrow 0$ as t grows larger, $R = Z_c = 50 \Omega$. From this measurement, C could also be determined from the time constant of the reflected waves. From these curves, an accurate measure of τ is difficult, but approximating $\tau = 0.1 \mu\text{s}$.

$$\tau = RC \Rightarrow C = \frac{\tau}{R} \cong \frac{0.1 \mu\text{s}}{50 \Omega} = 2 \text{ nF}$$

- iii) In this case, d is known to be 400 m.

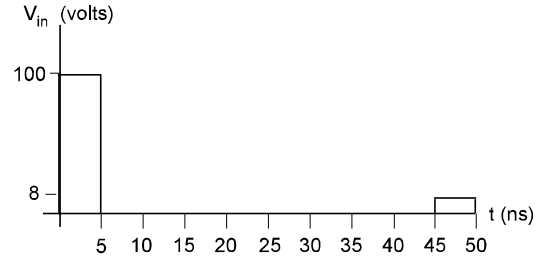
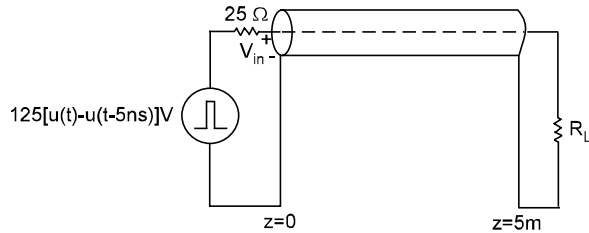


- From v^+ , $Z_c = 25 \Omega$.
- Since there are no reflections, $Z_L = Z_c = 25 \Omega$.

Example 13

Given the lossless transmission line below connecting the 25Ω source to Z_L . Find

- i) Z_c ii) \tilde{Z}_L iii) v_p iv) \mathcal{L} v) \mathcal{E}



Solution

- i) Z_c can be calculated from the voltage of the initial pulse.

$$v^+ = 100 \text{ V} = 125 \text{ V} \frac{Z_c}{Z_c + 25 \Omega} \quad \Rightarrow \quad Z_c = 100 \Omega$$

- ii) \tilde{Z}_L can be determined from the reflection coefficient. From the shape of the reflected pulse, it can be seen that the load must be real. Also, since the reflected wave is positive, $Z_L > Z_c$.

$$\Gamma_L(1 + \Gamma_s) = \Gamma_L(1 - 0.6) = 0.08$$

$$\Gamma_L = 0.2 = \frac{Z_L - Z_c}{Z_L + Z_c} \quad \Rightarrow \quad Z_L = 150 \Omega$$

$$\text{iii) } v_p = \frac{2d}{t_r} = \frac{2(5 \text{ m})}{50(10^{-9}) \text{ s}} = 2(10^8) \text{ m/s}$$

$$\text{iv) } v_p = \sqrt{\frac{1}{\mathcal{L}\mathcal{E}}} \quad \text{and} \quad Z_c = \sqrt{\frac{\mathcal{L}}{\mathcal{E}}} \quad \Rightarrow \quad \mathcal{L} = \frac{Z_c}{v_p} \quad \text{and} \quad \mathcal{E} = \frac{1}{Z_c v_p}$$

$$\mathcal{E} = \frac{1}{Z_c v_p} = \frac{1}{(100 \Omega) 2(10^8) \text{ m/s}} = 50 \text{ pF/m}$$

$$\text{v) } \mathcal{L} = \frac{Z_c}{v_p} = \frac{100 \Omega}{2(10^8) \text{ m/s}} = 0.5 \mu\text{H/m}$$