ELECTROMAGNETIC WAVES

Uniform plane waves in lossless media ☼

Solutions – time-domain

$$\mathbf{E}(\mathbf{r},t) = \mathbf{a}_E \, E \cos(\omega t - \boldsymbol{\beta} \bullet \mathbf{r}) = \mathbf{a}_E \, E \cos(\omega t - \mathbf{a}_\beta \, \beta \bullet \mathbf{r})$$

$$\mathbf{E}(\mathbf{r},t) = \mathbf{a}_E \, E \cos\left[\omega t - \beta\left(x \cos\theta_x + y \cos\theta_y + z \cos\theta_z\right)\right]$$

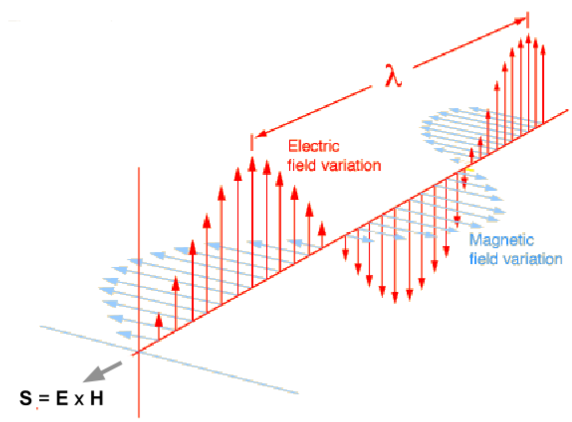
Solutions – frequency-domain

$$\mathbf{E}(\mathbf{r},t) = \text{Re}\, \tilde{\mathbf{E}} \, e^{-j\omega t}$$

$$\tilde{\mathbf{E}}(\mathbf{r},\omega) = \mathbf{a}_E E \, e^{-j\boldsymbol{\beta} \bullet \mathbf{r}}$$

The electric and magnetic vectors perpendicular and both are perpendicular to the direction of propagation.

$$\mathbf{a}_E \times \mathbf{a}_H = \mathbf{a}_\beta$$



$$\mathbf{H} = \mathbf{a}_\beta \times \frac{1}{\eta}\mathbf{E}$$ or use Faraday's law

$$\mathbf{E} = \eta \mathbf{H} \times \mathbf{a}_\beta$$ or use Ampere's law

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} \qquad \beta = \frac{2\pi}{\lambda}$$

$$v = f\lambda = \frac{\omega}{\beta}$$ (general)

$$v = \frac{1}{\sqrt{\varepsilon \, \mu}}$$ (lossless only)

All kinds of applets on transmission lines, waves etc.

http://www.educypedia.be/electronics/javatransmissinlines.htm

Example: ***TEM wave propagation*** ☼

An E-field is given by $\mathbf{E} = \mathbf{a}_z\ 50\cos\left[10^9 t - 5\left(\frac{x}{2} + \frac{\sqrt{3}y}{2}\right)\right]$ V/m . ($\mu = \mu_o$) Find

i) direction of travel
ii) velocity
iii) wavelength
iv) wave (or intrinsic) impedance
v) **H**

Energy, conduction current, displacement current ☼

From the point form of Ohm's law, $\mathbf{J} = \sigma\mathbf{E}$, two points can be appreciated:

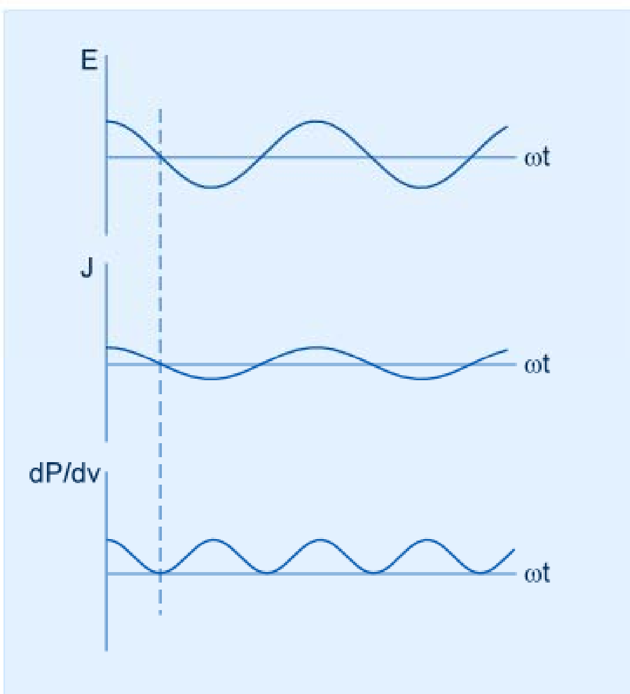
1. The conduction current and the electric field vary alike. Changes in the electric field produce like variations in the conduction current.
2. The conduction current and the electric field are in the same direction.

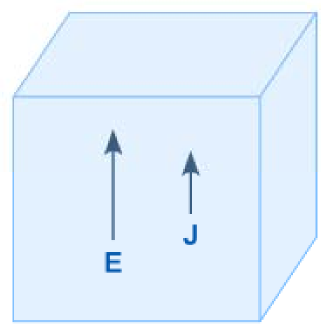
The result is that plots of **E** and **J** versus time are scaled versions of one another as shown below. From Joule's law,

$$\frac{dP}{dv} = \mathbf{E} \cdot \mathbf{J}$$

Given that E and J are in phase and parallel, the power dissipated per volume in conductive material is simply the product of the magnitudes of **E** and **J**.

$$\frac{dP}{dv} = E \cdot J$$





Displacement current density is the time rate of change of the electric flux density vector. The flux density can be broken into two pieces, the first being $\varepsilon_o \mathbf{E}$ which is present even in the absence of material being present (it's there when material is present too) and the second being **P** which is the polarization vector associated with the separation of bound change.

The displacement current is associated with the time rate of change of these two components.

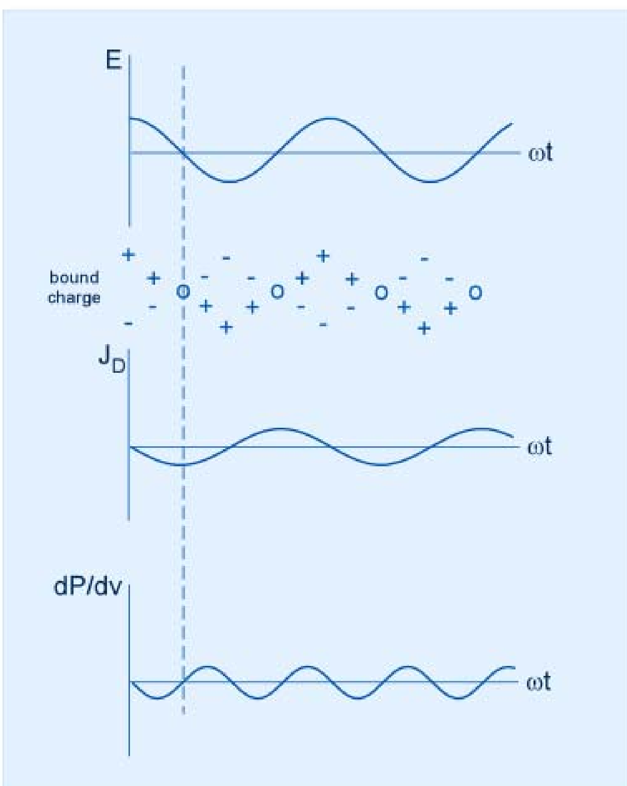
$$\mathbf{D} = \varepsilon \mathbf{E} = \varepsilon_r \varepsilon_o \mathbf{E} = \varepsilon_o \mathbf{E} + (\varepsilon_r - 1)\varepsilon_o \mathbf{E} \qquad \mathbf{D} = \varepsilon_o \mathbf{E} + \mathbf{P}$$

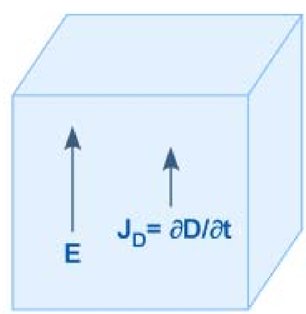
$$\mathbf{J}_D = \frac{\partial \mathbf{D}}{\partial t} = \varepsilon_o \frac{\partial \mathbf{E}}{\partial t} + \frac{\partial \mathbf{P}}{\partial t}$$

Note that both components of the displacement current have the same phase and direction (true at least for "low" frequencies—more on this below).

This allows us to frame the discussion of power absorbed by a dielectric in terms of the behavior of bound change with respect to changes in electric field.

Below, a sinusoidal variation in **E** is assumed and the corresponding bound charge separation is tracked with respect to time and correlated to the associated displacement current. The power per unit volume absorbed by a dielectric is $\mathbf{E} \cdot \mathbf{J}_D$

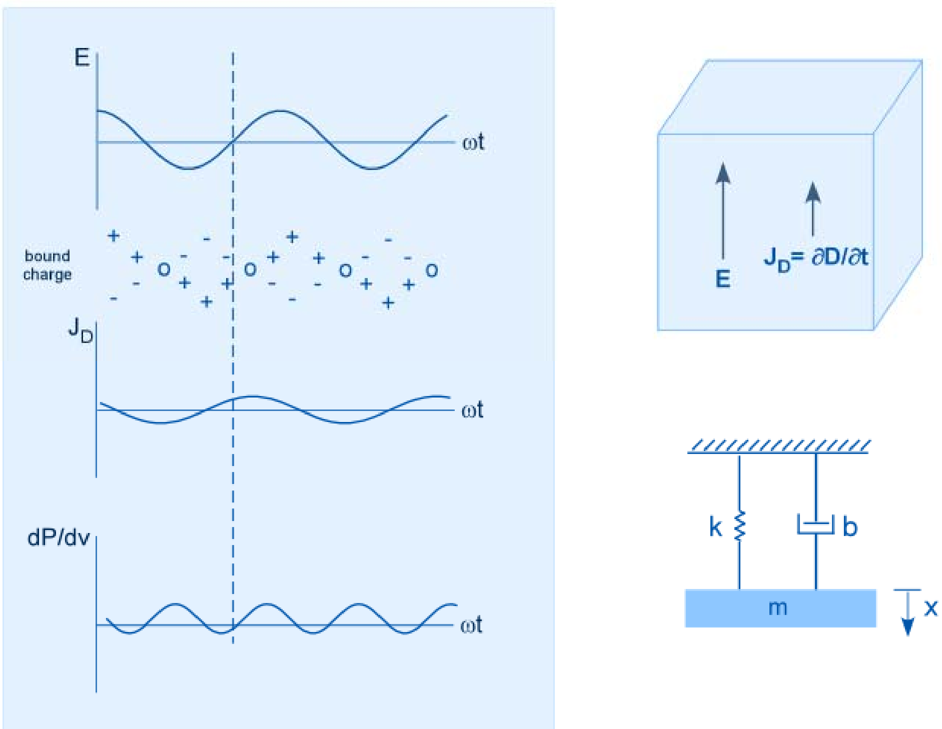




If the material under consideration have both displacement and conduction currents presence there will be both energy storage and average power dissipation. The net current will be between 0° and 90° out-of-phase with the electric field.

This situation can also occur in dielectrics at high frequencies in which the time constants associated with the dynamics of the bound charge mass can no longer be neglected with respect to the time rate of change of the field.

Displacement current at high-frequencies



The result in either case, whether the material has both conduction and displacement current or whether the frequency is sufficiently high so that the displacement current is not in phase with the electric field, the result is that there is an average power lost from the field in the material.

Another loss mechanism, to be treated later, is radiation loss which occurs anytime charge undergoes acceleration. This loss can be significant if the frequency and geometry are such to lead to efficient radiation. Radiation appears in the treatment of antennas.

All these loss mechanisms lead to the same result as does the simple case of conductivity and an equivalent conductivity can be assigned to each.

Waves in lossy material ☼

Taking Ampere and Faraday (with no charge or current sources)

Note: no current sources *DOES NOT* mean there are no currents. In conductive materials ($\sigma \neq 0$), currents exist in response to an electric field in conductive material.

$$\nabla \times \mathbf{H} = \mathbf{J} + \varepsilon \frac{\partial \mathbf{E}}{\partial t} = \sigma \mathbf{E} + \varepsilon \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}$$

Combining these equations

$$\nabla \times \nabla \times \mathbf{E} = \nabla(\nabla \bullet \mathbf{E}) - \nabla^2 \mathbf{E} = -\mu \frac{\partial}{\partial t}(\nabla \times \mathbf{H}) = -\mu \frac{\partial}{\partial t}\left(\sigma \mathbf{E} + \varepsilon \frac{\partial \mathbf{E}}{\partial t}\right)$$

$$\nabla^2 \mathbf{E} - \mu\sigma \frac{\partial \mathbf{E}}{\partial t} - \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

In the frequency domain

$$\nabla^2 \tilde{\mathbf{E}} - j\omega\mu\sigma \tilde{\mathbf{E}} + \omega^2 \mu\varepsilon \tilde{\mathbf{E}} = \nabla^2 \tilde{\mathbf{E}} + \omega^2 \mu\varepsilon (1 - j\frac{\sigma}{\omega\varepsilon})\tilde{\mathbf{E}} = \nabla^2 \tilde{\mathbf{E}} + \omega^2 \mu\varepsilon (1 - j\frac{\sigma}{\omega\varepsilon})\tilde{\mathbf{E}} = 0$$

$$\nabla^2 \tilde{\mathbf{E}} + \omega^2 \mu\tilde{\varepsilon}\, \tilde{\mathbf{E}} = 0 \qquad \left[\text{where } \tilde{\varepsilon} = \varepsilon\left(1 - j\frac{\sigma}{\omega\varepsilon}\right) \text{ is the complex permittitivty}\right]$$

In the frequency domain, solutions are

$$\tilde{\mathbf{E}} = \mathbf{a}_E \tilde{E} e^{\pm \tilde{\gamma} \bullet \mathbf{r}}$$

where $\tilde{\gamma} = \gamma \mathbf{a}_\gamma = \sqrt{-\omega^2 \mu \tilde{\varepsilon}}\ \mathbf{a}_\gamma = \pm j\omega\sqrt{\varepsilon\mu}\sqrt{1 - j\frac{\sigma}{\omega\varepsilon}}$

$$\tilde{\gamma} = (\alpha + j\beta)\mathbf{a}_\gamma$$

Terminology

$\tilde{\gamma}$	complex propagation vector
$\tilde{\gamma}$	complex propagation constant
α	attenuation constant (equal to zero for lossless materials)
β	phase constant

Details of solution

The solution must satisfy the wave equation, $\nabla^2\tilde{\mathbf{E}} + \omega^2\mu\tilde{\varepsilon}\ \tilde{\mathbf{E}} = 0$.

$$\tilde{\mathbf{E}} = \mathbf{a}_E\,\tilde{E}\,e^{\pm\tilde{\boldsymbol{\gamma}}\bullet\mathbf{r}}$$

$$\tilde{\mathbf{E}} = \mathbf{a}_E\,E\,e^{j\phi^{\mp}}\ e^{\pm(\alpha + j\beta)(\mathbf{a}_\gamma\bullet\mathbf{r})}$$

$$\tilde{\mathbf{E}} = \mathbf{a}_E\,E\,e^{j\phi^{\mp}}\ e^{\pm(\alpha + j\beta)(x\cos\theta_x + y\cos\theta_y + z\cos\theta_z)}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(\mathbf{a}_E\,E\,e^{j\phi^{\mp}}\ e^{\pm(\alpha + j\beta)(x\cos\theta_x + y\cos\theta_y + z\cos\theta_z)}\right)$$
$$+\ \omega^2\mu\tilde{\varepsilon}\left(\mathbf{a}_E\,E\,e^{j\phi^{\mp}}\ e^{\pm(\alpha + j\beta)(x\cos\theta_x + y\cos\theta_y + z\cos\theta_z)}\right) = 0.$$

Each derivative pulls down $\pm(\alpha + j\beta)\cos\theta_i$ from the exponential. The derivatives hear are second derivatives.

$$\left\{\left[\pm(\alpha + j\beta)\cos\theta_x\right]^2 + \left[\pm(\alpha + j\beta)\cos\theta_y\right]^2 + \left[\pm(\alpha + j\beta)\cos\theta_z\right]^2\right\}\ \tilde{\mathbf{E}}\ + \omega^2\mu\tilde{\varepsilon}\ \tilde{\mathbf{E}}\ = 0$$

$$\left\{\left[\pm(\alpha + j\beta)\right]^2\left(\cos^2\theta_x + \cos^2\theta_y + \cos^2\theta_z\right)\right\}\ \tilde{\mathbf{E}} + \omega^2\mu\tilde{\varepsilon}\ \tilde{\mathbf{E}}\ = 0$$

The sum of the squares of the directional cosines must equal zero.

$$\left\{\left[\pm(\alpha + j\beta)\right]^2\left(\cos^2\theta_x + \cos^2\theta_y + \cos^2\theta_z\right)\right\}\ \tilde{\mathbf{E}} + \omega^2\mu\tilde{\varepsilon}\ \tilde{\mathbf{E}}\ = 0$$

$$\tilde{\gamma}^2\ \tilde{\mathbf{E}} + \omega^2\mu\tilde{\varepsilon}\ \tilde{\mathbf{E}} = 0 \qquad \rightarrow \qquad \tilde{\gamma}^2 = -\omega^2\mu\tilde{\varepsilon}$$

The result is the propagation <u>constant</u> is complex $\tilde{\gamma} = \alpha + j\beta$. The propagation <u>vector</u> also indicates direction of travel, $\tilde{\boldsymbol{\gamma}} = \tilde{\gamma}\ \mathbf{a}_\gamma = (\alpha + j\beta)\mathbf{a}_\gamma$.

Propagation along $+\mathbf{a}_\gamma$

$$\tilde{\mathbf{E}} = \mathbf{a}_E\,\tilde{E}\,e^{\pm\tilde{\boldsymbol{\gamma}}\bullet\mathbf{r}}$$

$$\mathbf{E} = \mathbf{a}_E\,E\,e^{\pm\alpha(\mathbf{a}_\gamma\bullet\mathbf{r})}\ \cos\left[\omega t - \beta(\mathbf{a}_\gamma\bullet\mathbf{r}) + \phi^+\right]$$

Propagation along $-\mathbf{a}_\gamma$

$$\tilde{\mathbf{E}} = \mathbf{a}_E\,\tilde{E}\,e^{\tilde{\boldsymbol{\gamma}}\bullet\mathbf{r}}$$

$$\mathbf{E} = \mathbf{a}_E\,E\,e^{\alpha(\mathbf{a}_\gamma\bullet\mathbf{r})}\ \cos\left[\omega t + \beta(\mathbf{a}_\gamma\bullet\mathbf{r}) + \phi^-\right]$$

Plane waves in lossy materials

$$\tilde{\mathbf{E}} = \mathbf{a}_E \tilde{E} e^{\pm\tilde{\gamma}\bullet\mathbf{r}}$$

$$\tilde{\mathbf{E}} = \mathbf{a}_E E e^{j\phi^{\mp}} e^{\pm(\alpha + j\beta)(x\cos\theta_x + y\cos\theta_y + z\cos\theta_z)}$$

$$\tilde{\mathbf{E}} = \mathbf{a}_E E e^{\pm\alpha(\mathbf{a}_\gamma\bullet\mathbf{r})} e^{\pm\beta(\mathbf{a}_\gamma\bullet\mathbf{r})} e^{j\phi^{\mp}}$$

In the time domain

$$\mathbf{E} = \mathbf{a}_E E e^{\pm\alpha(\mathbf{a}_\gamma\bullet\mathbf{r})} \cos\left[\omega t \pm \beta\left(\mathbf{a}_\gamma\bullet\mathbf{r}\right) + \phi^{\mp}\right]$$

$$\mathbf{E} = \mathbf{a}_E E e^{\pm\alpha(x\cos\theta_x + y\cos\theta_y + z\cos\theta_z)} \cos\left[\omega t \pm \beta\left(x\cos\theta_x + y\cos\theta_y + z\cos\theta_z\right) + \phi^{\mp}\right]$$

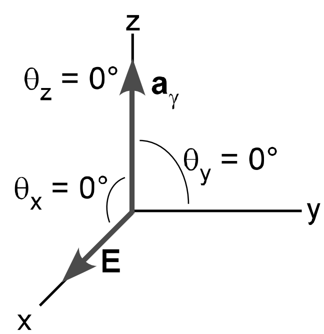
EM waves in lossy material are exponentially decaying sinusoids ☼

Consider an electromagnetic wave, pointing in $\mathbf{a}_x$ direction, traveling in a lossy material in the $\mathbf{a}_z$ direction (assume $\phi = 0$).

$$\tilde{\mathbf{E}} = \mathbf{a}_x \tilde{E} e^{-\tilde{\gamma}z}$$

$$-\tilde{\gamma}^2 = \omega^2\mu\left(\varepsilon - j\frac{\sigma}{\omega}\right)$$

$$\tilde{\gamma} = j\omega\sqrt{\mu}\sqrt{\left(\varepsilon - j\frac{\sigma}{\omega}\right)} = \alpha + j\beta$$



In the time domain,

$$\mathbf{E} = \mathbf{a}_x E e^{-\alpha z}\cos(\omega t - \beta z)$$

The intrinsic impedance is complex (**E** and **H** are NOT in phase for lossy materials)

$$\tilde{\eta} = \sqrt{\frac{\mu}{\tilde{\varepsilon}}} = \sqrt{\frac{\mu}{\varepsilon\left(1 - j\frac{\sigma}{\omega\varepsilon}\right)}} = \sqrt{\frac{\mu}{\varepsilon - j\frac{\sigma}{\omega}}} = \eta e^{j\theta_\eta}$$

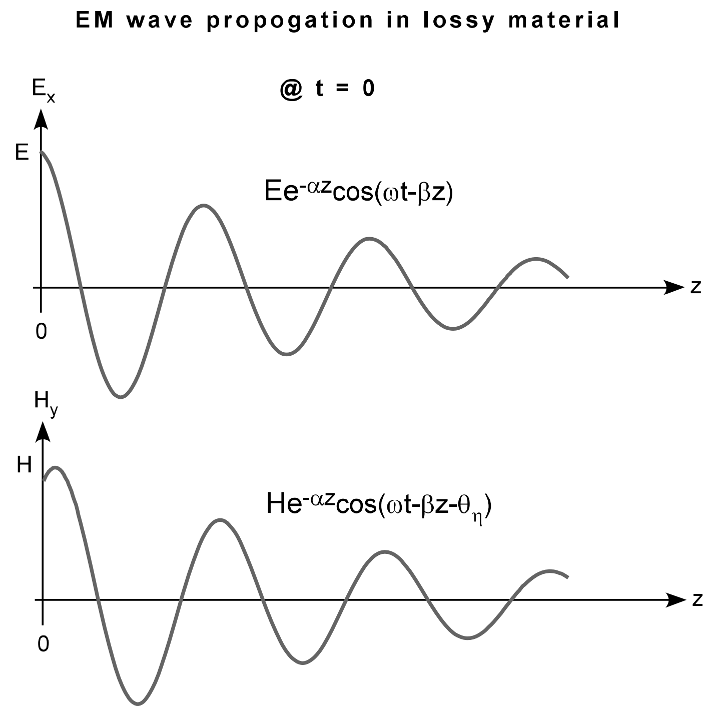
$\tilde{\mathbf{H}} = \mathbf{a}_\beta \times \frac{1}{\tilde{\eta}}\tilde{\mathbf{E}}$ or use Faraday's law

$\tilde{\mathbf{E}} = \tilde{\eta}\tilde{\mathbf{H}} \times \mathbf{a}_\beta$ or use Ampere's law

$\beta = \frac{2\pi}{\lambda}$ $\quad$ $v = f\lambda = \frac{\omega}{\beta}$ (general) $\quad$ $v \neq \frac{1}{\sqrt{\varepsilon\mu}}$ (lossless only)

Wave impedance in a lossy material

What is the relation between **E** and **H** in the diagram shown? ☼



Example: ***Wave propagation in lossy material*** ☼

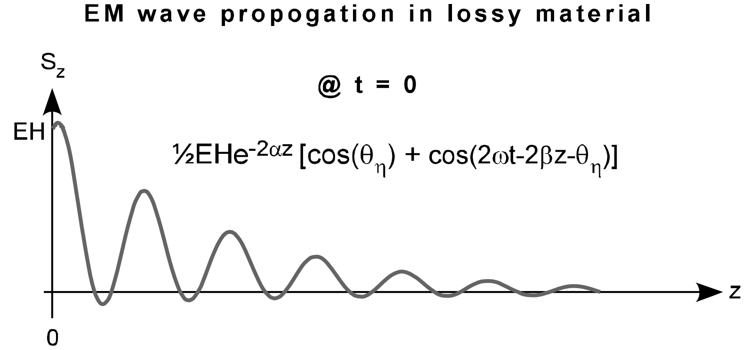
An electric field is given as $\tilde{\mathbf{E}} = \mathbf{a}_x 100e^{-\tilde{\gamma}z}$ V/m is traveling through material with σ=0.1/Ωm, μ_r=1, ε_r=4. The frequency is 2.45 GHz. Find α and β and the dB/m attenuation in the material.

Power flow in a lossy material ☼

E points along $\mathbf{a}_x$, traveling in $\mathbf{a}_z$ direction. Take phase of $\mathbf{E}\big|_{\substack{z=0\\t=0}} = 0°$.

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} = \mathbf{a}_x E e^{-\alpha z}\cos(\omega t - \beta z) \times \mathbf{a}_y H e^{-\alpha z}\cos(\omega t - \beta z - \theta_\eta)$$

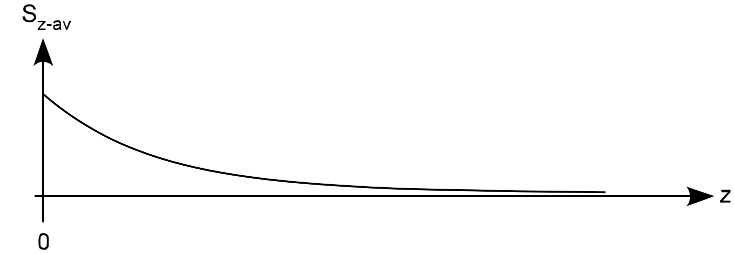
$$= \mathbf{a}_z \frac{EH}{2} e^{-2\alpha z}\left[\cos\left(\theta_\eta\right) + \cos(2\omega t - 2\beta z - \theta_\eta)\right]$$



time-average Poynting vector

$$\langle \mathbf{S} \rangle = \frac{1}{2}\mathrm{Re}\left(\tilde{\mathbf{E}} \times \tilde{\mathbf{H}}^*\right) = \frac{1}{2}\mathrm{Re}\left(\mathbf{a}_x E e^{-\alpha z} e^{-j\beta z} \times \mathbf{a}_y H\, e^{j\theta_\eta} e^{-\alpha z} e^{j\beta z}\right)$$

$$= \mathbf{a}_z \mathrm{Re}\frac{1}{2} EH e^{j\theta_\eta} e^{-2\alpha z} = \mathbf{a}_z \frac{1}{2} EH \cos\left(\theta_\eta\right)\, e^{-2\alpha z}$$



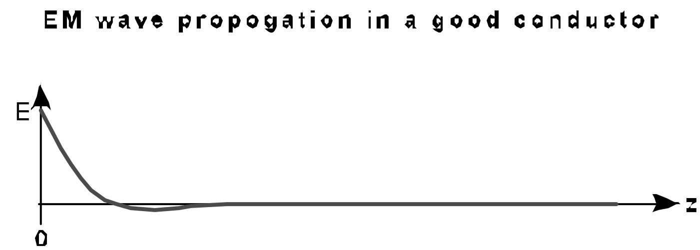
Special Cases ☼

High loss (good conductors) ~ $\sigma \gg \omega\varepsilon$

$$\tilde{\gamma} = j\omega\sqrt{\mu\varepsilon}\sqrt{\left(1 - j\frac{\sigma}{\omega\varepsilon}\right)} \cong j\omega\sqrt{\mu\varepsilon}\sqrt{-j\frac{\sigma}{\omega\varepsilon}}$$

$$= \omega\sqrt{\mu\varepsilon}\sqrt{\frac{j^2\sigma}{j\omega\varepsilon}} = \sqrt{\sigma\mu\omega}\sqrt{j} = \sqrt{\sigma\mu\omega}\sqrt{e^{j\frac{\pi}{2}}}$$

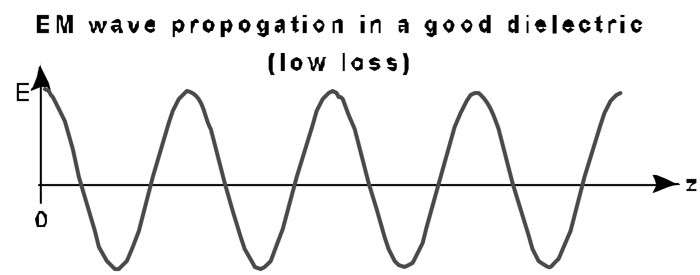
$$= \sqrt{\sigma\mu\omega}\,\frac{1+j}{\sqrt{2}} = \sqrt{\frac{\sigma\mu\omega}{2}}(1+j) = \alpha + j\beta$$



Low loss (good dielectric) ~ $\sigma \ll \omega\varepsilon$

$$\tilde{\gamma} = j\omega\sqrt{\mu\varepsilon}\sqrt{\left(1 - j\frac{\sigma}{\omega\varepsilon}\right)} \cong j\omega\sqrt{\mu\varepsilon}\left(1 - j\frac{\sigma}{2\omega\varepsilon}\right)$$

$$= \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}} + j\omega\sqrt{\mu\varepsilon} = \alpha + j\beta$$



Good dielectrics (low loss) and good conductors (high loss) ☼

Note that, in the expression for complex permittivity, the imaginary term within the parentheses gives the ratio between conduction current and displacement current.

$$\tilde{\varepsilon} = \varepsilon\left(1 - j\frac{\sigma}{\omega\varepsilon}\right)$$

$$\mathbf{J}_c = \sigma\mathbf{E} \;\rightarrow\; \tilde{\mathbf{J}}_c = \sigma\tilde{\mathbf{E}}$$

$$\mathbf{J}_d = \frac{\partial\mathbf{D}}{\partial t} = \varepsilon\frac{\partial\mathbf{E}}{\partial t} \;\rightarrow\; \tilde{J}_d = j\omega\tilde{\mathbf{D}} = j\omega\varepsilon\tilde{\mathbf{E}}$$

$$\frac{\tilde{\mathbf{J}}_c}{\tilde{\mathbf{J}}_d} = \frac{\sigma\tilde{\mathbf{E}}}{j\omega\varepsilon\tilde{\mathbf{E}}} = \frac{\sigma}{j\omega\varepsilon} = -j\frac{\sigma}{\omega\varepsilon}$$

"good dielectrics" are defined by $\left|\tilde{\mathbf{J}}_d\right| >> \left|\tilde{\mathbf{J}}_c\right|$

"good conductors" are defined by $\left|\tilde{\mathbf{J}}_c\right| >> \left|\tilde{\mathbf{J}}_d\right|$

Example: ***Good conductor? Good dielectric?***

Find the propagation constant for f = 100Hz, σ = 10/Ωm, μ_r=1, ε_r=10. Is the material a good conductor or a good dielectric? Neither?

Example: ***Copper***

At what frequency does Cu cease to be a good conductor?

σ=5.8(10^7)/Ωm, μ_r=1, ε_r=1.

Skin depth and its effects in circuits ☼

The distance a wave must travel into a good conductor for its amplitude to be reduced by a factor of e^{-1} is called skin depth, δ.

Recall that, for a good conductor, conduction current is much greater than displacement current ($\sigma >> \omega\varepsilon$). $\mathbf{J}_c = \sigma\mathbf{E}$, $\mathbf{J}_d = d\mathbf{D}/dt = \varepsilon\, d\mathbf{E}/dt$

$$-\tilde{\gamma}^2 = \omega^2\mu\varepsilon(1 - j\frac{\sigma}{\omega\varepsilon})$$

$$\tilde{\gamma} \cong \alpha + j\beta = \sqrt{\frac{\sigma\omega\mu}{2}} + j\sqrt{\frac{\sigma\omega\mu}{2}}$$

α is the attenuation constant and is the reciprocal of skin depth.

Consider an electric wave pointing in the x-direction and traveling in the z-direction:

Wave impedance

This implies that the magnetic field must lag the electric field

$(\tilde{E} = \tilde{\eta}\,\tilde{H})$

$$\tilde{\eta} = \sqrt{\frac{\mu}{\tilde{\varepsilon}}} = \sqrt{\frac{\mu}{\varepsilon\left(1 - j\frac{\sigma}{\omega\varepsilon}\right)}} \cong \sqrt{\frac{\mu}{\varepsilon\left(-j\frac{\sigma}{\omega\varepsilon}\right)}} = \sqrt{\frac{\mu\omega}{\sigma}}\angle 45°$$

Example: *Skin effect* ☼

The distance in a good conductor that is required to reduce the amplitude of the field vectors by a factor of e^{-1} is called the skin depth, δ.

Assuming copper:

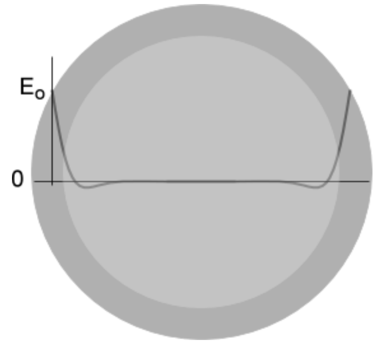
$$E(z) = |\mathbf{E}(z)| = E_o e^{-\alpha z} = E_o e^{-\frac{z}{\delta}} = E_o e^{-1} \text{ for } z = \delta = \alpha^{-1}$$

$$\delta = \alpha^{-1} = \sqrt{\frac{2}{\sigma\mu\omega}} = \frac{1}{2\pi}\sqrt{\frac{1}{5.8f}} \cong \frac{0.0661}{\sqrt{f}}$$

copper, non-magnetic $\mu = \mu_o = 4\pi(10^{-7})$ H/m $\sigma = 5.8(10^{7})$ S/m				
frequency (Hz)	60 Hz	1 KHz	1 MHz	1 GHz
skin depth δ	8.5 mm	2.1 mm	66.1 μm	2.1 μm

Skin effect in circuits ☼

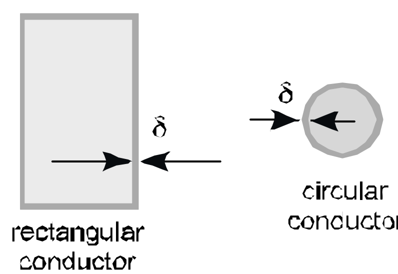
Consider the important special case in which the skin depth is much smaller than the cross sectional dimensions of the conductor.



In this case, it is convenient to suppose the current density is uniformly distributed over a skin depth.

$$R_{\square} = \frac{\Delta V}{\Delta I}$$

The AC resistance of conductors is a function of frequency since the skin depth is a function of frequency.



i) circular conductors

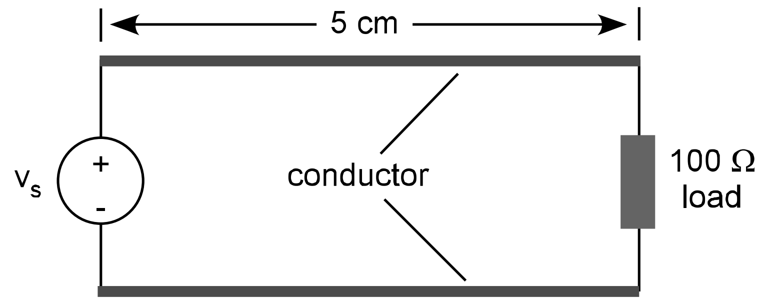
ii) square conductors

iii) stranded wire

Example: ***Skin effect*** ☼

A voltage, $\mathbf{v_s(t) = 10 \cos (2\pi ft)}$ **V** is applied to a 100 Ω load and a solid copper conductor as shown. The radius of the conductors is 2 mm and their length is 5 cm.

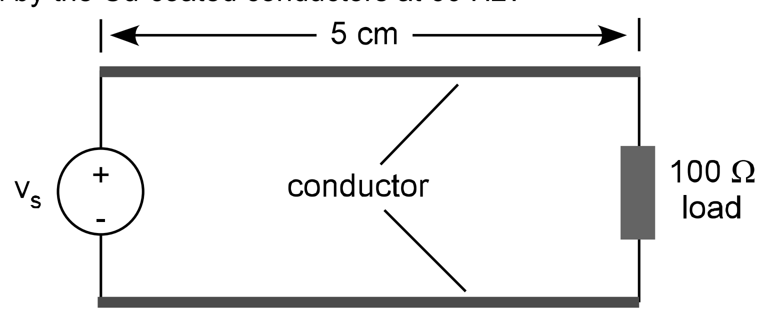
a) What power is dissipated in the copper conductor at 60 Hz?



b) What is the resistance of the conductors at 4 GHz? What power is dissipated in the copper conductors at 4 GHz?

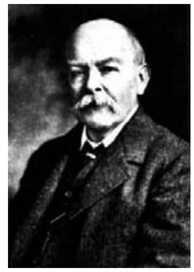
A voltage, $\mathbf{v_s(t) = 10 \cos (2\pi ft)}$ **V** is applied to a similar conductor-load, this time the conductors consist of a 10 μm copper coating with the interior of the conductors being a good insulator like Teflon.

a) What is the resistance of the Cu-coated conductors at 60 Hz? What power is dissipated by the Cu-coated conductors at 60 Hz?



b) What is the resistance of the Cu-coated conductors at 4 GHz? What power is dissipated by the Cu-coated conductors at 4 GHz?

Poynting's theorem ☼



John Henry Poynting

$$\mathbf{H} \bullet (\nabla \times \mathbf{E}) = \mathbf{H} \bullet \left(-\frac{\partial \mathbf{B}}{\partial t} \right)$$

$$\mathbf{E} \bullet (\nabla \times \mathbf{H}) = \mathbf{E} \bullet \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right)$$

Using this and the vector identity

$$\nabla \bullet (\mathbf{E} \times \mathbf{H}) = \mathbf{H} \bullet (\nabla \times \mathbf{E}) - \mathbf{E} \bullet (\nabla \times \mathbf{H})$$

Poynting's theorem ☼

Looking at the integral form of Poynting's theorem, it is readily seen that the theorem is an expression of conservation of energy IF Poynting's vector is power flux density in W/m^2.

Conservation of energy is the justification for the interpretation of $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ as power flux density.

Example: *units*

What are the units of a) $\mathbf{E} \cdot \mathbf{D}$ b) $\mathbf{H} \cdot \mathbf{B}$ c) $\mathbf{S}$?

Example: *Calculations of power flow*

An electric vector, pointing in the y-direction, travels in the z-direction. At z=0, the electric vector has an amplitude of 2 V/m. f=10 MHz with material properties $\mu_r=1$, $\varepsilon_r=4$, $\sigma=10^{-5}/\Omega m$.

Calculate the terms in the integral form of Poynting's theorem for the cube $0 < x < 2$ m, $0 < y < 2$ m, $0 < z < 2$ m.

Example: ***Power density*** ☼

Calculate Poynting's vector ***and*** the time-averaged Poynting's vector for the plane wave below, traveling in air.

$$\tilde{\mathbf{E}} = \mathbf{a}_x \; 2e^{-j\frac{\pi}{2}y} \text{ V/m}$$

Example: ***wave propagation in lossy material*** ☼

An EM wave travels along $\mathbf{a}_z$ in nonmagnetic material with a complex permittivity @ $\omega = 10^{10}$ r/s of $\tilde{\varepsilon} = \varepsilon_o(20 - j15) = 25\varepsilon_o e^{-j36.9^\circ}$. It is known that $\tilde{\mathbf{E}}$ points in the $\mathbf{a}_x$ direction and that $\langle \mathbf{S} \rangle\big|_{z=0} = 62.9\ \mathbf{a}_z\ \mathrm{W/m^2}$.

i) **a)** What is the material's conductivity? **b)** Can the material be classified as a good conductor or as a good dielectric?

ii) What is the material's complex propagation constant? (give in rectangular form)

iii) What is the material's intrinsic impedance? (give in polar form)

(Example: *cont.)* An EM wave travels along $\mathbf{a}_z$ in nonmagnetic material with a complex permittivity @ $\omega = 10^{10}$ r/s of $\tilde{\varepsilon} = \varepsilon_o(20 - j15) = 25\varepsilon_o e^{-j36.9°}$. It is known that $\tilde{\mathbf{E}}$ points in the $\mathbf{a}_x$ direction and that $\langle \mathbf{S} \rangle\big|_{z=0} = 62.9\, \mathbf{a}_z\ \mathrm{W}/\mathrm{m}^2$.

iv) Give the electric and magnetic vectors in the *frequency domain*. (assume the phase of $\tilde{\mathbf{E}}$ is 0° at z=0).

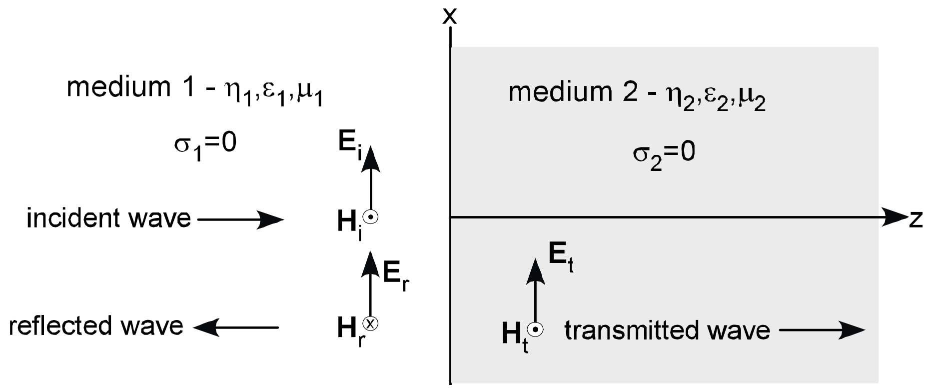
v) Give the *time-domain* electric and magnetic vectors.

vi) Find the Poynting's vector, **S**, and the time-averaged Poynting vector, $\langle \mathbf{S} \rangle$.

vii) **a)** What is the wave's wavelength? **b)** What is the wave's phase velocity?

EM waves incident on boundaries (normal incidence) ☼

How are EM waves reflected and transmitted when a plane dielectric boundary is encountered? Consider the situation below:



Define a transmission coefficient, T, and reflection coefficient, Γ.

$$T = \frac{E_t}{E_i} \qquad \Gamma = \frac{E_r}{E_i}$$

Let $\tilde{\mathbf{E}}_i = \mathbf{a}_x E_i e^{-j\beta_1 z}$

Boundary conditions: $E_{1tan}\big|_{z=0} = E_{2tan}\big|_{z=0}$ $\qquad H_{1tan}\big|_{z=0} = H_{2tan}\big|_{z=0}$

Example: *transmission and reflection* ☼

Find the reflection and transmission coefficients for a wave traveling in a printed circuit board made from FR-4 (most common PCB material—fiberglass epoxy base) into air.

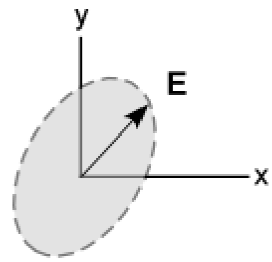
FR-4	ε_r=4.4,	μ_r=1
air	ε_r=1,	μ_r=1

Polarization ☼

Polarization refers to the path traced by the electric vector in a traveling electromagnetic wave. Assume a wave in traveling in the $+a_z$ direction and take the point-of-view seen in front of the oncoming EM wave. For a given value of z, what path is traced by the electric vector in the x-y plan as time advances?

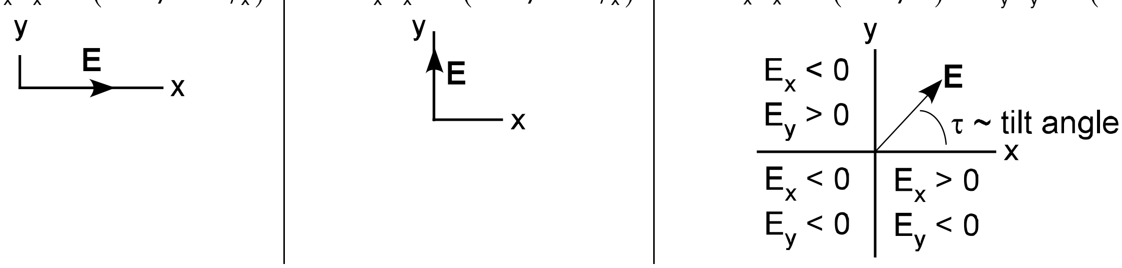
In general, a wave traveling in the a_z direction can have an x-component and a y-component. The resultant vector traces an ellipse—***elliptical*** polarization—in the x-y plan as it travels, the shape of which depends on E_x, E_y, ϕ_x, and ϕ_y.

$$\mathbf{E} = \mathbf{a}_x E_x \cos(\omega t - \beta z + \phi_x) + \mathbf{a}_y E_y \cos(\omega t - \beta z + \phi_y)$$

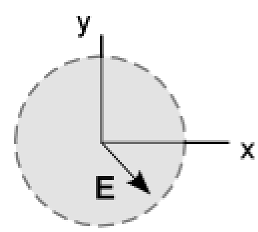


Linear polarization is a special case

$\mathbf{E} = \mathbf{a}_x E_x \cos(\omega t - \beta z + \phi_x)$ | $\mathbf{E} = \mathbf{a}_x E_x \cos(\omega t - \beta z + \phi_x)$ | $\mathbf{E} = \mathbf{a}_x E_x \cos(\omega t - \beta z) + \mathbf{a}_y E_y \cos(\omega t - \beta z)$



When $E_x = E_y$ and ϕ_x and ϕ_y differ by ±90°, the result is ***circular*** polarization.



For circular and elliptical polarization, the sense of the rotation is referred to as left-handed or right-handed, depending on which hand, if the fingers follow the rotation of **E**, the thumb is in the direction of propagation.

Examples ☼

What is the polarization for $\mathbf{E} = \mathbf{a}_x 2\cos(\omega t - \beta z + 45°) - \mathbf{a}_y 10\cos(\omega t - \beta z - 90°)$? (LHEP)

What is the polarization for $\mathbf{E} = \mathbf{a}_x 2\cos(\omega t - \beta z + 45°) + \mathbf{a}_y 2\cos(\omega t - \beta z - 45°)$? (RHCP)

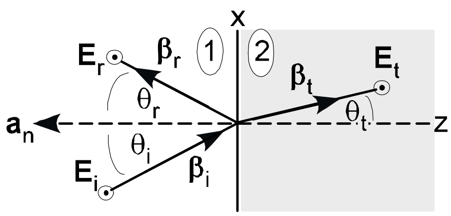
What is the polarization for $\mathbf{E} = \mathbf{a}_x 3\cos(\omega t - \beta z + 45°) + \mathbf{a}_y \cos(\omega t - \beta z + 45°)$? (LP, τ = 18.43°)

EM waves incident on boundaries (oblique incidence) ☼

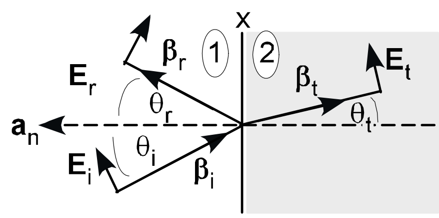
Now consider an EM wave, traveling in region 1, which encounters a second region at an angle other than normal. In this case, we must consider two types of polarization.

1. Perpendicular polarization: **E** $\perp$ to plane of incidence (POI).
2. Parallel polarization: **E** $\parallel$ to POI

E $\perp$ POI



E $\parallel$ to POI



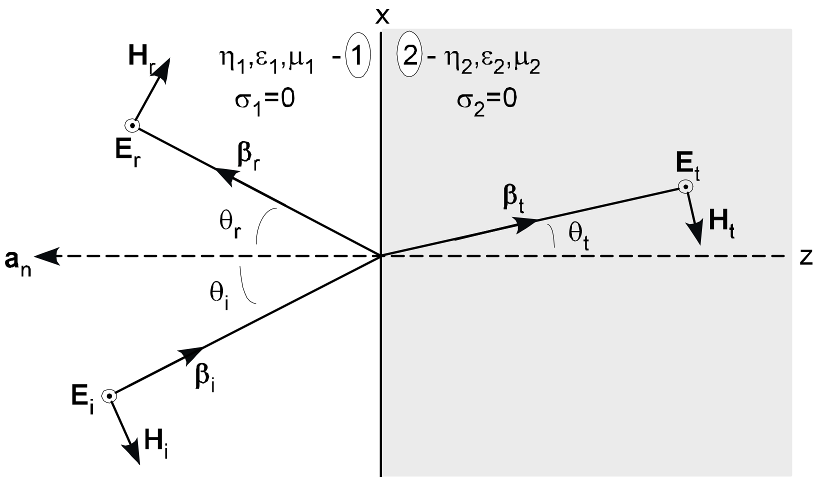
Definitions

POI – plane formed by propagation vector, β, and the vector normal to the boundary between regions 1 and 2.

Polarization: in general, any non-random orientation of an electric and magnetic field. In particular, polarization more usually describes the path the electric vector takes, in planes of constant phase as the wave travels.

> *Note:* We're considering only *linear* polarization, where the electric vector points in one direction. It reaches its maximum positive, grows smaller, becomes zero then negative, becomes maximum negative, grows smaller, becomes zero, etc.
>
> That is the vector remains on a straight line—thus, linear polarization.
>
> Any TEM wave can be expressed in terms of linearly polarized waves.

Refection and transmission at dielectric boundaries (E $\perp$ POI) ☼

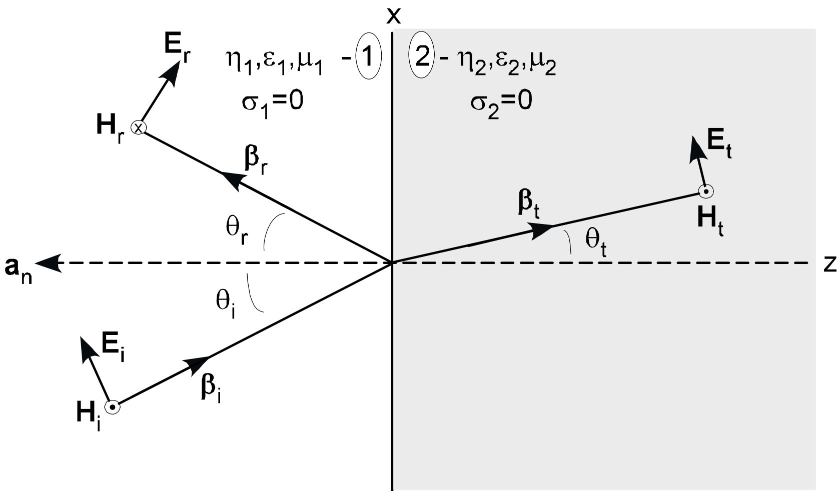


1st, give expressions for the electric and magnetic vectors.

Then, to find reflection and transmission coefficients, require tangential **H** and **E** to be continuous at the boundary (z = 0).

Refection and transmission at dielectric boundaries (E $\perp$ POI)

Refection and transmission at dielectric boundaries (E ∥ POI) ☼



Example: *Reflection and transmission* ☼

A uniform plane wave, in air, impinges on a dielectric material at $\theta_i = 45°$. The transmitted wave propagates with $\theta_t = 30°$. The frequency is 300 MHz and the electric vector is perpendicular to the plane of incidence.

i) Find ε_2 in terms of ε_0 (assume $\mu_1=\mu_2=\mu_o$)

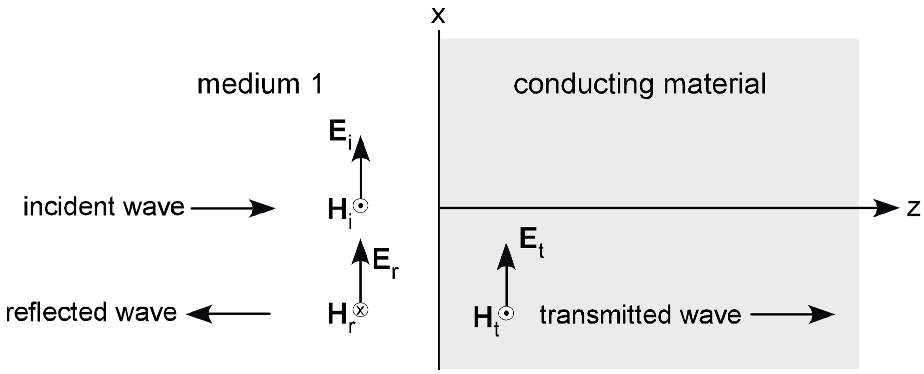
ii) Find the reflection coefficient, Γ.

iii) Find the transmission coefficient, T.

iv) Give expressions for the incident $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{H}}$-fields, the reflected $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{H}}$ fields, and the transmitted $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{H}}$-fields.

EM waves normally incident on good conductors ☼

Consider EM waves encountering a conducting material



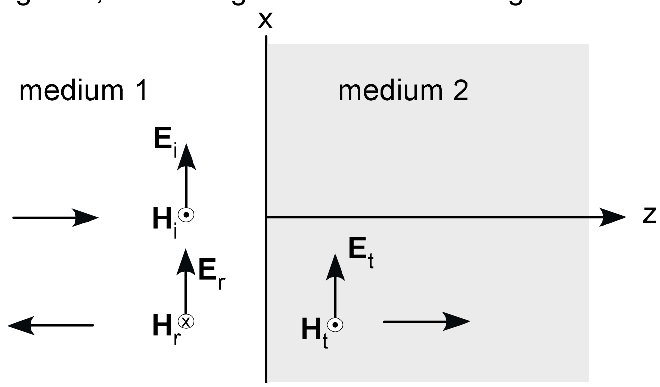
$$\tilde{\Gamma} = \frac{\tilde{\eta}_2 - \tilde{\eta}_1}{\tilde{\eta}_2 + \tilde{\eta}_1} \qquad \tilde{T} = \frac{2\tilde{\eta}_2}{\tilde{\eta}_2 + \tilde{\eta}_1}$$

Example: ***Reflection and transmission from a metal*** ☼

A 1-MHz plane wave, in air is incident on a copper sheet. The amplitude of the incident **E**-field is 100 V/m. Find Γ, T, δ, and the **|E|** and **|H|** fields at a distance of δ into the copper.

Standing Waves ☼

Consider normal incidence. In steady-state, the incident and reflected waves interact to produce, in region 1, a traveling wave and a standing wave.



$\tilde{\mathbf{E}}_1 = \tilde{\mathbf{E}}_i + \tilde{\mathbf{E}}_r$ choose reference phase $\tilde{\mathbf{E}}_i = \mathbf{E}_i e^{-j\frac{\varphi_\Gamma}{2}} e^{-j\beta z}$

$$\tilde{\mathbf{E}}_1 = \mathbf{E}_i e^{-j\frac{\varphi_\Gamma}{2}} e^{-j\beta z} + \Gamma e^{j\varphi_\Gamma} \mathbf{E}_i e^{-j\frac{\varphi_\Gamma}{2}} e^{j\beta z}$$

$$\tilde{\mathbf{E}}_1 = \mathbf{E}_i e^{-j\frac{\varphi_\Gamma}{2}} e^{-j\beta z} + \Gamma \mathbf{E}_i e^{j\frac{\varphi_\Gamma}{2}} e^{j\beta z} + \Gamma \mathbf{E}_i e^{-j\frac{\varphi_\Gamma}{2}} e^{-j\beta z} - \Gamma \mathbf{E}_i e^{-j\frac{\varphi_\Gamma}{2}} e^{-j\beta z}$$

Standing Wave Ratio (SWR) ☼

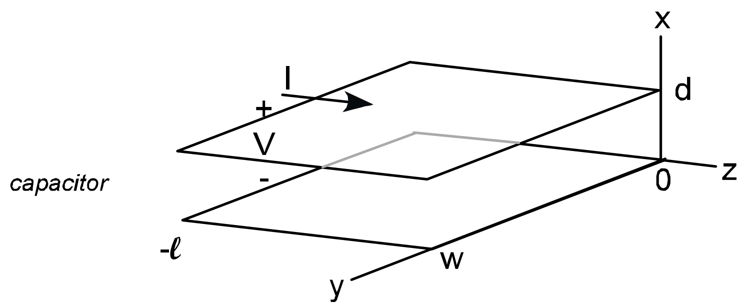
$$\text{SWR} = \frac{E_{max}}{E_{min}} = \frac{1 + \Gamma}{1 - \Gamma}$$

$$\Gamma = \frac{\text{SWR} - 1}{\text{SWR} + 1} \qquad \left(\Gamma = \left|\tilde{\Gamma}\right| \right)$$

The SWR ratio can readily be measured—so too $\Gamma = \left|\tilde{\Gamma}\right|$.

For no reflection, SWR=1. As more of the incident wave is reflected, the SWR ratio grows.

Capacitor ☼



If a DC voltage, **V**, is placed across the two plates shown, the electrostatic field is

$$\mathbf{E} = -\mathbf{a}_x \frac{V}{d}$$

The ratio of the charge stored on the plates, Q, and V is the capacitance of the structure.

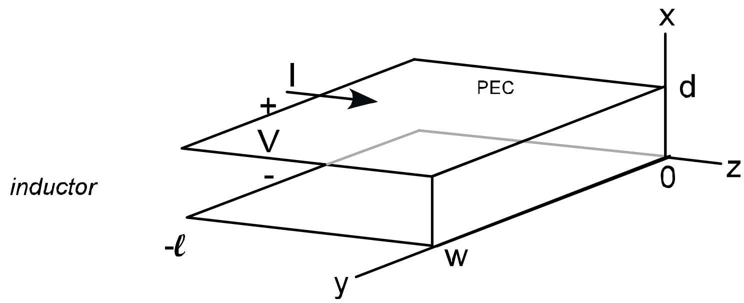
$$C_{dc} = \frac{Q}{V}$$

In this model, no current flows (I = 0) from the DC voltage source. Since I = 0 and $\frac{dV}{dt} = 0$, there is no reason that the parameter, C_{dc}, should be suspected of having any particular relation to the capacitance from lumped element circuit theory, defined by

$$i(t) = C_{ac} \frac{dv(t)}{dt}$$

- ✓ Are C_{dc} and C_{ac} really the same?
- ✓ If so, are there limits to the validity? For instance, does the capacitor operate at high frequencies as it does at low frequencies?
- ✓ What are low frequencies?
- ✓ What are high frequencies?
- ✓ If there are changes at high frequencies, what are they?

Inductor ☼



If the plates are shorted, and a DC current, I, is input into as shown the magnetostatic field is

$$\mathbf{H} = -\mathbf{a}_y \frac{I}{w}$$

The ratio of the magnetic flux, ψ_m, to the current, I is the inductance of the one-turn structure.

$$L_{dc} = \frac{\psi_m}{I}$$

In this model, no voltage exists (V = 0) across the PEC plates due to the DC current. Since V = 0 and $\frac{dI}{dt} = 0$, there is no reason that the parameter, L_{dc}, should be suspected of having any particular relation to the inductance from lumped element circuit theory, defined by

$$v(t) = L_{ac} \frac{di(t)}{dt}$$

- ✓ Are L_{dc} and L_{ac} really the same?
- ✓ If so, are there limits to the validity? For instance, does the inductor operate at high frequencies as it does at low frequencies?
- ✓ What are low frequencies?
- ✓ What are high frequencies?
- ✓ If there are changes at high frequencies, what are they?

Maxwell's equations

Gauss' law	$\nabla \cdot \mathbf{D} = \rho_v$
Conservation of magnetic flux	$\nabla \cdot \mathbf{B} = 0$
Ampere's law	$\nabla \times \mathbf{H} = \mathbf{J} + \dfrac{\partial \mathbf{D}}{\partial t}$
Faraday's law	$\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$
Constituent relations	$\mathbf{J} = \sigma \mathbf{E}$
	$\mathbf{D} = \varepsilon \mathbf{E}$
	$\mathbf{B} = \mu \mathbf{H}$

In EM Fields (340) and in the beginning of EM Waves (341), wave propagation effects were assumed to be negligible. For example, when discussing the magnetic field due to a changing current, it was assumed the field appears identically everywhere at the same instant. This cannot be so. EM waves travel fast, but their speed is finite.

For the past couple of weeks in EM waves, we've been discussing electromagnetic waves. When does the transition occur? When must one begin accounting for wave phenomena? What are the effects?

These important questions are answered by quasi-statics, which is the study of the nether region between statics and waves.

Quasi-Statics ☼

Maxwell's equations can be expanded in a power series in ω (about $\omega = 0$). The first two terms (the static solution, and the 1^{st}-order term) are often referred to as the quasi-static solution.

There are several motivations for developing the quasi-static technique:

1) finding the solution to Maxwell's equations can be a daunting task, and, using the techniques that have been developed for the quasi-static field, we can use solutions from statics to approximate the solution for the dynamic situations,

2) what is meant by "low frequency" can be defined.

3) the emerging effects seen when leaving the "low frequency" regime can be found.

Power series expansion of Maxwell's equations ☼

$$\mathbf{B} = \mathbf{B}\Big|_{\omega=0} + \omega\frac{\partial \mathbf{B}}{\partial \omega}\Big|_{\omega=0} + \frac{\omega^2}{2!}\frac{\partial^2 \mathbf{B}}{\partial \omega^2}\Big|_{\omega=0} + \cdots = \sum_{n=0}^{\infty}\frac{\omega^n}{n!}\frac{\partial^n \mathbf{B}}{\partial \omega^n}\Big|_{\omega=0} = \sum_{n=0}^{\infty}\omega^n \mathbf{B}_n$$

Faraday's law, $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \rightarrow \quad \nabla \times \tilde{\mathbf{E}} = -j\omega\, \tilde{\mathbf{B}}$

Looking at the two sides of the equation for Faraday's law, consider them in the form of power series in ω. Since each power of ω is independent, once can write a series of equations, one for each power of ω.

$$\nabla \times \sum_{n=0}^{\infty}\omega^n\, \tilde{\mathbf{E}}_n = -j\omega \sum_{n=0}^{\infty}\omega^n\, \tilde{\mathbf{B}}_n$$

Since each power of ω is independent, equations for each power must hold. This gives rise sets of equations in the zeroth-order, first-order, etc.

In the frequency-domain,

$$\nabla \times \omega^n \tilde{\mathbf{E}}_n = -j\omega\omega^{n-1}\tilde{\mathbf{B}}_{n-1} \qquad \left(\text{for } n = 0,\ \nabla \times \tilde{\mathbf{E}}_0 = 0\right)$$

In the time-domain, $\nabla \times \mathbf{E}_n = -\frac{\partial \mathbf{B}_{n-1}}{\partial t} \qquad \left(\text{for } n = 0,\ \nabla \times \mathbf{E}_0 = 0\right)$

zeroth-order

coefficients of ω^0

$\nabla \times \mathbf{E}_0 = 0$

$\nabla \times \mathbf{H}_0 = \mathbf{J}_0$

$\nabla \cdot \mathbf{D}_0 = \rho_0$

$\nabla \cdot \mathbf{B}_0 = 0$

$\nabla \cdot \mathbf{J}_0 = 0$

first-order

coefficients of ω^1

$\nabla \times \mathbf{E}_1 = -\frac{\partial \mathbf{B}_0}{\partial t}$

$\nabla \times \mathbf{H}_1 = \mathbf{J}_1 + \frac{\partial \mathbf{D}_0}{\partial t}$

$\nabla \cdot \mathbf{D}_1 = \rho_1$

$\nabla \cdot \mathbf{B}_1 = 0$

$\nabla \cdot \mathbf{J}_1 = -\frac{\partial \rho_0}{\partial t}$

second-order

coefficients of ω^2

$\nabla \times \mathbf{E}_2 = -\frac{\partial \mathbf{B}_1}{\partial t}$

$\nabla \times \mathbf{H}_2 = \mathbf{J}_2 + \frac{\partial \mathbf{D}_1}{\partial t}$

$\nabla \cdot \mathbf{D}_2 = \rho_2$

$\nabla \cdot \mathbf{B}_2 = 0$

$\nabla \cdot \mathbf{J}_2 = -\frac{\partial \rho_1}{\partial t}$

boundary conditions

$\mathbf{a}_n \times \left(\mathbf{E}_1 - \mathbf{E}_2\right)_m = 0$

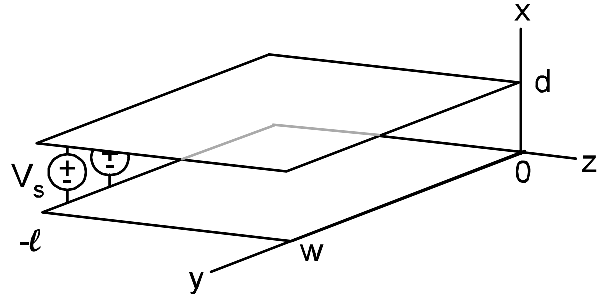
$\mathbf{a}_n \times \left(\mathbf{H}_2 - \mathbf{H}_2\right)_m = \mathbf{K}_m$

$\mathbf{a}_n \cdot \left(\mathbf{D}_2 - \mathbf{D}_2\right)_m = \rho_{sm}$

$\mathbf{a}_n \cdot \left(\mathbf{B}_2 - \mathbf{B}_2\right)_m = 0$

Capacitor: quasi-static solution ☼

parallel plate model with distributed source along y at z = $-\ell$ $\frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0$



0th-order fields ☼

$$\nabla \times \mathbf{E}_0 = \left(\frac{\partial E_{0z}}{\partial y} - \frac{\partial E_{0y}}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial E_{0x}}{\partial z} - \frac{\partial E_{0z}}{\partial x}\right)\mathbf{a}_y + \left(\frac{\partial E_{0y}}{\partial x} - \frac{\partial E_{0x}}{\partial y}\right)\mathbf{a}_z = 0$$

$$\nabla \bullet \mathbf{D}_0 = \nabla \bullet \varepsilon_o \mathbf{E}_0 = \varepsilon_o \left(\frac{\partial E_{0x}}{\partial x} + \frac{\partial E_{0y}}{\partial y} + \frac{\partial E_{0z}}{\partial z}\right) = \rho_{0v}$$

$$\nabla \times \mathbf{H}_0 = \left(\frac{\partial H_{0z}}{\partial y} - \frac{\partial H_{0y}}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial H_{0x}}{\partial z} - \frac{\partial H_{0z}}{\partial x}\right)\mathbf{a}_y + \left(\frac{\partial H_{0y}}{\partial x} - \frac{\partial H_{0x}}{\partial y}\right)\mathbf{a}_z = \mathbf{J}_o$$

$$\nabla \bullet \mathbf{B}_0 = \nabla \bullet \mu_o \mathbf{H}_0 = \mu_o \left(\frac{\partial H_{0x}}{\partial x} + \frac{\partial H_{0y}}{\partial y} + \frac{\partial H_{0z}}{\partial z}\right) = 0$$

1st-order fields ☼

$$\nabla \times \mathbf{E}_1 = \left(\frac{\partial E_{1z}}{\partial y} - \frac{\partial E_{1y}}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial E_{1x}}{\partial z} - \frac{\partial E_{1z}}{\partial x}\right)\mathbf{a}_y + \left(\frac{\partial E_{1y}}{\partial x} - \frac{\partial E_{1x}}{\partial y}\right)\mathbf{a}_z = -\mu_o \frac{\partial \mathbf{H}_0}{\partial t}$$

$$\nabla \bullet \mathbf{D}_1 = \nabla \bullet \varepsilon_o \mathbf{E}_1 = \varepsilon_o \left(\frac{\partial E_{1x}}{\partial x} + \frac{\partial E_{1y}}{\partial y} + \frac{\partial E_{1z}}{\partial z}\right) = \rho_{1v}$$

$$\nabla \times \mathbf{H}_1 = \left(\frac{\partial H_{1z}}{\partial y} - \frac{\partial H_{1y}}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial H_{1x}}{\partial z} - \frac{\partial H_{1z}}{\partial x}\right)\mathbf{a}_y + \left(\frac{\partial H_{1y}}{\partial x} - \frac{\partial H_{1x}}{\partial y}\right)\mathbf{a}_z = \mathbf{J}_1 + \varepsilon_o \frac{\partial \mathbf{E}_0}{\partial t}$$

$$\nabla \bullet \mathbf{B}_1 = \nabla \bullet \mu_o \mathbf{H}_1 = \mu_o \left(\frac{\partial H_{1x}}{\partial x} + \frac{\partial H_{1y}}{\partial y} + \frac{\partial H_{1z}}{\partial z}\right) = 0$$

2nd-order fields ☼

$$\nabla \times \mathbf{H}_2 = \left(\frac{\partial H_{2z}}{\partial y} - \frac{\partial H_{2y}}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial H_{2x}}{\partial z} - \frac{\partial H_{2z}}{\partial x}\right)\mathbf{a}_y + \left(\frac{\partial H_{2y}}{\partial x} - \frac{\partial H_{2x}}{\partial y}\right)\mathbf{a}_z = \mathbf{J}_2 + \varepsilon_o \frac{\partial \mathbf{E}_1}{\partial t}$$

$$\nabla \bullet \mathbf{B}_2 = \nabla \bullet \mu_o \mathbf{H}_2 = \mu_o \left(\frac{\partial H_{2x}}{\partial x} + \frac{\partial H_{2y}}{\partial y} + \frac{\partial H_{2z}}{\partial z}\right) = 0$$

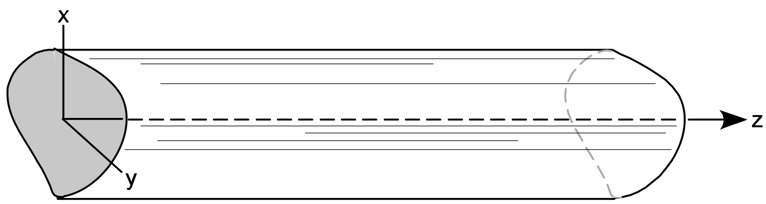
$$\nabla \times \mathbf{E}_2 = \left(\frac{\partial E_{2z}}{\partial y} - \frac{\partial E_{2y}}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial E_{2x}}{\partial z} - \frac{\partial E_{2z}}{\partial x}\right)\mathbf{a}_y + \left(\frac{\partial E_{2y}}{\partial x} - \frac{\partial E_{2x}}{\partial y}\right)\mathbf{a}_z = -\mu_o \frac{\partial \mathbf{H}_1}{\partial t}$$

$$\nabla \bullet \mathbf{D}_2 = \nabla \bullet \varepsilon_o \mathbf{E}_2 = \varepsilon_o \left(\frac{\partial E_{2x}}{\partial x} + \frac{\partial E_{2y}}{\partial y} + \frac{\partial E_{2z}}{\partial z}\right) = \rho_{2v}$$

capacitor: quasi-static solution and 2^{nd}-order corrections ☼

Transmission lines ☼

Transmission lines are a special case of solutions to Maxwell's equations when the properties of the physical system do not change in one direction (typically taken to be the z-direction).



Maxwell in the frequency domain (current and charge-free)

$$\nabla \times \tilde{\mathbf{E}} = -j\omega\tilde{\mathbf{B}}$$
$$\nabla \times \tilde{\mathbf{H}} = j\omega\tilde{\mathbf{D}}$$
$$\nabla \cdot \tilde{\mathbf{D}} = 0$$
$$\nabla \cdot \tilde{\mathbf{B}} = 0$$

Solutions are solutions to the Helmholtz wave equation

$$\left(\nabla^2 + \omega^2\mu\varepsilon\right)\begin{Bmatrix}\tilde{\mathbf{E}} \\ \tilde{\mathbf{H}}\end{Bmatrix} = 0$$

Plane waves are solutions.

$$\left.\begin{matrix}\tilde{\mathbf{E}}(x,y,z,\omega) \\ \tilde{\mathbf{H}}(x,y,z,\omega)\end{matrix}\right\} = \left\{\begin{matrix}\tilde{\mathbf{E}}(x,y,\omega)e^{\pm j\beta z} \\ \tilde{\mathbf{H}}(x,y,\omega)e^{\pm j\beta z}\end{matrix}\right.$$

$$\left(\nabla_t^2 + \omega^2\mu\varepsilon - \beta^2\right)\begin{Bmatrix}\tilde{\mathbf{E}} \\ \tilde{\mathbf{H}}\end{Bmatrix} = 0 \qquad \text{where } \nabla_t^2 = \nabla^2 - \frac{\partial^2}{\partial z^2}$$

TEM solutions exist in transmission lines (coaxial lines, stripline, twin-lead) for which the $\mathbf{a}_z$ component of **E** and **H** are zero. The microstrip transmission line carries waves that are TEM to a good approximation.

TEM waves

The fields are of the form

$$\left.\begin{matrix}\tilde{\mathbf{E}}_t(x,y,z,\omega)\\ \tilde{\mathbf{H}}_t(x,y,z,\omega)\end{matrix}\right\} = \left\{\begin{matrix}\tilde{\mathbf{E}}_t(x,y,\omega)e^{\pm j\beta z}\\ \tilde{\mathbf{H}}_t(x,y,\omega)e^{\pm j\beta z}\end{matrix}\right.$$

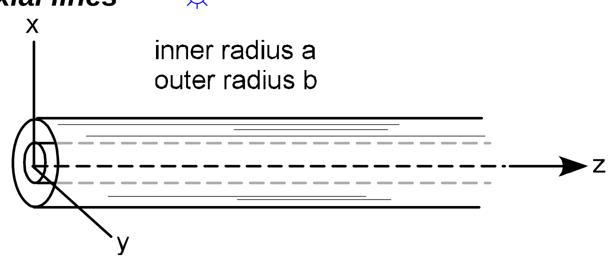
The key is that the Helmholtz equation splits into two pieces

$$\left(\nabla_t^2 + \omega^2\mu\varepsilon - \beta^2\right)\begin{Bmatrix}\tilde{\mathbf{E}}_t\\ \tilde{\mathbf{H}}_t\end{Bmatrix} = 0 \qquad \text{where } \nabla_t^2 = \nabla^2 - \frac{\partial^2}{\partial z^2}$$

$$\left.\begin{matrix}\nabla_t^2\tilde{\mathbf{E}}_t\\ \nabla_t^2\tilde{\mathbf{H}}_t\end{matrix}\right\} = 0 \qquad \text{and} \qquad \left(\omega^2\mu\varepsilon - \beta^2\right)\begin{Bmatrix}\tilde{\mathbf{E}}_t\\ \tilde{\mathbf{H}}_t\end{Bmatrix} = 0$$

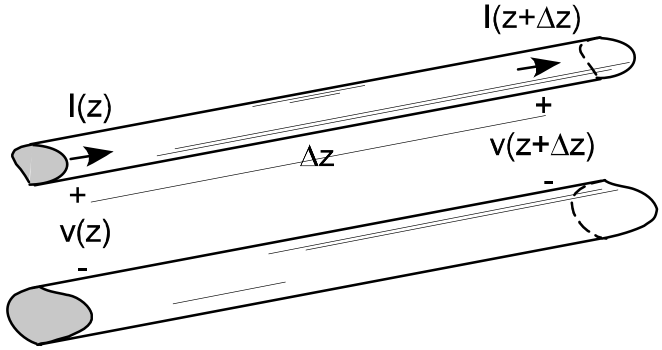
The result is that the form of the waves in transverse directions are solutions are static solutions ($\tilde{\mathbf{E}}_t$ and $\tilde{\mathbf{H}}_t$ are solutions to Laplace).

Example 1: *coaxial lines*



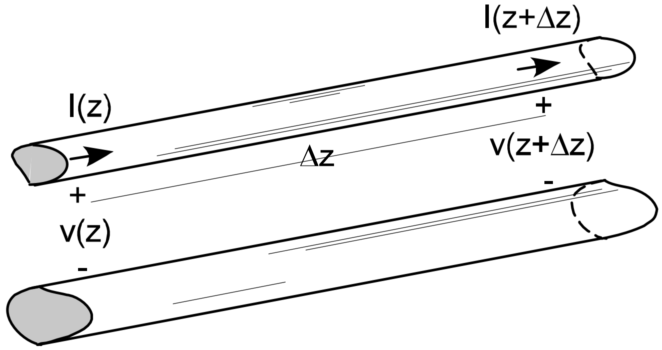
Example 1: *coaxial lines (continued)*

Voltage & Current Waves: *Telegrapher's equations* ☼



From Faraday

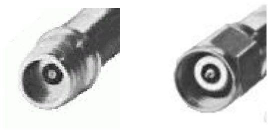
Voltage & Current Waves: *Telegrapher's equations* ☼



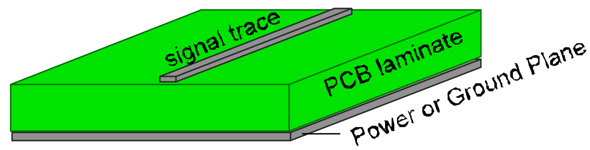
From charge conservation

Coaxial lines: Source for table is E. Kuester, U. of Colorado (2000)

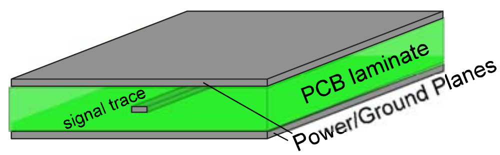
Type	Graphic	f_{max}	Notes
Phone jacks	 	100 KHz	Phone jacks and plugs are also known as TS (Tip-Sleeve) for two-conductor connections or TRS (Tip-Ring-Sleeve) for three-conductor connections. They are widely used with musical instruments and audio equipment. These are really only suitable for audio frequencies.
RCA jacks	 	10 MHz	A round, press-on connector commonly used for consumer-grade audio and composite video connections. The jacks are sometimes color-coded: red (audio-right), black or white (audio-Left) and yellow (composite video). Generally not a constant characteristic impedance connector.
F (video)	 	250 MHz to 1 GHz	The "F" series connectors are primarily utilized in television cable and antenna applications. Normally these are used at 75 ohm characteristic impedance. 3/8-32 coupling thread is standard, but push-on designs are also available.
BNC	 	2 GHz or higher	Bayonet Neil-Concelman or British Navy Connector. The BNC is used in video and RF applications to 2 GHz. Above 4 GHz, the slots may radiate signals, so the connector is usable, but not necessarily mechanically stable up to about 10 GHz. Both 50 ohm and 75 ohm versions are available.
Type N	 	12 GHz or higher	The N-type connector was designed by Paul Neill of Bell Laboratories, and offers high performance RF performance with a constant impedance. The connector has a threaded coupling interface to ensure that it mates correctly to provide the optimum performance. Both 50 and 75 ohm versions are available, the 50 ohm version being by far the most widely used. T The connector able to withstand relatively high powers when compared to the BNC or TNC connectors. The standard versions are specified for operation up to 11 GHz, although precision versions are available for operation to 18GHz. Type N uses an internal gasket to seal out the environment, and is hand tightened. There is an air gap between center and outer conductor. A 75 ohm version, with a reduced center pin is available and is used by the cable-TV industry.
SMA, 3.5 mm, or APC-3.5	 	12 GHz or higher	The SMA (Subminiature A) connector is intended for use on semi-rigid cables and in components which are connected infrequently. It takes the cable dielectric directly to the interface without air gaps. A few hundred interconnect cycles are possible if performed carefully. Care should be taken to join connectors straight-on.
APC-7 or 7 mm		18 GHz	The APC-7 (Amphenol Precision Connector - 7 mm) offers the lowest reflection coefficient and most repeatable measurement of all 18 GHz connectors. These connectors are designed to perform repeatably for thousands of interconnect cycles as long as the mating surfaces are kept clean.

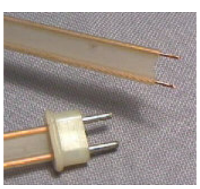
2.4mm		50 GHz and higher	This design eliminates the fragility of the SMA and 2.92-mm connectors by increasing the outer wall thickness and strengthening the female fingers. The inside of the outer conductor is 2.4 mm in diameter, and the outside is 4.7 mm. Because they are not mechanically compatible with SMA, 3.5-mm and 2.92-mm, precision adapters are required in order to mate to those types. The 2.4-mm product is offered in three quality grades; general purpose, instrument, and metrology. General purpose grade is intended for economy use on components, cables and microstrip, where limited connections and low repeatability is acceptable. The higher grades are appropriate for their respective applications.

Microstrip: Widely used in PCBs.
Resource: http://qucs.sourceforge.net/tech/node44.html



Stripline: Widely used in PCBs. Stripline provides some self-shielding.
Resource:
http://obiwan.cs.ndsu.nodak.edu/~ekhan/mes/programs/strip.htm

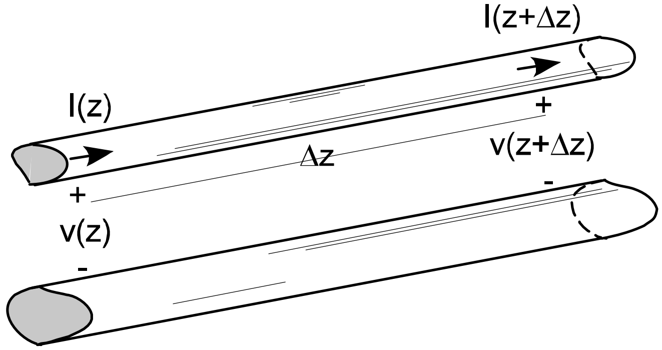


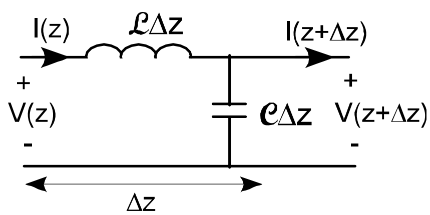


Twin lead: Twin lead characteristic impedance is commonly 300 Ω.
Resource: http://www.technick.net/public/code/cp_dpage.php?aiocp_dp=util_inductance_wire_2

All kinds of applets on transmission lines, waves etc.
http://www.educypedia.be/electronics/javatransmissinlines.htm

Voltage & Current Waves: Telegrapher's equations ☼





Voltage and current are related via the Telegrapher's equations

$$\frac{\partial \tilde{V}}{\partial z} = -j\omega\mathcal{L}\ \tilde{I} \qquad\qquad \frac{\partial \tilde{I}}{\partial z} = -j\omega\mathcal{C}\ \tilde{V}$$

$$\frac{\partial^2 \tilde{V}}{\partial z^2} = -j\omega\mathcal{L}\frac{\partial \tilde{I}}{\partial z} = -j\omega\mathcal{L}\left(-j\omega\mathcal{C}\tilde{V}\right)$$

$$\frac{\partial^2 \tilde{V}}{\partial z^2} + \omega^2\mathcal{L}\mathcal{C}\ \tilde{V} = 0 \qquad\qquad (V = V^+e^{j\varphi^+}e^{-j\beta z} + V^-e^{j\varphi^-}e^{j\beta z} = \tilde{V}^+e^{-j\beta z} + \tilde{V}^-e^{j\beta z})$$

$$\frac{\partial^2 \tilde{I}}{\partial z^2} = -j\omega\mathcal{C}\frac{\partial \tilde{V}}{\partial z} = -j\omega\mathcal{C}\left(-j\omega\mathcal{L}\ \tilde{I}\right)$$

$$\frac{\partial^2 \tilde{I}}{\partial z^2} + \omega^2\mathcal{L}\mathcal{C}\ \tilde{I} = 0 \qquad\qquad (\tilde{I} = \frac{\tilde{V}^+}{Z_o}e^{-j\beta z} - \frac{\tilde{V}^-}{Z_o}e^{j\beta z})$$

These are both wave equations with $\beta = \omega\sqrt{\mathcal{L}\mathcal{C}}$

Although technically NOT a general result, for transmission lines this formula can typically be used since most transmission lines, if practical, ARE low-loss. The low-loss case gives this result.

.

Consider a voltage wave traveling in the +z direction and determine the relation between voltage and current.

This relation defines the **characteristic impedance**.
http://obiwan.cs.ndsu.nodak.edu/~ekhan/mes/programs/strip.htm

Lossy Transmission Lines ☼

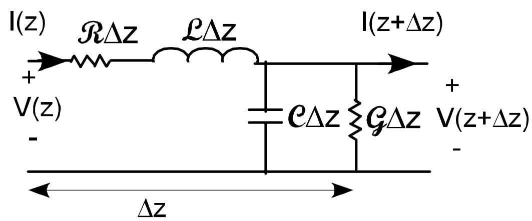
Real transmission lines are lossy. First, there is loss due to the current which is modeled as a series resistance/meter.

For coaxial lines, this resistance can be found using techniques already developed.

$$\mathscr{R} = \frac{1}{\sigma\delta}\left(\frac{1}{P_{inner}} + \frac{1}{P_{outer}}\right)$$

The other loss is associated with the voltage between conductors and is due to dielectric loss.

This loss is modeled as a conductance/meter.



circuit model for lossy line

Physical basis for $\mathscr{G}$

Review of mechanism for dielectric current

loss due to radiation

loss due to lattice coupling

Transmission line equations ☼

$$\frac{\partial \tilde{V}}{\partial z} = -(\mathscr{R} + j\omega\mathscr{L})\tilde{I} \qquad \frac{\partial \tilde{I}}{\partial z} = -(\mathscr{G} + j\omega\mathscr{C})\tilde{V}$$

$$\frac{\partial^2 \tilde{V}}{\partial z^2} = -(\mathscr{R} + j\omega\mathscr{L})\frac{\partial \tilde{I}}{\partial z} = (\mathscr{R} + j\omega\mathscr{L})(\mathscr{G} + j\omega\mathscr{C})\tilde{V}$$

Defining impedance/meter and admittance/meter.

$$\tilde{\mathscr{Z}} = \mathscr{R} + j\omega\mathscr{L} \qquad \tilde{\mathscr{Y}} = \mathscr{G} + j\omega\mathscr{C}$$

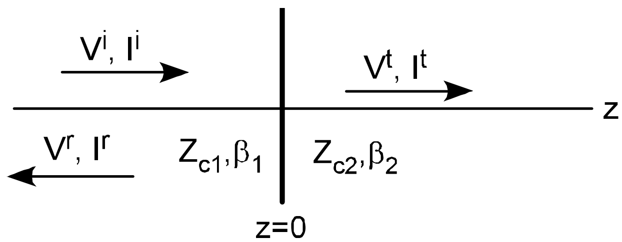
The following form is obtained.

$$\frac{\partial^2 V}{\partial z^2} = \tilde{\mathscr{Z}}\tilde{\mathscr{Y}}\, V$$

$$\tilde{\gamma} = \sqrt{\tilde{\mathscr{Z}}\tilde{\mathscr{Y}}} \qquad \tilde{Z}_C = \sqrt{\frac{\tilde{\mathscr{Z}}}{\tilde{\mathscr{Y}}}} = \sqrt{\frac{\mathscr{R} + j\omega\mathscr{L}}{\mathscr{G} + j\omega\mathscr{C}}}$$

Low-loss $(\mathscr{R} << \omega\mathscr{L} \qquad \mathscr{G} << \omega\mathscr{C})$

Reflected Waves ☼



$$\Gamma = \left.\frac{V^r}{V^i}\right|_{z=0} \qquad T = \left.\frac{V^t}{V^i}\right|_{z=0}$$

The “boundary conditions” are KVL and KCL

$$\Gamma = \frac{Z_{c2} - Z_{c1}}{Z_{c2} + Z_{c1}} \qquad T = \frac{2Z_{c2}}{Z_{c2} + Z_{c1}}$$

Lumped-element loads ☼

The reflection coefficient would be unchanged if, instead of a second medium with characteristic impedance Z_{c2}, there were a lumped element with this same impedance, $Z_L = Z_{c2}$.

Matched load ($Z_L = Z_{c1}$)

$\Gamma = 0$, no reflections.

Short circuit ($Z_L = 0$)

$\Gamma = -1$, reflected voltage cancels incident voltage ($V_1 = 0$)

Open circuit ($Z_L = \infty$)

$\Gamma = 1$, total voltage is twice incident voltage

(Or, why you don't want to be at the end of the power line in an area with frequent lightning storms)

Γ for a passive load is a complex number with a magnitude less than or equal to one.

Transient waves on lossy lines ☼

When loss must be taken into account, high frequencies see a different environment due to the frequency dependence of the loss mechanism (skin effect and dielectric loss). This complicates the analysis for the transient solution.

The most important things here are its **qualitative characteristics** and the **physical reasons** for these characteristics.

Dispersion and attenuation ☼

In general, α and v_p are functions of ω in a lossy transmission line.

It is true that, in the low-loss approximation, v_p is not a function of ω, but, strictly speaking, v_p is a function of ω, even in the low-loss case (it's a second-order effect in the low-loss case).

Dispersion occurs when v_p is a function of ω. If one looks sufficiently close, v_p is almost always dependent on ω. Aside from effects due to loss, $\mathcal{C}$ and $\mathcal{L}$ vary with frequency, or, more fundamentally, ε and μ vary with frequency.

For transmission lines, α typically increases with ω (R_{ac} increases as ω increases)

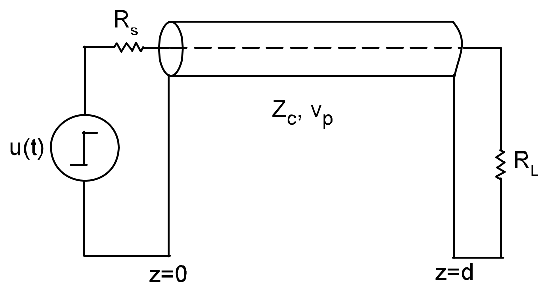
From Fourier analysis, the harmonic functions form a complete basis set, and we can use superposition to construct any waveform. That is, we can expand any signal in terms of harmonic functions.
Looking at the problem in this light, there are two things going one simultaneously: 1^{st}, the time-delay suffered by components varies with their frequency, and 2^{nd}, the high-frequency components of waveforms suffer greater attenuation as they propagate.

The net result is that waves (any wave that is not a pure sinusoid) changes shape as it propagates. Sharp corners become rounded, rise times become longer, fall times lengthen.

Transient Waves on lossless lines ☼

With no loss, all harmonics experience the same propagation environment.

The result is that waves retain their shape as they propagate.

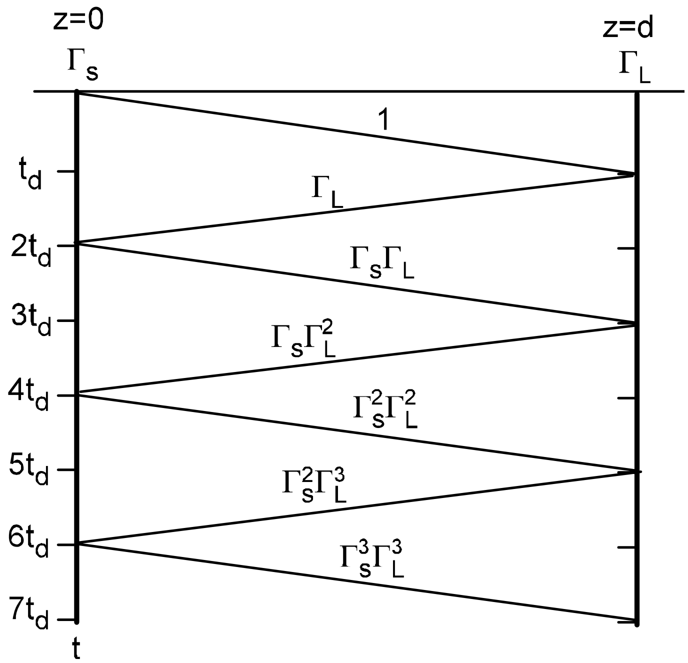


To track the transient response in this case, two reflection coefficients can be defined—Γ_s (at the source end), and Γ_L (at the load end).

There is also a time delay associated with propagation, $t_d = d/v_p$.

Bounce diagrams ☼

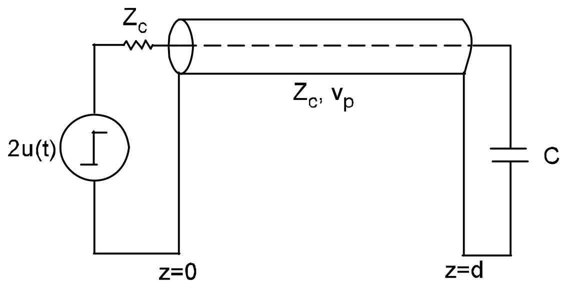
This process is often analyzed using bounce diagrams.



Example

Transient Waves on lossless lines ☼

How are voltage waves reflected when transmission lines are terminated in reactive elements?



For this case, the capacitor, the reflection will vary with time. Initially when the positively-traveling voltage wave encounters the capacitance, it will be uncharged.

At that instant, the capacitor will act as a short circuit, so will require the sum of the incident wave and reflected wave—at that instant—to be zero. For $t=t_d$, $\Gamma_{cap}=-1$.

For long times, the capacitor will be fully charged and so act as an open circuit. For large t, $\Gamma_{cap}=1$.

How does the reflection coefficient move between these two points?

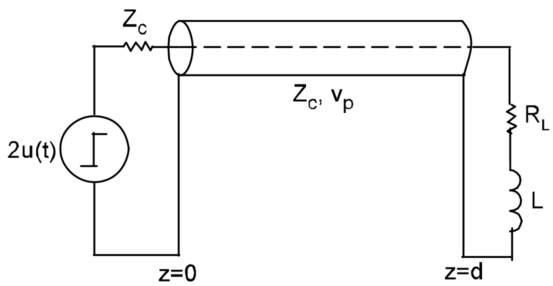
Answer: It moves between them as a first-order circuit with a time constant $\tau = Z_cC$.

Analysis

What would the situation if the line were terminated with an inductance?

Example ☼

Find the reflected wave, in the time-domain, at z=d.



Time-domain reflectometry ☼

TDR is a widely-used experimental probe to investigate material properties and transmission line discontinuities.

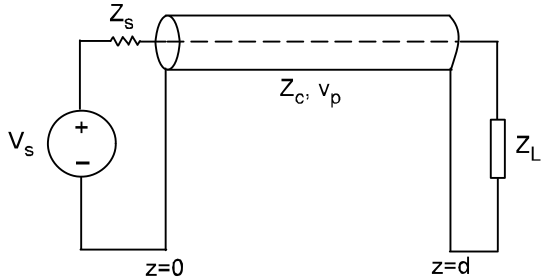
Although workers in the field have developed many specialized techniques, the basic idea is simple. The standard experiment in TDR for transmission lines is to inject a pulse and observe the reflected wave. For meaningful location data to be obtained with TDR, time must be measured very accurately.

The available information includes 1) location of discontinuity, and 2) the character of the discontinuity.

Example

A 10 volt pulse is introduced into a 75Ω transmission line having $v_p = 10^8$ m/s. 11.3 μs later, a minus two (-2) volt pulse is observed at the TDR unit. What information is available about the discontinuity and its location?

Steady-State Waves ☼



The source introduces a positively traveling wave of amplitude and phase $\tilde{V}_0$.
The frequency-domain components (sinusoidal steady-state) are:

$$\tilde{V}_0^+(z) = \tilde{V}_0 e^{-j\beta z}$$

$$\tilde{V}_1^+(z) = \tilde{V}_0 e^{-j\beta z}\tilde{\Gamma}_L\tilde{\Gamma}_s e^{-j2\beta d}$$

$$\vdots$$

$$\tilde{V}_n^+(z) = \tilde{V}_0 e^{-j\beta z}\left(\tilde{\Gamma}_L\tilde{\Gamma}_s e^{-j2\beta d}\right)^n$$

$$\tilde{V}^+(z) = \sum_{n=0}^{\infty}\tilde{V}_n^+ = \sum_{n=0}^{\infty}\tilde{V}_0 e^{-j\beta z}\left(\tilde{\Gamma}_L\tilde{\Gamma}_s e^{-j2\beta d}\right)^n = \tilde{V}_0 e^{-j\beta z}\sum_{n=0}^{\infty}\left(\tilde{\Gamma}_L\tilde{\Gamma}_s e^{-j2\beta d}\right)^n = \frac{\tilde{V}_0 e^{-j\beta z}}{1 - \tilde{\Gamma}_L\tilde{\Gamma}_s e^{-j2\beta d}}$$

$$\tilde{V}^+(z) = \frac{\tilde{V}_0}{1 - \tilde{\Gamma}_L\tilde{\Gamma}_s e^{-j2\beta d}} e^{-j\beta z} = \tilde{V}^+ e^{-j\beta z}$$

Similarly, it can be shown that

$$\tilde{V}^-(z) = \tilde{V}^+\tilde{\Gamma}_L e^{-j2\beta d} e^{j\beta z}$$

The ratio between the positively-going voltage wave to the negatively-going voltage wave, as a function of z is

$$\tilde{\Gamma}(z) = \tilde{\Gamma}_L e^{-j2\beta d} e^{j2\beta z} \qquad \tilde{\Gamma}_{in} = \tilde{\Gamma}(z=0) = \tilde{\Gamma}_L e^{-j2\beta d}$$

What does this mean?

It means the ratio of the voltage going in the +z direction to that going in the –z direction is constant in magnitude but that their relative phases vary as z varies (we're neglecting loss here). This should not be surprising—after all, the two voltage waves are traveling in opposite directions.

$$\tilde{V}(z) = \tilde{V}^+(z) + \tilde{V}^-(z) = \tilde{V}^+\left(e^{-j\beta z} + \tilde{\Gamma}_L e^{-j2\beta d} e^{j\beta z}\right)$$

$$\tilde{I}(z) = \tilde{I}^+(z) - \tilde{I}^-(z) = \frac{\tilde{V}^+}{Z_c}\left(e^{-j\beta z} - \tilde{\Gamma}_L e^{-j2\beta d} e^{j\beta z}\right)$$

$$\tilde{Z}(z) = \frac{\tilde{V}(z)}{\tilde{I}(z)} = \frac{\tilde{V}^+\left(e^{-j\beta z} + \tilde{\Gamma}_L e^{-j2\beta d} e^{j\beta z}\right)}{\frac{\tilde{V}^+}{Z_c}\left(e^{-j\beta z} - \tilde{\Gamma}_L e^{-j2\beta d} e^{j\beta z}\right)} = Z_c \frac{e^{-j\beta z} + \tilde{\Gamma}_L e^{-j2\beta d} e^{j\beta z}}{e^{-j\beta z} - \tilde{\Gamma}_L e^{-j2\beta d} e^{j\beta z}} = Z_c \frac{e^{-j2\beta z} + \tilde{\Gamma}_L e^{-j2\beta d}}{e^{-j2\beta z} - \tilde{\Gamma}_L e^{-j2\beta d}}$$

$$\tilde{Z}_{in} = \tilde{Z}(z=0) = Z_c \frac{1 + \tilde{\Gamma}_L e^{-j2\beta d}}{1 - \tilde{\Gamma}_L e^{-j2\beta d}}$$

What are some consequences for this expression for $\tilde{Z}_{in}$?

It means, for instance, that a short circuit ($\tilde{\Gamma}_L$ = -1) can look like a open circuit when $e^{-j2\beta d}e^{j2\beta z}$ = -1, that is when $2\beta(z\text{-}d) = 2(2\pi/\lambda)(z\text{-}d) = -\pi$.

The input impedance for a transmission line of length $\lambda/4$, when terminated with a **short** circuit looks like an **open** circuit!

The input impedance for a transmission line of length $\lambda/2$, when terminated with a **short** circuit looks like a **short** circuit!

Similarly,

The input impedance for a transmission line of length $\lambda/4$, when terminated with an **open** circuit looks like a **short** circuit!

The input impedance for a transmission line of length $\lambda/2$, when terminated with an **open** circuit looks like an **open** circuit!

VSWR

Just as with fields, there are standing waves present anytime there are reflections. The voltage standing wave ratio (VSWR).

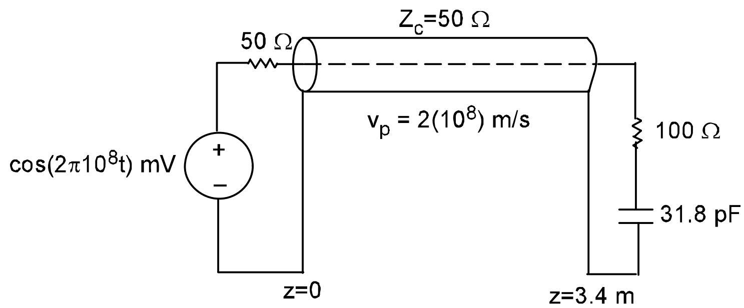
$\tilde{V}^+(z) = \tilde{V}^+e^{-j\beta z}$ $\qquad$ $\tilde{V}^-(z) = \tilde{V}^+\tilde{\Gamma}_L e^{-j2\beta d}e^{j\beta z}$

$$\text{VSWR} = \frac{V_{max}}{V_{min}}$$

$$\text{VSRW} = \frac{1 + \left|\tilde{\Gamma}_L\right|}{1 - \left|\tilde{\Gamma}_L\right|}$$

Voltage minima, spaced $\lambda/2$ apart, become more pronounced and narrower as $|\tilde{\Gamma}_L|$ grows. (see page 52 in text).

Steady-State Voltages and Currents ☼



Find β, λ, d, $\tilde{Z}_L$, $\tilde{Z}_{in}$, $\tilde{\Gamma}_{in}$, $\tilde{\Gamma}_L$, $\tilde{V}_{in}$, $\tilde{I}_{in}$, $\tilde{V}_L$, $\tilde{I}_L$, P_{in}, P_L

Transmission Line Examples

A 1 volt drop occurs across a load of $\tilde{Z}_L$ = 130 + j80 Ω which is connected to a 12.7 m length of lossless transmission line. The line has Z_C = 53 Ω and v_P = 250 m/μs; the signal generator has a Thevénin resistance R_S = 100 Ω and operates at 250 MHz. Calculate

i) the power delivered to the load.

ii) the transmission line input reflection coefficient.

iii) the transmission line input impedance.

iv) the transmission line input voltage.

v) the open-circuit voltage of the generator.

vi) the power delivered by the generator to the input of the transmission line.

Example ☼

i) Calculate the input impedance of an open-circuited transmission line of $\lambda/8$ length.

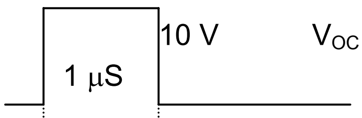
ii) Calculate the input impedance of a short-circuited transmission line of $\lambda/8$ length.

iii) Calculate the input impedance of a $3\lambda/8$ line with Z_C=50 Ω which is terminated by a load of $\tilde{Z}_L$ =100+j150 Ω.

iv) A 50 Ω transmission line of length 0.225 λ has an input impedance of $\tilde{Z}_{in}$ = 75 - j75 Ω. Calculate the load impedance.

Example ☼

A generator with V_{OC}=10 V and R_{th}=50 Ω is used to test various transmission line/load combinations. The generator is connected to the input to the line and the input voltage is observed on an oscilloscope. The oscilloscope patterns for several tests are shown below. Fill in the table.

	Z_C	Z_L what is the load?