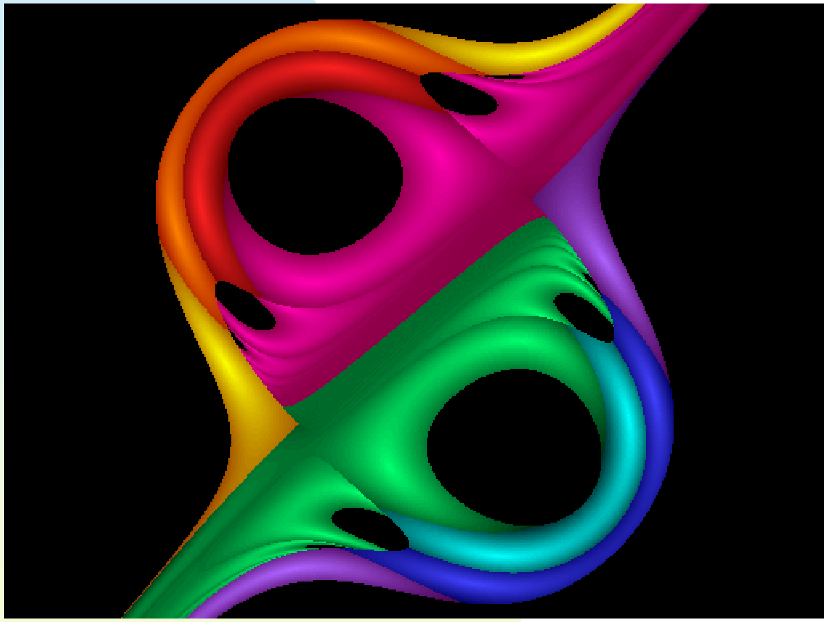
Session overview



- Strange attractors
- Please turn in *Controlling Chaos* explorations
- HW4 (Complex Number review) due Monday
  - Reference: http://www.clarku.edu/~djoyce/complex/

Application

- So far we've considered 1D dynamical systems
- What if we generalized to 2D, and considered *dissipative* dynamical systems?
- Applied to fluids, brain activity, meteorology, chemical reactions
- Yet little is understood of them

How cool is chaos?

"Never in the annals of science and engineering has there been a phenomenon so ubiquitous, a paradigm so universal, or a discipline so multidisciplinary as that of chaos. Yes chaos represents only the tip of an awesome iceberg, far beneath it lies a much finer structure of immense complexity, a geometric labyrinth of endless convolutions, and a surreal landscape of enchanting beauty. The bedrock which anchors these local and global bifurcation terrains is the omnipresent nonlinearity that was once wantonly linearized by the engineers and applied scientists of yore, thereby forfeiting their only chance to grapple with reality.

Leon Chua, quoted in PJS, ch 12 introduction

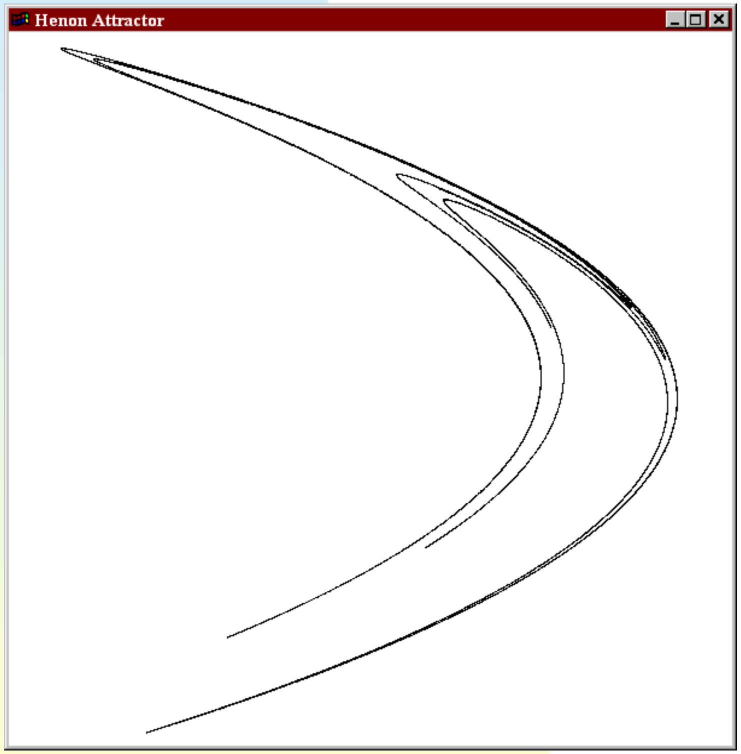
A dynamical system in two dimensions

- Consider a two-dimensional dynamical system consisting of the following steps:
  - Bend up - a non-linear bending in the *y*-coordinate given by $H_1(x, y) = (x, y + 1 - ax^2)$
  - Contract in *x* - a contraction in the *x*-direction given by $H_2(x, y) = (bx, y)$
  - Reflect - a reflection across the diagonal, given by $H_3(x, y) = (y, x)$
- The resulting system is $H(x, y) = H_3(H_2(H_1(x,y)) =$ ________________

What does this look like?

- Download from Angel and run *HenonAttractor.cpp*.

Hénon's Attractor



- $H(x,y)=(y + 1 - ax^2, bx)$
- Let $a = 1.4$ and $b = 0.3$ in $H(x, y) = (y + 1 - ax^2, bx)$
- Let the initial points of the orbit be (0, 0)
- The resulting image is shown to the left
- The region shown is $-1.5 \leq x \leq 1.5$, $-0.4 \leq y \leq 0.4$

What about other initial points?

- Do next quiz question now: describe the orbit of (1.5,0)

Not all points are attracted

- Consider the orbit of (1.5, 0):
  - (1.5, 0)
  - (-2.15, 0.45)
  - (-5.0215, -0.645)
  - (-34.94664715, -1.50645)
  - …
- Clearly this orbit is going to $-\infty$

The trapping region

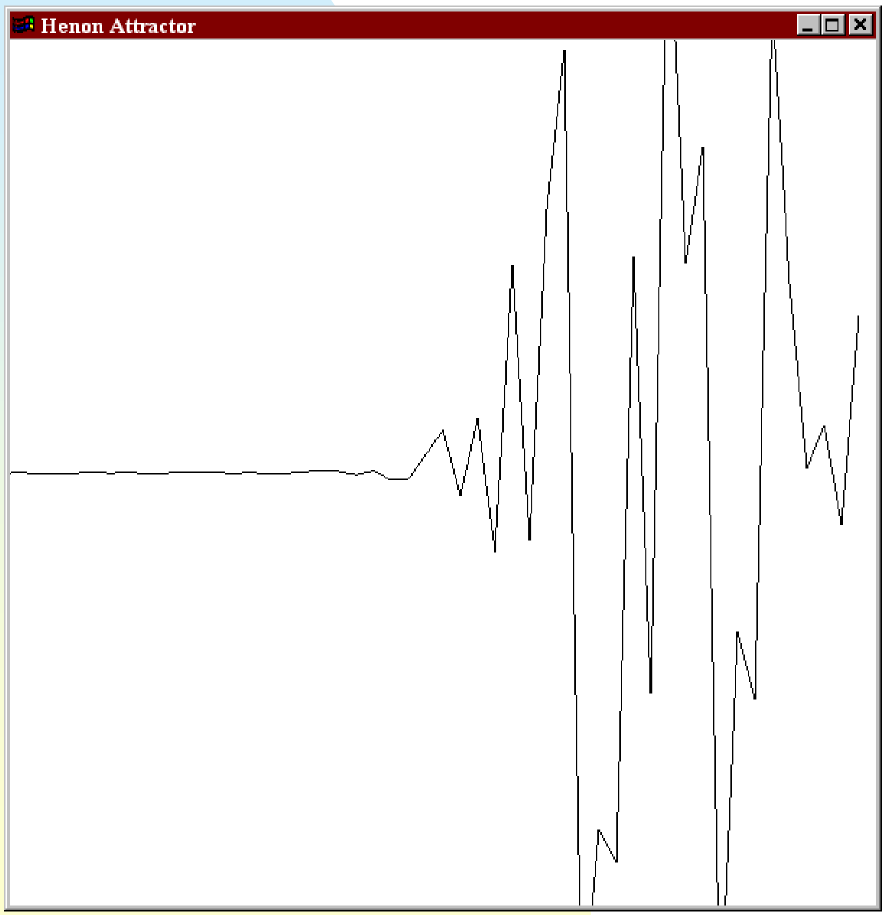
- Even though many orbits escape to infinity, we can still speak of an attractor
- There is a trapping region, *R*, from which no orbit can escape
  - It is a quadrilateral with vertices (-1.33, 0.42), (1.32, 0.133), (1.245, -0.14), (-1.06, -0.5)
- The image of *R* under *H* lies entirely within *R*
- Thus, repeated application of *H* must always produce subsets of the region and thus no orbits can escape

Basin of attraction

- Are there points outside the trapping region for which their orbits are eventually caught by the trapping region?
- The set of all points in the plane eventually caught by the trapping region is called the *basin of attraction*
- The trapping region itself is contained in the basin of attraction

- How does this relate to what we know about chaos and fractals?

Sensitivity to initial conditions

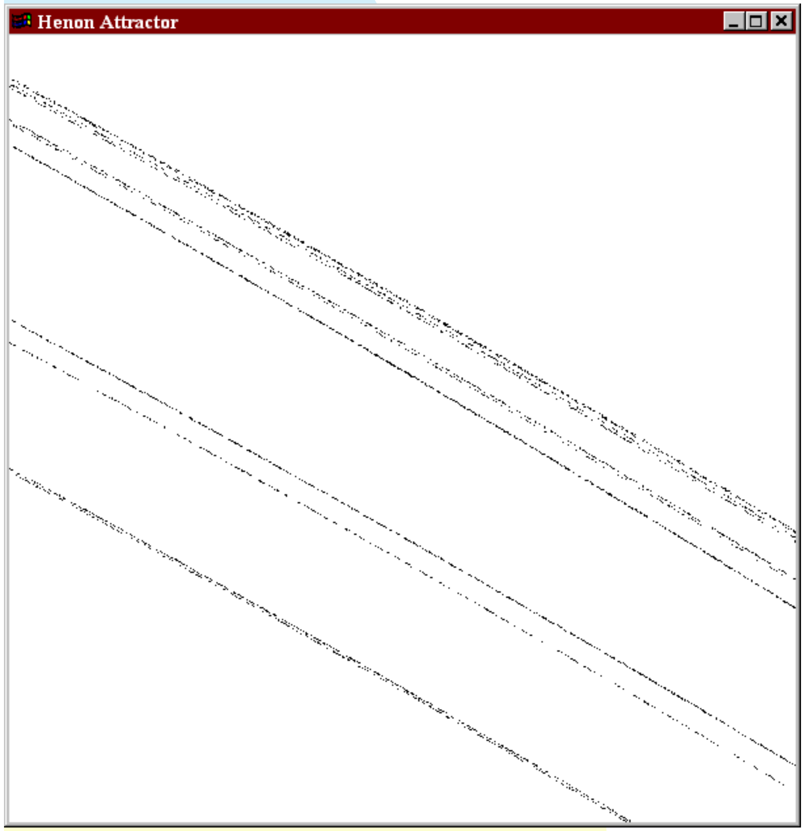


- Once we get onto the attractor, the orbits of two different points will have the same limit set
- However, the orbits follow very different paths
  - One exception:
- The plot to the left is the difference between the first 50 *x* values of the orbits of (0, 0) and (0.00001, 0)
- The vertical range is [-1.5, 1.5], so note that the difference is as big as the orbit itself

Fractal nature

- Zoom in on the parabola
- Describe what you see.

Fractal nature



- The Hénon attractor is a fractal
- Zooming in on a region of the attractor shows additional curves, not visible before
- The region at the left is $0.7 \leq x \leq 0.8$ and $0.15 \leq y \leq 0.18$

What is the area of the Henon attractor?

- 1. How much does the area of the trapping region change after one iteration of it
  - Recall that the factor by which the area changes after undergoing an affine transformation specified by a matrix M is |det(M)|

What is the area of the Henon attractor?

- 2. So after n iterations, the area is scaled by a factor of __________
- 3. As n→inf, then
- 4. This is Cantor-like; Fractal dimension ~ 1.28

Fixed points?

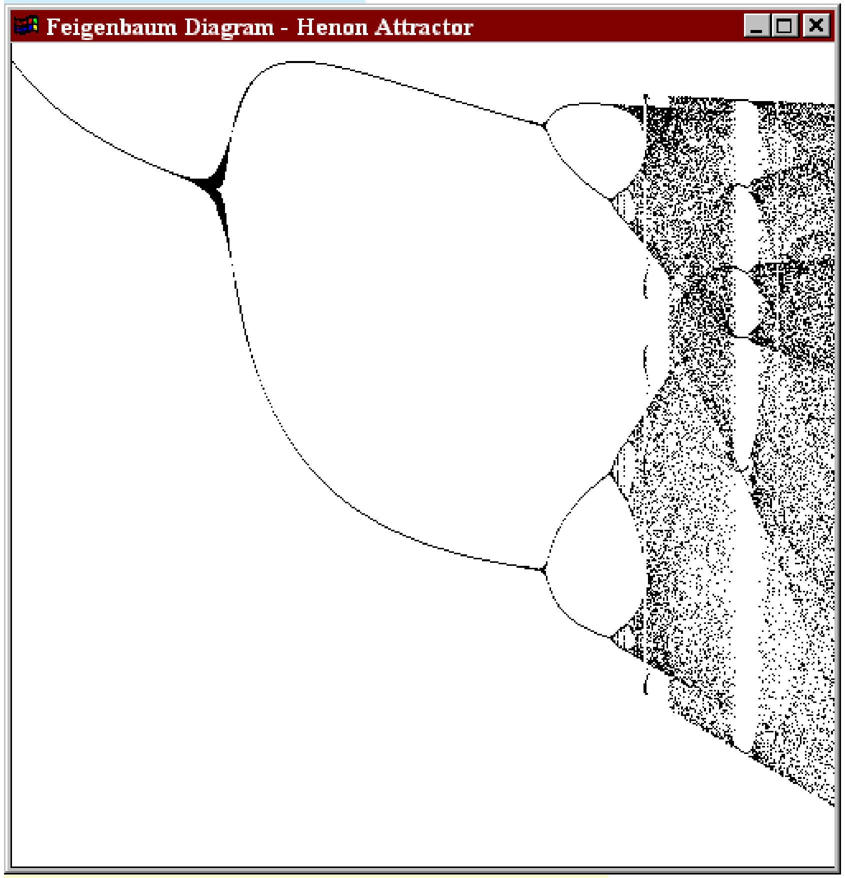
- Solve $x = 1 + y - ax^2$

$$y = bx$$

$$x_{1,2} = \frac{b - 1 \pm \sqrt{(b-1)^2 + 4a}}{2a}$$

$$y = bx_{1,2}$$

The Feigenbaum diagram



- The Hénon attractor exhibits period doubling behavior
- The Feigenbaum diagram at the left is for b=0.3, $0<a<1.4$, and $-1.5<x<1.5$

Properties of strange attractors

- Let $T(x, y)$ be a transformation in the plane
- A bounded subset A of the plane is a chaotic and strange attractor for T if there exists a set R with the following properties:
  - Attractor - R is a neighborhood of A. R is a trapping region. Each orbit in R remains in R for all iterations. Moreover, the orbit becomes close to A and stays as close to it as we desire. Thus, A is an *attractor*.

Properties of strange attractors (cont.)

- Sensitivity - Orbits started in *R* exhibit sensitive dependence on initial conditions. This makes *A* a *chaotic attractor*.
- Fractal - The attractor has a fractal structure and is therefore called a *strange attractor*.
- Mixing - *A* cannot be split into two different attractors. There are initial points in *R* with orbits that get arbitrarily close to any point of *A*