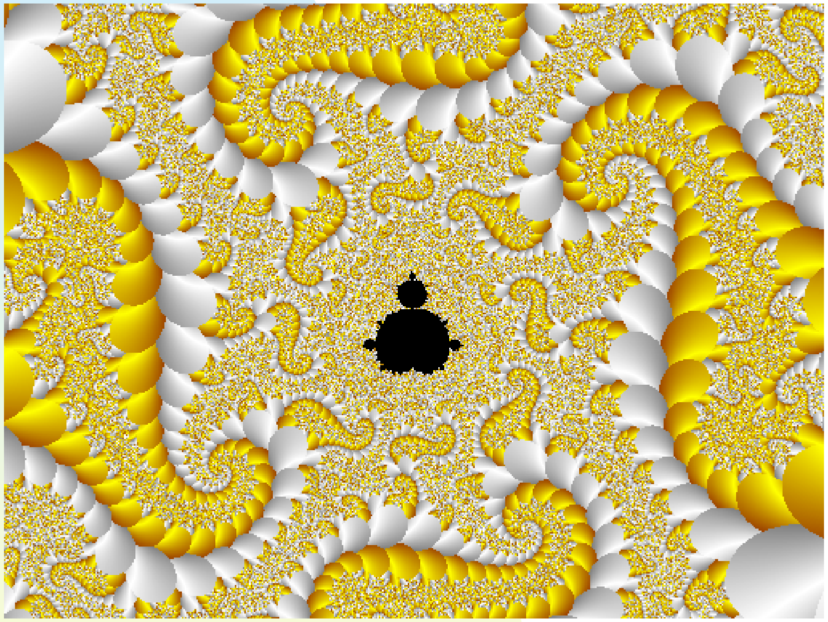
Session overview



- Attraction and repulsion

Characterization of Linear Systems

- Last question on quiz. Answers?

Attracting fixed points

- x_0 is an ***attracting fixed point*** if there exists an interval I (containing x_0) $\ni F^n(x) \in I \ \forall\ n > 0$ and $x \in I$, and $F^n(x) \to x_0$ as $n \to \infty$

Repelling fixed points

- x_0 is a ***repelling fixed point*** if $\forall$ small intervals I (containing x_0) there exists $x \in I$ and $N > 0 \ni F^n(x) \notin I \ \forall \ n > N$

Neutral fixed points

- x_0 is a ***neutral fixed point*** if there exists an interval I (containing x_0) $\ni F^n(x) \in I \ \forall\ n > 0$ and ($x \in I$ but $F^n(x)$ does not approach x_0 as $n \to \infty$

Example

- For the linear map, $F(x) = ax + b$:
  - x_0 is attracting when $|a| < 1$
  - x_0 is repelling when $|a| > 1$
  - x_0 is neutral when $|a| = 1$

Linearization

- Let $\varepsilon > 0$ be a small number
- Let F(x) be differentiable on the interval $I = (x_0 - \varepsilon, x_0 + \varepsilon)$
- Then $F(x) \approx F(x_0) + F'(x_0)(x - x_0)$ for $x \in I$
- In other words, *all smooth curves look linear if looked at up close*

Local representation

- Suppose x_0 is a fixed point of F
- We can represent the smooth map locally on I as $F(x) \approx F'(x_0)x + F(x_0) - x_0F'(x_0)$
  - associate m with $F'(x_0)$
  - associate b with $F(x_0) - x_0F'(x_0)$
- Recall that
  - $|m| < 1 \Rightarrow$ ____________,
  - $|m| > 1 \Rightarrow$ ____________,
- If $|F'(x_0)| < 1$ then x_0 is an attracting fixed point, and if $|F'(x_0)| > 1$ then x_0 is a repelling fixed point

Periodic points

- Periodic points are also classified as attracting or repelling
- Suppose x_0 is a periodic point of period p
- If x_0 is an attracting fixed point of F^p, then…
  - x_0 is an attracting periodic point of period p
- Similarly for repelling and neutral

Slopes of periodic points

- $|(F^p)'(x_0)| < 1 \Rightarrow$
- x_0 is an attracting fixed point of $F^p \Rightarrow$
- x_0 is an attracting periodic point of F

- Similarly, $|(F^p)'(x_0)| > 1 \Rightarrow x_0$ is a repelling periodic point of F

How do you compute $(F^p)'(x_0)$?

- (Chain Rule; on board)

How do you compute $(F^p)'(x_0)$?

- Use the chain rule:

  $(F \circ G)'(x_0) = F'(G(x_0))G'(x_0)$

- So $(F^2)'(x_0) = F'(F(x_0))F'(x_0) = F'(x_1)F'(x_0)$
- …
- $(F^n)'(x_0) =$

  $F'(x_{n-1})F'(x_{n-2})\ldots F'(x_1)F'(x_0)$

Example

- $F(x) = -x^3$
- $x_0 = 1$ is a period-2 point
  - its orbit is { 1, -1, 1, -1, … }
- Is this point attracting or repelling?

Quiz

- Analyze the logistic map, $f(x) = ax(1-x)$
- More interesting than the linear map