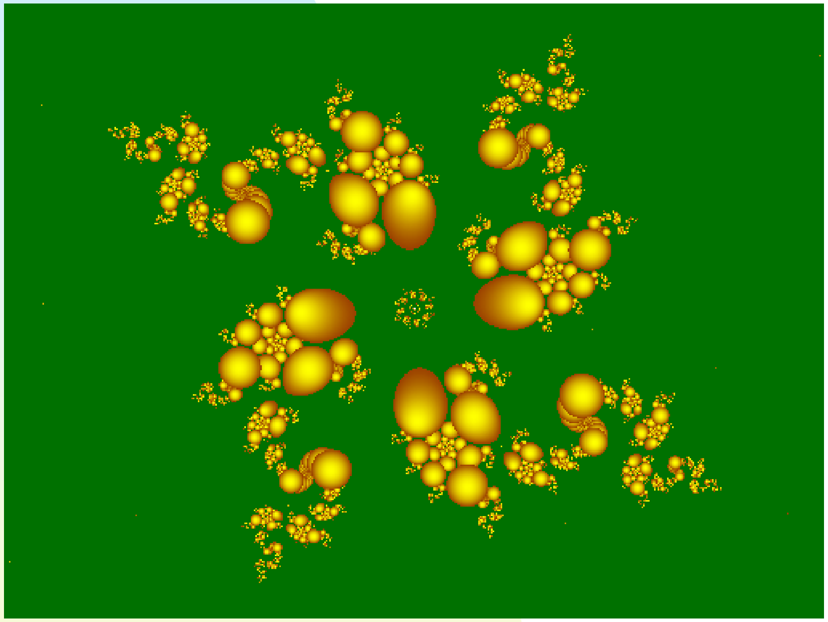
Session overview



- Next 2 weeks: Chaos
- Today: Dynamical systems and orbits
- Announcements:
  - All grades should be up-to-date on Angel
  - Let me know if not

Feedback on Feedback

- See summary

Example: Iteration with calculator keys

- Consider the square root key on a calculator
- Start with a number, press the square root key, record the number, press the square root key again, record the new number, press the square root key again, …
- Continue in this fashion

One set of data

x	√x	x	√x
.2	.447214	.999214	.999607
.447214	.668740	.999607	.999804
.668740	.817765	.999804	.999902
.817765	.904304	.999902	.999951
.904304	.950949	.999951	.999975
.950949	.975166	.999975	.999988
.975166	.987505	.999988	.999994
.987505	.993733	.999994	.999997
.993733	.996862	.999997	.999998
.996862	.998430	.999998	.999999
.998430	.999214	.999999	1

Dynamical systems

- A dynamical system consists of a rule and a set
- Example:
  - rule: square root
  - set: $[0, \infty)$
- We are interested in the long term behavior of the system
- Let $F(x) = \sqrt{x}$. We call $F(x)$ a ***map*** (from $[0, \infty) \to [0, \infty)$)

Orbits

- Given any x_0 in the domain of F, the sequence of points { x_0, $F(x_0)$, $F(F(x_0))$, $F(F(F(x_0)))$, … } is called the ***orbit*** of x_0 under F
- The initial point, x_0, is sometimes called the ***seed***
- For the example worked out earlier, the orbit would be

$$\left\{ .2, \sqrt{.2}, \sqrt{\sqrt{.2}}, \sqrt{\sqrt{\sqrt{.2}}}, \ldots \right\}$$

Notation

- $x_0 = F^0(x_0) = x_0$
- $x_1 = F^1(x_0) = F(x_0)$
- $x_2 = F^2(x_0) = F(x_1) = F(F(x_0))$
- $x_3 = F^3(x_0) = F(x_2) = F(F(F(x_0)))$
- …

Quiz

- Describe qualitatively all orbits of $F(x) = \sqrt{x}$

Fixed points

- If $F(x_0) = x_0$ then x_0 is a ***fixed point***
- The orbit is $\{ x_0, x_0, x_0, \ldots \}$
- For $F(x) = \sqrt{x}$,

  there are two fixed points:

  $x_0 = 0$

  $x_0 = 1$

Periodic points

- If $F^n(x_0) = x_0$ for n > 1, but not for n = 1, then x_0 is a ***periodic point***
- The smallest n for which $F^n(x_0) = x_0$ is the ***period***
- Example:
  - $F(x) = (7/2)x(1-x)$
  - $x_0 = 3/7$ is a period 2 periodic point
  - Orbit: { 3/7, 6/7, 3/7, 6/7, 3/7, … }

Quiz

- Find all fixed and period 2 points of $F(x) = (7/2)x(1-x)$
  - Fixed:
    - set $F(x) = x$ and solve
  - Period 2:
    - set $F(F(x)) = x$ and solve

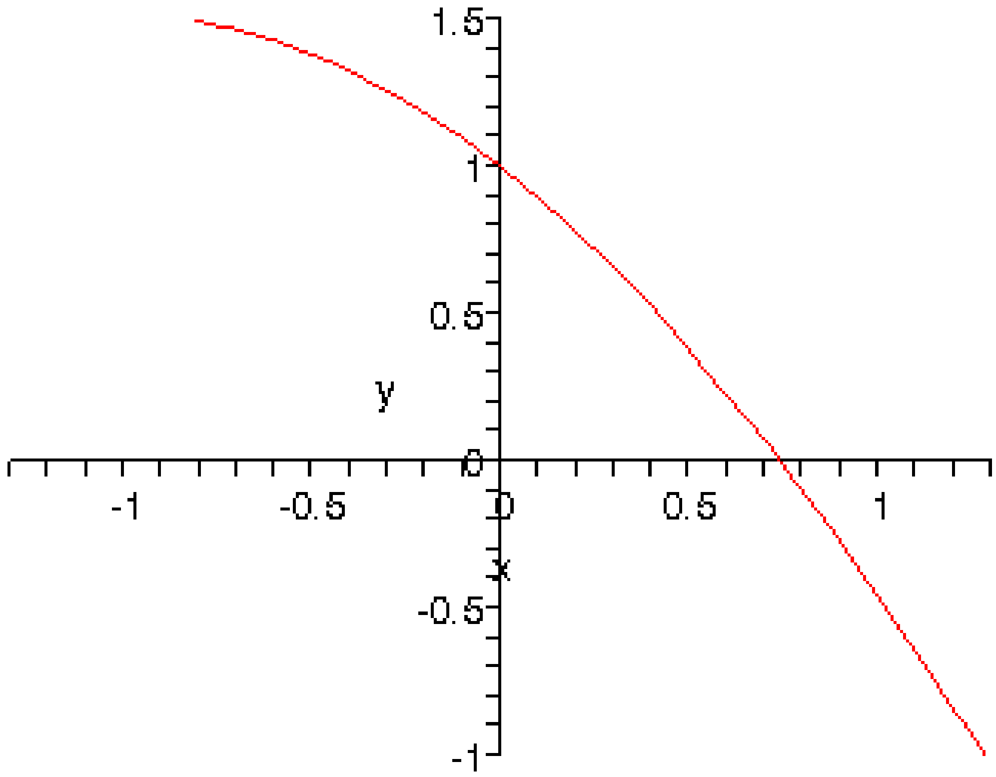
Newton's method

- Used to solve equations in the form $f(x) = 0$
- Newton's method is really a dynamical system
- Example: solve $\cos x = x$
- Let $f(x) = \cos x - x$
- Guess x_0 (close to what you believe the answer to be)
- $x_1 = x_0 - f(x_0)/f'(x_0)$
- $x_2 = x_1 - f(x_1)/f'(x_1)$
- …
- $x_{n+1} = x_n - f(x_n)/f'(x_n)$

Orbit of $x_0 = 1$

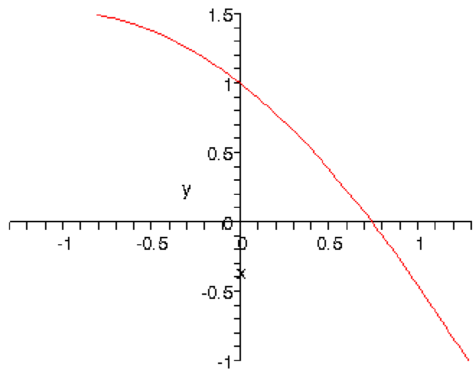
- $x_0 = 1$
- $x_1 = 0.75036\ 38678$
- $x_2 = 0.73911\ 28909$
- $x_3 = 0.73908\ 51334$
- $x_4 = 0.73908\ 51332$
- $x_5 = 0.73908\ 51332$
- Note that $x_5 = x_4$ to 10 decimal places
- Thus, 0.73908 51332 is a fixed point

Graph of cos x - x

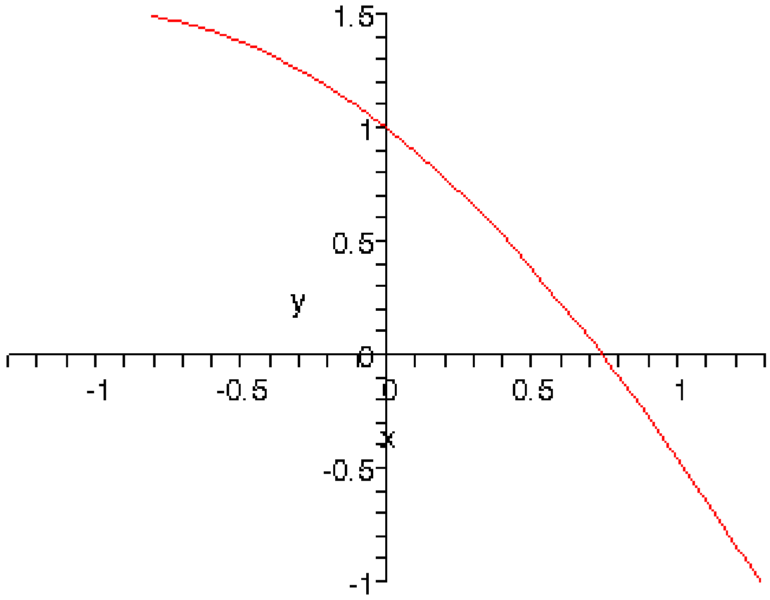


How Newton's method works

- Guess x_0 close to $\underline{x}$ (the solution)
- Go vertically from x_0 on x-axis to $f(x_0)$
- Follow the tangent line to $f(x)$ at $(x_0, f(x_0))$ back to the x-axis; call where it hits x_1
- Repeat the previous two steps until either:
  - completed too many loops (diverges)
  - $|x_n - x_{n-1}| < \text{tolerance}$ (converges)



Derivation of Newton's method (on board)

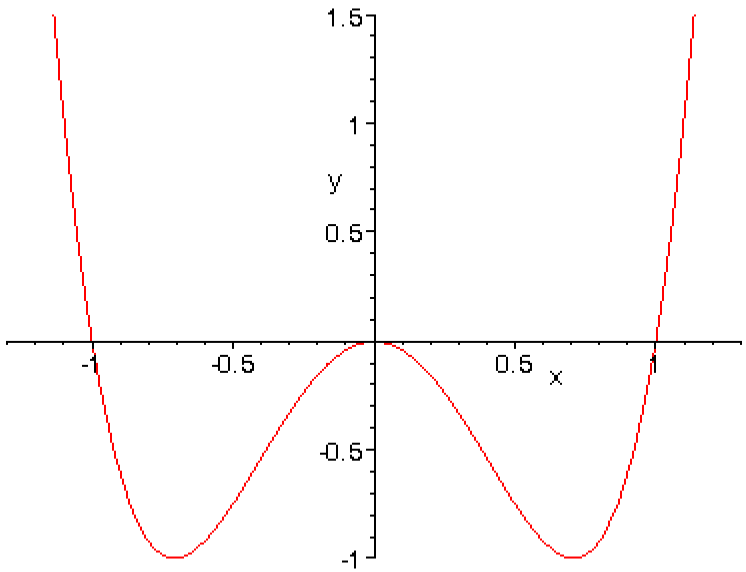


Derivation of Newton's method

- The line through $(x_0, f(x_0))$ of slope $f'(x_0)$ is:

  $$y - f(x_0) = f'(x_0)\,(x - x_0)$$
- This hits the x-axis at $(x_1, 0)$
- $0 - f(x_0) = f'(x_0)\,(x_1 - x_0)$
- $- f(x_0) / f'(x_0) = x_1 - x_0$
- $x_1 = x_0 - f(x_0) / f'(x_0)$

Quiz



- Study all possible orbits of Newton's method for $f(x) = 4x^4 - 4x^2$
- Are there any "bad" initial values for x_0?