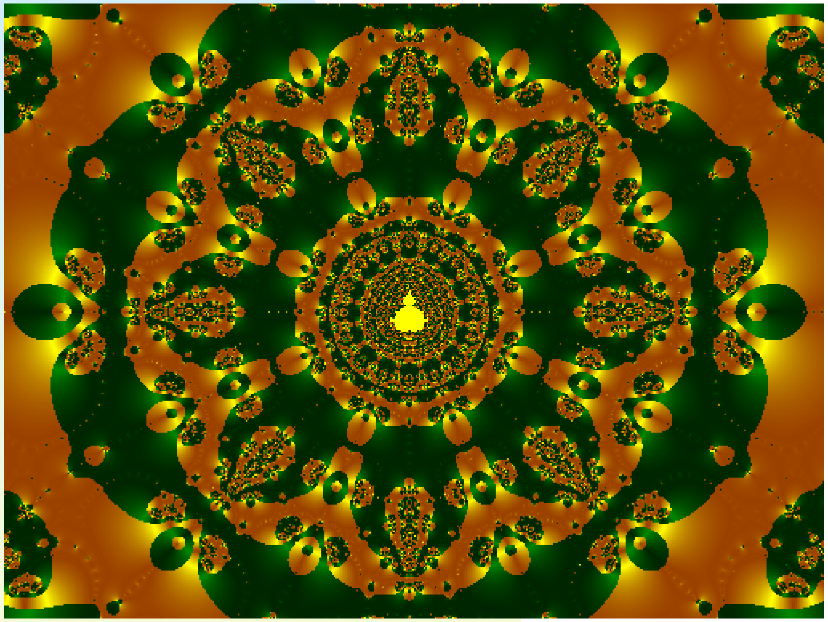
Session overview



- Review of vectors
- Similarities in affine maps
- Announcements:
  - Project 1 due tomorrow
  - Homework 1 due Thurs.

Wrapping up Hausdorff dimension

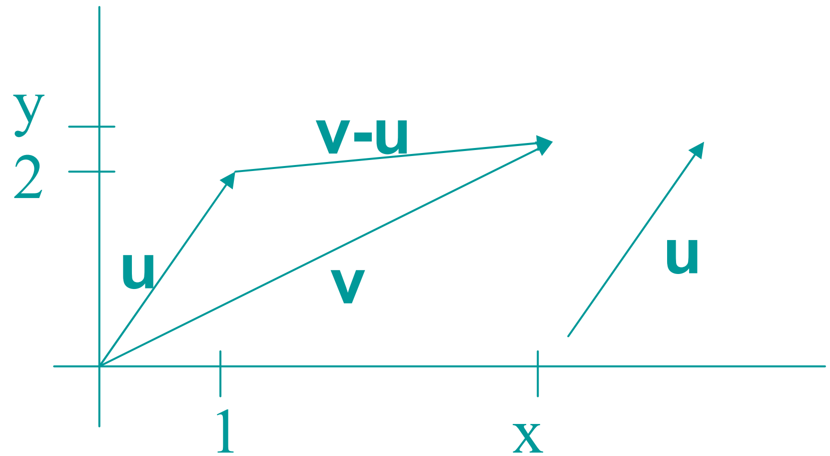
- Recall Hausdorff dimension of:
  - Koch curve: log(4)/log(3)
  - Sierpinski Gasket: log(3)/log(2)
  - Middle-thirds Cantor set: log(2)/(log(3)
- Familiar?

- The Hausdorff dimension is **equivalent** to the self-similarity dimension for the case of self-similar fractals

Definition of fractal and fractal dimension

- Mandelbrot coined the term *fractal* in 1977
- A set $X \subseteq \Re^n$ is said to be a ***fractal*** if its Hausdorff dimension, D_H, strictly exceeds its topological dimension, D_T
- The number D_H is the ***fractal dimension*** of the set

Vectors in $\Re^2$



- $\mathbf{u} = (1\ 2)^T$, $\mathbf{v} = (x\ y)^T$
- Since **u** + (**v-u**) = **v**, we know which way **v-u** points
- Remember, we usually put tails of vectors at the origin, but we don't have to
- A vector is defined by only two things: direction and length
- Both vectors named **u** above are identical

Direction

- Two non-zero vectors **u** and **v** are in the same direction if **u** = k**v** for some positive scalar k
- **u** = $(1\ 2)^T$ and **v** = $(3\ 6)^T$ are in the same direction
- **u** and **w** = $(-2\ -4)^T$ are on the same line but in opposite directions
- **u** and **x** = $(1\ 3)^T$ are in different directions since there is no k for which **u** = k**x**

Length and distance

- The length of a vector **u** = (u_1 u_2) is

$$\|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2}$$

- The distance between two vectors is the length of their difference:

$$\|\mathbf{u} - \mathbf{v}\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2}$$

Similarity

- A map w(**u**):$\Re^2 \rightarrow \Re^2$ is a ***similarity*** (on $\Re^2$) if for every **u** and **v** $\in \Re^2$, ‖w(**u**)-w(**v**)‖ = s‖**u**-**v**‖ for some constant scalar, s, called the ***similarity constant***

Example 1

- The scale and translate affine map is a similarity

$$w\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$$

- s, e, f are any real constants
- **u**, **v** are any vectors in $\Re^2$

$$\left\| w(\mathbf{u}) - w(\mathbf{v}) \right\| = \left\| \begin{pmatrix} s(u_1 - v_1) \\ s(u_2 - v_2) \end{pmatrix} \right\| = s\left\| \mathbf{u} - \mathbf{v} \right\|$$

- In this case, s is the similarity constant

Example 2

- The rotation and translate affine map is a similarity

$$w\begin{pmatrix}u_1\\u_2\end{pmatrix}=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix}+\begin{pmatrix}e\\f\end{pmatrix}$$

$$\|w(\mathbf{u})-w(\mathbf{v})\|=\left\|\begin{pmatrix}(\cos\theta u_1-\sin\theta u_2)-(\cos\theta v_1-\sin\theta v_2)\\(\sin\theta u_1+\cos\theta u_2)-(\sin\theta v_1+\cos\theta v_2)\end{pmatrix}\right\|$$

$$=\left\|\begin{pmatrix}\cos\theta(u_1-v_1)-\sin\theta(u_2-v_2)\\ \sin\theta(u_1-v_1)+\cos\theta(u_2-v_2)\end{pmatrix}\right\|$$

$$=\sqrt{\cos^2\theta(u_1-v_1)^2+\sin^2\theta(u_2-v_2)^2+\sin^2\theta(u_1-v_1)^2+\cos^2\theta(u_2-v_2)^2}$$

$$=\sqrt{(u_1-v_1)^2[\cos^2\theta+\sin^2\theta]+(u_2-v_2)^2[\cos^2\theta+\sin^2\theta]}$$

$$=\sqrt{(u_1-v_1)^2+(u_2-v_2)^2}$$

$$=\|\mathbf{u}-\mathbf{v}\|$$

A composition

- Note that we can combine a scale, a rotation, and a translation into one affine map that is a similarity

$$w\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} s\cos\theta & -s\sin\theta \\ s\sin\theta & s\cos\theta \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$$

Exercise

- Determine whether the reflection and translation map is a similarity

$$w\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$$

Not all affine maps are similarities

- Non-uniform scaling is not a similarity ($s_1 \neq s_2$)

$$w\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} s_1 u_1 \\ s_2 u_2 \end{pmatrix}$$

$$\|w(\mathbf{u}) - w(\mathbf{v})\| = \left\| \begin{pmatrix} s_1 u_1 - s_1 v_1 \\ s_2 u_2 - s_2 v_2 \end{pmatrix} \right\| = \sqrt{s_1^2 (u_1 - v_1)^2 + s_2^2 (u_2 - v_2)^2}$$

- If w is a similarity, there must be a number k satisfying

$$\sqrt{s_1^2 (u_1 - v_1)^2 + s_2^2 (u_2 - v_2)^2} = k\sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2}$$

$$\Leftrightarrow s_1^2 (u_1 - v_1)^2 + s_2^2 (u_2 - v_2)^2 = k^2 (u_1 - v_1)^2 + k^2 (u_2 - v_2)^2$$

$$\Leftrightarrow s_1 = k \,\&\, s_2 = k$$

Exercise on similarity

- Try your skill at determining the number of similarities and the similarity constants for the fractals on the handout
- All the fractals shown can be made by a technique known as a multiple reduction copy machine (MRCM)